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Error Analysis of Legendre-Gauss-Radau Collocation for Linear and Non-linear Volterra Integro-differential Equations of the Second Kind

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Abstract
In this paper, we propose an efficient numerical method for Volterra-type integro-differential equations, based on Legendre-Gauss-Radau interpolation, which is easy to be implemented for both linear and nonlinear problems and possesses the spectral accuracy. Error estimate is conducted to show the spectral accuracy of the method. We also develop a multi-step version of this approach. Numerical results coincide well with the theoretical analysis and demonstrate the effectiveness of these approaches

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1. Introduction

In recent years there has been a growing interest in the Volterra integral equation. Volterra integral equation arises in many physical applications, e.g., potential theory and Dirichlet problems, electrostatics, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat transfer problems [39], [24], [5], [37].

In this paper we are interested in numerically solving Volterra integro-differential equation of the second kind in the following form:

$$\begin{cases} u'(t)+\int_0^t K(t,\tau)u(\tau)d\tau = g(t), & 0<t\le T \\ u(0)=U_0 \end{cases} \tag{1.1}$$

where the source function $g(t)$ and the kernel function $K(t,\tau)$ are given, and $u(t)$ is the unknown function to be determined. A general form of (1.1) reads as:

$$\begin{cases} u'(t)+\int_0^t K(t,\tau)F(u(\tau))d\tau = g(t), & 0<t\le T \\ u(0)=U_0 \end{cases} \tag{1.2}$$

where $F()$ is a nonlinear function with certain smoothness. For the existence and uniqueness of nonlinear integral equation, one can consult with [17], [40].

We will consider the case that the solutions of (1.2)(or linear case (1.1)) are sufficiently smooth - in this case it is necessary to consider very high-order numerical methods such as spectral methods for approximating the solutions. There are many existing numerical methods for solving the Volterra equation and a large variety of numerical methods have been developed to rapidly and accurately obtain approximations to $u(t)$. Overviews and references to the literature for many existing methods are available in [1], [6], [11]. Collocation methods [6], [3], [33], Sinc methods [25], Newton-Gregory methods [11], Runge-Kutta methods [26], [27], qualocation methods [21], [31], [32] are several of the many approaches that have previously been considered. In [23], spectral collocation method was analyzed by authors for linear fractional integro-differential equations.

As we know, spectral method employs global orthogonal polynomials as trial functions. It often provides exceedingly accurate numerical results with relatively less degree of freedoms, and thus has been widely used for scientific computation, see, e.g., Bernardi and Maday [2], Boyd [4], Canuto, Hussaini, Quarteroni and Zang [7], Funaro [14], Gottlieb and Orsag [15], Guo [16], Shen and Tang [28] and Shen, Tang and Wang [29].

There are some publications about spectral for numerical solutions to Volterra integral equation, such as [22], [8], [9], [41]. Recently, a Legendre-Gauss-Radau collocation method is proposed for initial value problems of ordinary differential equations [38]. Motivated by the idea in [38], we propose a Legendre-Gauss-Radau collocation method for Volterra integro-differential equation (1.2) as well as (1.1).

In this paper, we investigate an efficient numerical method for Volterra integro-differential equation of the second kind (1.2) as well as (1.1).By the next section, we derive a new collocation method, which approximates exact solutions directly by Legendre-Gauss-Radau interpolation with $N+1$ nodes. Then we analyze the numerical accuracy in Sect. 3. This process has several fascinating advantages. Firstly, it is easier to be implemented for nonlinear problems. Next, it possesses the spectral accuracy. In other words, for any fixed mode N, the smoother the exact solutions become, the more accurate the numerical results will be. In Sect. 4, we provide the multi-step version of Legendre-Gauss-Radau collocation method. By using this algorithm with moderate mode N, we could evaluate numerical solutions step by step as time increases. This feature simplifies actual calculation and saves much work. In Sect. 5, we present some numerical results and compare the proposed new methods with several commonly used methods. They indicate the high accuracy of suggested algorithms, and coincide well with theoretical analysis. The final section is for some concluding remarks.

2. Legendre-Gauss-Radau Collocation Method

In this section, we derive the Legendre-Gauss-Radau collocation method with the numerical implementation.

Let $L_l(x)$ be the standard Legendre polynomial of degree l. The shifted Legendre polynomials $L_{T,l}(x)$ are defined by:

$$L_{T,l}(t) = L_l\left(\frac{2t}{T}-1\right) = \frac{(-1)^l}{l!}\partial_t^l\left(t^l\left(1-\frac{t}{T}\right)^l\right), \qquad l = 0,1,2,....$$

The properties of the shifted Legendre polynomials can be deduced by using the corresponding properties of the standard Legendre polynomials(cf. [16]).

Denote by $\xi_j, 0 \le j \le N$ the nodes of the standard Legendre-Gauss-Radau interpolation on the interval $[-1,1)$. In particular, $\xi_0 = -1$. The corresponding Christoffel numbers are $\rho_j, 0 \le j \le N$. The nodes of the shifted Legendre-Gauss-Radau interpolation on the interval $[0,T)$ are the distinct zeros of $L_{T,N}(t) + L_{T,N+1}(t)$, denoted by $t_{T,j}^N, 0 \le j \le N$. Clearly,

$$t_{T,j}^N = \frac{T}{2}(\xi_j + 1).$$

The corresponding Christoffel numbers are $\omega_{T,j}^{N} = \frac{T}{2}\rho_j, 0 \le j \le N.$

We need Legendre-Gauss-Radau type quadrature. Let $P_N(0,T)$ be the set of polynomials of degree at most N. The Legendre-Gauss-Radau quadrature, in the shifted case, reads as for any $\psi \in P_{2N}(0,T)$,

$$\int_0^T \psi(t)dt = \sum_{j=0}^{N} \omega_{T,j}^{N} \psi(t_{T,j}^{N}) \tag{2.1}$$

For any $v \in C[0,T)$, the shifted Legendre-Gauss-Radau interpolation $\Pi_N v(t) \in P_N(0,T)$ is determined uniquely by:

$$\Pi_N v(t_{T,j}^{N}) = v(t_{T,j}^{N}), \qquad 0 \le j \le N.$$

For any $K(t,s) \in C(0,T)^2$, notation $\tilde{\Pi}_N^k$ used later means the shifted Legendre-Gauss-Radau interpolation on $[0, t_{T,k}^{N})$ with respect to s, which satisfies that for any $t \in [0,T)$:

$$\tilde{\Pi}_N^k K(t,\cdot) \in P_N(0, t_{T,k}^{N}) \text{ and } \tilde{\Pi}_N^k K(t, t_{t_k,j}^{N}) = K(t, t_{t_k,j}^{N}), \quad j = 0,1,\ldots,N$$

here $t_{t_k,j}^{N}$ is the nodes of the shifted Legendre-Gauss-Radau interpolation on the interval $[0, t_{T,k}^{N})$ (The corresponding Christoffel numbers are $\omega_{t_k,j}^{N}, 0 \le j \le N$).

The Legendre-Gauss-Radau collocation method for solving linear case (1.1) is to seek $u^N(t) \in P_N(0,T)$, such that:

$$\frac{d}{dt}u^N(t_{T,k}^{N}) + \int_0^{t_{T,k}^{N}} \tilde{\Pi}_N^k K(t_{T,k}^{N}, \tau) u^N(\tau) d\tau = g(t_{T,k}^{N}), \qquad 1 \le k \le N, \tag{2.2}$$

and

$$u^N(t_{T,0}^{N}) = U_0. \tag{2.3}$$

Because of $\tilde{\Pi}_N^k K(t_{T,k}^{N}, \tau) u^N(\tau) \in P_{2N}(0, t_{T,k}^{N})$ and (2.1), scheme (2.2) is equivalent to:

$$\frac{d}{dt}u^N(t_{T,k}^{N}) + \sum_{j=0}^{N} K(t_{T,k}^{N}, t_{t_k,j}^{N}) u^N(t_{t_k,j}^{N}) \omega_{t_k,j}^{N} = g(t_{T,k}^{N}), \qquad 1 \le k \le N. \tag{2.4}$$

The Legendre-Gauss-Radau collocation method for solving nonlinear case (1.2) is to seek $u^N(t) \in P_N(0,T)$, such that:

$$\frac{d}{dt}u^N(t_{T,k}^{N}) + \int_0^{t_{T,k}^{N}} \tilde{\Pi}_N^k K(t_{T,k}^{N}, \tau) \Pi_N F(u^N(\tau)) d\tau = g(t_{T,k}^{N}), \qquad 1 \le k \le N, \tag{2.5}$$

and

$$u^N(t_{T,0}^{N}) = U_0. \tag{2.6}$$

Because of $\tilde{\Pi}_N^k K(t_{T,k}^{N}, \tau) \Pi_N F(u^N(\tau)) \in P_{2N}(0, t_{T,k}^{N})$ and (2.1), scheme (2.5) is equivalent to:

$$\frac{d}{dt}u^N(t_{T,k}^{N}) + \sum_{j=0}^{N} K(t_{T,k}^{N}, t_{t_k,j}^{N}) \left[\Pi_N F(u^N)\right](t_{t_k,j}^{N}) \omega_{t_k,j}^{N} = g(t_{T,k}^{N}), \qquad 1 \le k \le N. \tag{2.7}$$

By taking $F(z) = z$ the nonlinear case (1.2) falls into the linear case (1.1). So does (2.5) and (2.7)(they fall into (2.2) and (2.4), respectively).

2.1 Implementation of the Collocation Method

We next describe the numerical implementation for (2.5). Indeed, the set of $L_{T,l}(t)$ is a complete $L^2(0,T)$ -orthogonal system, namely,

$$\int_0^T L_{T,l}(t)L_{T,m}(t)dt = \frac{T}{2l+1}\delta_{l,m} \tag{2.8}$$

where $\delta_{l,m}$ is the Kronecker symbol. Thus for any $v \in L^2(0,T)$,

$$v(t) = \sum_{l=0}^{\infty} \hat{v}_l L_{T,l}(t), \qquad \hat{v}_l = \frac{2l+1}{T}\int_0^T v(t)L_{T,l}(t)dt. \tag{2.9}$$

We first express the approximate solution as:

$$u^N(t) = \sum_{l=0}^{N} \hat{u}_l L_{T,l}(t), \qquad 0 \le t \le T \tag{2.10}$$

Obviously, $u^N(t)L_{T,l}(t) \in P_{2N}(0,T)$ for $0 \le l \le N$. Therefore, by multiplying (2.10) by $L_{T,l}(t)$ and integrating the result over the interval $(0,T)$, we use orthogonality (2.8) and (2.1) to verify that:

$$\hat{u}_l = \frac{2l+1}{T}\sum_{j=0}^{N} u^N(t_{T,j}^N)L_{T,l}(t_{T,j}^N)\omega_{T,j}^N, \qquad 0 \le l \le N, \tag{2.11}$$

Namely,

$$u^N(t) = \sum_{j=0}^{N}\sum_{l=0}^{N}\frac{(2l+1)}{T}L_{T,l}(t_{T,j}^N)u^N(t_{T,j}^N)\omega_{T,j}^N L_{T,l}(t) \tag{2.12}$$

Hence, we obtain:

$$\begin{aligned}
\frac{d}{dt}u^N(t_{T,k}^N) &= \sum_{j=0}^{N}\sum_{l=0}^{N}\left(L_{T,l}'(t_{T,k}^N)\right)\left(\frac{(2l+1)}{T}L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)u^N(t_{T,j}^N) \\
&= \sum_{j=1}^{N}\sum_{l=0}^{N}\left(L_{T,l}'(t_{T,k}^N)\right)\left(\frac{(2l+1)}{T}L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)u^N(t_{T,j}^N) \\
&\quad + \sum_{l=0}^{N}\left(L_{T,l}'(t_{T,k}^N)\right)\left(\frac{(2l+1)}{T}L_{T,l}(t_{T,0}^N)\omega_{T,0}^N\right)U_0
\end{aligned} \tag{2.13}$$

For the nonlinear term under integral in (2.5), we have:

$$\begin{aligned}
\Pi_N F(u^N(t)) &= \sum_{l=0}^{N}\sum_{j=0}^{N}\left(\frac{(2l+1)}{T}F(u^N(t_{T,j}^N))L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)L_{T,l}(t) \\
&= \sum_{j=1}^{N}\sum_{l=0}^{N}\frac{(2l+1)}{T}L_{T,l}(t_{T,j}^N)F(u^N(t_{T,j}^N))\omega_{T,j}^N L_{T,l}(t) \\
&\quad + \sum_{l=0}^{N}\frac{(2l+1)}{T}L_{T,l}(t_{T,0}^N)F(U_0)\omega_{T,0}^N L_{T,l}(t)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{t_{T,k}^N}\tilde{\Pi}_N^k K(t_{T,k}^N,\tau)\Pi_N F(u^N(\tau)d\tau \\
&= \sum_{j=0}^{N}\sum_{i=0}^{N}\left(\int_0^{t_{T,k}^N}\tilde{\Pi}_N^k K(t_{T,k}^N,\tau)L_{T,l}(\tau)d\tau\right)\left(\frac{2l+1}{T}L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)F(u^N(t_{T,j}^N)) \\
&= \sum_{j=1}^{N}\sum_{i=0}^{N}\left(\int_0^{t_{T,k}^N}\tilde{\Pi}_N^k K(t_{T,k}^N,\tau)L_{T,l}(\tau)d\tau\right)\left(\frac{2l+1}{T}L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)F(u^N(t_{T,j}^N)) \\
&\quad + \sum_{l=0}^{N}\left(\int_0^{t_{T,k}^N}\tilde{\Pi}_N^k K(t_{T,k}^N,\tau)L_{T,l}(\tau)d\tau\right)\left(\frac{2l+1}{T}L_{T,l}(t_{T,0}^N)\omega_{T,0}^N\right)F(U_0)
\end{aligned} \tag{2.14}$$

Further, we set:

$$\mathbf{u} = (u^N(t_{T,1}^N), u^N(t_{T,2}^N), ..., u^N(t_{T,N}^N))^T, \quad \mathbf{g} = (g(t_{T,1}^N), g(t_{T,2}^N), ..., g(t_{T,N}^N))^T,$$
$$\mathbf{F(u)} = (F(u^N(t_{T,1}^N)), F(u^N(t_{T,2}^N)), ..., F(u^N(t_{T,N}^N)))^T,$$
$$\mathbf{A} = (a_{kl})_{N\times(N+1)} = \left(L_{T,l}(t_{T,k}^N)\right)_{1\le k\le N; 0\le l\le N}$$
$$\mathbf{B} = (b_{kl})_{N\times(N+1)} = \left(\sum_{j=0}^{N} K(t_{T,k}^N, t_{t_k,j}^N) L_{T,l}(t_{t_k,j}^N)\omega_{t_k j}^N\right)_{1\le k\le N; 0\le l\le N}$$
$$\mathbf{C} = (c_{lj})_{(N+1)\times N} = \left(\frac{2l+1}{T} L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)_{0\le l\le N; 1\le j\le N}$$
$$\mathbf{h} = (h_l)_{(N+1)\times 1} = \left(\frac{2l+1}{T} L_{T,l}(t_{T,0}^N)\omega_{T,0}^N\right)_{0\le l\le N}.$$

Then we can rewrite (2.5) as the following compact matrix form,

$$\mathbf{ACu} + \mathbf{BCF(u)} = \mathbf{g} - \mathbf{Ah}U_0 - \mathbf{Bh}F(U_0) \tag{2.15}$$

In actual computation, we first use (2.15) to evaluate $u^N(t_{T,k}^N), 1\le k\le N$. Then we use (2.12) to obtain:

$$u^N(T) = \sum_{l=0}^{N} \left(\frac{(2l+1)}{T}\sum_{j=0}^{N} L_{T,l}(t_{T,j}^N)u^N(t_{T,j}^N)\omega_{T,j}^N\right) L_{T,l}(T) \tag{2.16}$$

Expression above can be in matrix-vector form as:

$$u^N(T) = \mathbf{wDv} \tag{2.17}$$

where:

$$\mathbf{w} = (w_l)_{1\times(N+1)} = \left(L_{T,l}(T)\right)_{0\le l\le N}$$
$$\mathbf{D} = (d_{jl})_{(N+1)\times(N+1)} = \left(\frac{2l+1}{T} L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)_{0\le j\le N; 0\le l\le N}$$
$$\mathbf{v} = (v_l)_{(N+1)\times 1} = \left(u^N(t_{T,j}^N)\right)_{0\le j\le N}.$$

Remark 1. The numerical scheme (2.15) is a nonlinear system for $u^N(t_{T,k}^N), 1\le k\le N$ unless function $F(.)$ is linear. In order to solve it, a proper solver should be used, such as Newton method. For the linear case, equation (2.15) is:

$$\mathbf{u} = (\mathbf{AC} + \mathbf{BC})^{-1}\left(\mathbf{g} - \mathbf{Ah}U_0 - \mathbf{Bh}U_0\right) \tag{2.18}$$

3. Error Analysis of the Collocation Method

In this section, we present error analysis of the collocation method for (2.2) and (2.5). In particular, we shall prove the spectral accuracy of numerical solution $u^N(t)$.

Let $(u,v)_T$ and $\|v\|_T$ be the inner product and the norm of space $L^2(0,T)$, respectively. $|v|_{r,T} = \left\|\frac{d^r v}{dt^r}\right\|_T$. We also introduce the following discrete inner product and norm,

$$(u,v)_{T,N} = \sum_{j=0}^{N} u(t_{T,j}^N)v(t_{T,j}^N)\omega_{T,j}^N, \qquad \|u\|_{T,N} = (u,u)_{T,N}^{\frac{1}{2}} \tag{3.1}$$

Thanks to (2.1), for any $\psi, \phi \in P_N(0,T)$,

$$(\psi,\phi)_T = (\psi,\phi)_{T,N}, \qquad \|\psi\|_T = \|\psi\|_{T,N}. \tag{3.2}$$

We introduce an approximation result for use later.

Lemma 1. (see [38]) For any $v \in H^r(0,T)$ with integers $1 \le r \le N+1$, there holds the following approximate estimate:

$$\|\Pi_N v - v\|_T \le cT^r N^{-r} |v|_{r,T}, \tag{3.3}$$

$$\left\|\frac{d}{dt}(\Pi_N v - v)\right\|_T \le cT^{r-1} N^{3/2-r} |v|_{r,T}, \tag{3.4}$$

We introduce the stability of Legendre-Gauss-Radau interpolation.

Lemma 2. For any $v \in H^1(0,T)$, there holds the following estimate:

$$\|\Pi_N v\|_T \le c_s \left(\|v\|_T + TN^{-1} |v|_{1,T}\right). \tag{3.5}$$

Proof. Because of:

$$\|\Pi_N v\|_T \le \|v\|_T + \|\Pi_N v - v\|_T,$$

then (3.5) comes from Lemma 1 with $r = 1$.

Lemma 3. (see [34]) Assume that $F_j(x)$ is the N th Lagrange interpolation polynomials associated with the Gauss, or Gauss-Radau, or Gauss-Lobatto points. Then:

$$\max_{x \in (-1,1)} \sum_{j=0}^{N} |F_j(x)| = 1 + \frac{2^{3/2}}{\sqrt{\pi}} N^{1/2} + B_0 + O(N^{-1/2}), \tag{3.6}$$

where B_0 is a bounded constant.

Lemma 4. For any $v \in C[0,T)$, there holds the following estimate:

$$\|\Pi_N v\|_{L^\infty(0,T)} \le c_s N^{1/2} \|v\|_{L^\infty(0,T)}. \tag{3.7}$$

Proof. The estimate is a direct result of Lemma 3.

Lemma 5. Let $D = \{(t,s) : 0 \le s, t \le T\}$, for any $K(t,s) \in C(D)$ and $f \in L^2(0,T)$, there holds the following estimate:

$$\left\|\int_0^t K(t,s) f(s)\right\|_T \le \|K\|_{L^2(D)} \|f\|_T. \tag{3.8}$$

Proof. Making use of Hölder inequality, we can obtain:

$$\int_0^T \left(\int_0^t K(t,s) f(s) ds\right)^2 dt \le \int_0^T \left(\int_0^T |K(t,s) f(s)| ds\right)^2 dt$$

$$\le \int_0^T \left(\int_0^T |K(t,s)|^2 ds \int_0^T |f(s)|^2 ds\right) dt \le \int_0^T \left(\left(\int_0^T |K(t,s)|^2 ds\right) dt\right)\left(\int_0^T |f(s)|^2 ds\right)$$

$$= \left(\int_D |K(t,s)|^2 dt ds\right)\left(\int_0^T |f(s)|^2 ds\right) = \|K\|_{L^2(D)}^2 \|f\|_T^2.$$

This is the desired result.

3.1 The Linear Case

As usual, we first compare the numerical solution $u^N(t)$ with the interpolation $\Pi_N u(t)$ of solution to (1.1). For this purpose, let:

$$G_{T,1}^N(t) = \Pi_N \frac{d}{dt} u(t) - \frac{d}{dt} \Pi_N u(t),$$

$$G_{T,2}^N(t) = \Pi_N \left(\int_0^t K(t,\tau) u(\tau) - \tilde{\Pi}_N^k K(t,\tau) u^N(\tau) d\tau\right).$$

Then it follows from (1.1) that for $1 \le k \le N$,

$$\frac{d}{dt}\Pi_N u(t_{T,k}^N)+\int_0^{t_{T,k}^N} K(t_{T,k}^N,\tau)u(\tau)d\tau = g(t_{T,k}^N)-G_{T,1}^N(t_{T,k}^N) \tag{3.9}$$

Further, let $E^N(t)=u^N(t)-\Pi_N u(t)$. Subtracting (2.2) from (3.9) yields:

$$\frac{d}{dt}E^N(t_{T,k}^N)=G_{T,1}^N(t_{T,k}^N)+G_{T,2}^N(t_{T,k}^N), \quad 1\le k\le N, \qquad E^N(0)=0. \tag{3.10}$$

Now, by multiplying the $k-$th equation of (3.10) by $2E^N(t_{T,k}^N)\omega_{T,k}^N$, for $1\le k\le N$, and putting the resulting equations together, we obtain:

$$2\sum_{k=1}^N E^N(t_{T,k}^N)\frac{d}{dt}E^N(t_{T,k}^N)\omega_{T,k}^N = 2\sum_{k=1}^N E^N(t_{T,k}^N)(G_{T,1}^N(t_{T,k}^N)+G_{T,2}^N(t_{T,k}^N))\omega_{T,k}^N.$$

Since $G_{T,1}^N(t)\in P_N(0,T), |G_{T,1}^N(0)|$ is finite. Besides, $G_{T,2}^N(0)=E^N(0)=0$. The previous statements lead to:

$$2\left(E^N,\frac{d}{dt}E^N\right)_{T,N} = 2(G_{T,1}^N,E^N)_{T,N}+2(G_{T,2}^N,E^N)_{T,N}. \tag{3.11}$$

Obviously, $\frac{d}{dt}E^N(t)\in P_{N-1}(0,T)$. Thus by (3.2),

$$2\left(E^N,\frac{d}{dt}E^N\right)_{T,N} = 2\left(E^N,\frac{d}{dt}E^N\right)_T = |E^N(T)|^2. \tag{3.12}$$

This means that:

$$|E^N(T)|^2 = 2(G_{T,1}^N,E^N)_{T,N}+2(G_{T,2}^N,E^N)_{T,N} \le 2\|E^N\|_T\left(\|G_{T,1}^N\|_{T,N}+\|G_{T,2}^N\|_{T,N}\right) \tag{3.13}$$

We next estimate $\|G_{T,1}^N\|_T$. Thus, by virtue of (3.3) with $v=\frac{du}{dt}$ and $r-1$, we have that for integers $2\le r\le N+2$,

$$\left\|\Pi_N\frac{d}{dt}u-\frac{d}{dt}u\right\|_T \le cT^{r-1}N^{1-r}|u|_{r,T}.$$

The above result, together with (3.4), gives that for $2\le r\le N+2$,

$$\|G_{T,1}^N\|_T \le \left\|\frac{d}{dt}(\Pi_N u-u)\right\|_T+\left\|\Pi_N\frac{d}{dt}u-\frac{d}{dt}u\right\|_T \le cT^{r-1}N^{3/2-r}|u|_{r,T}. \tag{3.14}$$

Now we come to estimate $\|G_{T,2}^N\|_T$. Firstly, we have:

$$|G_{T,2}^N(t)| \le \left|\Pi_N\int_0^t\left[K(t,\tau)-\tilde{\Pi}_N^k K(t,\tau)\right]u(\tau)d\tau\right|+\left|\Pi_N\int_0^t\tilde{\Pi}_N K(t,\tau)\left[u(\tau)-\Pi_N u(\tau)\right]d\tau\right|$$

$$+\left|\Pi_N\int_0^t\tilde{\Pi}_N^k K(t,\tau)\left[\Pi_N u(\tau)-u^N(\tau)\right]d\tau\right| =: |\Pi_N J_1|+|\Pi_N J_2|+|\Pi_N J_3|.$$

If we assume that:

$$K(t,s)\in C^{r+1}(D), \quad D=\{(t,s):0\le s\le t\le T\},$$

then we have:

$$\|J_1\|_T \le \|K-\tilde{\Pi}_N^k K\|_{L^2(D)}\|u\|_T \le ct^r N^{-r}\left\|\frac{\partial^r K}{\partial\tau^r}\right\|_{L^2(D)}\|u\|_T \le cT^r N^{-r}\left\|\frac{\partial^r K}{\partial\tau^r}\right\|_{L^2(D)}\|u\|_T,$$

$$|J_1|_{1,T} \le \left\|\left[K(t,t)-\tilde{\Pi}_N^k K(t,t)\right]u(t)\right\|_T+\left\|\int_0^t\left[\frac{\partial}{\partial t}\left(K(t,\tau)-\tilde{\Pi}_N^k K(t,\tau)\right)\right]u(\tau)d\tau\right\|_T$$

$$\leq ct^r N^{-r} \left\| \frac{\partial^r}{\partial \tau^r} K \right\|_{L^2(D)} \|u\|_{L^\infty(0,T)} + ct^r N^{-r} \left\| \frac{\partial^{r+1}}{\partial \tau^r \partial t} K \right\|_{L^2(D)} \|u\|_T$$

$$\leq cT^r N^{-r} \left(\left\| \frac{\partial^r}{\partial \tau^r} K \right\|_{L^2(D)} \|u\|_{L^\infty(0,T)} + \left\| \frac{\partial^{r+1}}{\partial \tau^r \partial t} K \right\|_{L^2(D)} \|u\|_T \right),$$

$$\|J_2\|_T \leq \left\|\tilde{\Pi}_N^k K\right\|_{L^2(D)} \|u - \Pi_N u\|_T \leq cT^r N^{-r} \left(\|K\|_{L^2(D)} + TN^{-1} \left\| \frac{\partial}{\partial \tau} K \right\|_{L^2(D)} \right) |u|_{r,T},$$

$$|J_2|_{1,T} \leq \left\|\tilde{\Pi}_N^k K(t,t)[u(t) - \Pi_N u(t)]\right\|_T + \left\| \int_0^t \frac{\partial}{\partial t} \tilde{\Pi}_N^k K(t,\tau)[u(\tau) - \Pi_N u(\tau)] d\tau \right\|_T$$

$$\leq cT^r N^{-r} \left(N^{1/2} \|K\|_{L^\infty(D)} + \left\| \frac{\partial}{\partial t} K \right\|_{L^2(D)} + TN^{-1} \left\| \frac{\partial^2}{\partial \tau \partial t} K \right\|_{L^2(D)} \right) |u|_{r,T} \; .$$

$$\|J_3\|_T \leq \left\|\tilde{\Pi}_N^k K\right\|_{L^2(D)} \|u^N - \Pi_N u\|_T \leq \left(\|K\|_{L^2(D)} + TN^{-1} \left\| \frac{\partial}{\partial \tau} K \right\|_{L^2(D)} \right) \|E^N\|_T,$$

$$|J_3|_{1,T} \leq \left\|\tilde{\Pi}_N^k K(t,t)[u^N(t) - \Pi_N u(t)]\right\|_T + \left\| \int_0^t \frac{\partial}{\partial t} \tilde{\Pi}_N^k K(t,\tau)[u^N(\tau) - \Pi_N u(\tau)] d\tau \right\|_T$$

$$\leq \left(N^{1/2} \|K\|_{L^\infty(D)} + \left\| \frac{\partial}{\partial t} K \right\|_{L^2(D)} + TN^{-1} \left\| \frac{\partial^2}{\partial \tau \partial t} K \right\|_{L^2(D)} \right) \|E^N\|_T.$$

Now we assume that:

$$M = \max\left\{ \|K\|_{L^\infty(D)}, \|K\|_{L^2(D)}, \left\| \frac{\partial^{j+1}}{\partial \tau^j \partial t} K \right\|_{L^2(D)} (0 \leq j \leq r) \right\},$$

then we have:

$$\left\|G_{T,2}^N\right\|_T \leq cMT^r N^{-r} \left(\|u\|_{L^\infty(0,T)} + \|u\|_T + |u|_{r,T} \right) + c_e \|E^N\|_T, \tag{3.15}$$

with: $c_e = c_s M(1 + TN^{-1/2} + 2TN^{-1} + T^2 N^{-2})$. On the other hand, for any $t \in (0,T)$,

$$t^{1/2} |E^N(t)|^2 = T^{1/2} |E^N(T)|^2 - \int_t^T \frac{d}{dy} \left(y^{1/2} (E^N(y))^2 \right) dy$$

$$= T^{1/2} |E^N(T)|^2 - \frac{1}{2} \int_t^T y^{-1/2} (E^N(y))^2 dy - 2 \int_t^T y^{1/2} E^N(y) \frac{d}{dy} E^N(y) dy$$

$$\leq T^{1/2} |E^N(T)|^2 + 2 \|E^N\|_T \left\| t^{1/2} \frac{d}{dt} E^N \right\|_T .$$

Hence:

$$|E^N(t)|^2 \leq t^{-1/2} \left(T^{1/2} |E^N(T)|^2 + 2 \|E^N\|_T \left\| t^{1/2} \frac{d}{dt} E^N \right\|_T \right).$$

Integrating the inequality above with respect to t yields:

$$\|E^N\|_T^2 \leq 2T |E^N(T)|^2 + 4T^{1/2} \|E^N\|_T \left\| t^{1/2} \frac{d}{dt} E^N \right\|_T . \tag{3.16}$$

Due to $\frac{d}{dt} E^N(t) \in P_{N-1}(0,T)$ and $|G_{T,1}^N(0) + G_{T,2}^N(0)| < \infty$, we use (3.2) and (3.10) successively to deduce that:

$$\left\| t^{1/2} \frac{d}{dt} E^N \right\|_T = \left\| t^{1/2} \frac{d}{dt} E^N \right\|_{T,N} = \left\| t^{1/2} (G_{T,1}^N + G_{T,2}^N) \right\|_{T,N} \le T^{1/2} \left(\left\| G_{T,1}^N \right\|_{T,N} + \left\| G_{T,2}^N \right\|_{T,N} \right).$$

Now we have:

$$\left\| E^N \right\|_T \le 8T \left(\left\| G_{T,1}^N \right\|_{T,N} + \left\| G_{T,2}^N \right\|_{T,N} \right).$$

Then we come to the following results.

Theorem 1. (L^2 -convergence) If the kernel satisfies:

$$K(t,s) \in C^{r+1}(D), \quad D = \{(t,s) : 0 \le s \le t \le T\}$$

let

$$M = \max \left\{ \|K\|_{L^\infty(D)}, \|K\|_{L^2(D)}, \left\| \frac{\partial^{j+1}}{\partial \tau^j \partial t} K \right\|_{L^2(D)} (0 \le j \le r) \right\},$$

with $8c_e T \le \beta < 1$. And $u \in H^r(0,T) \cap L^\infty(0,T) (2 \le r \le N+1)$ is the solution to (2.1). Then we have:

$$\left\| u - u^N \right\|_T \le c_\beta T^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right), \tag{3.17}$$

$$| u(T) - u^N(T) | \le c_\beta T^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right), \tag{3.18}$$

where c_β is a positive constant depending on M and β.

Proof. Since $\left\| E^N \right\|_T \le c_\beta T^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right)$, we have:

$$\left\| u - u^N \right\|_T \le \left\| u - \Pi_N u \right\|_T + \left\| E^N \right\|_T \le c_\beta T^r N^{3/2-r} \left(|u|_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right).$$

Further, recall $E^N(0) = 0$,

$$\begin{aligned} | E^N(T) | &= \left| \int_0^T \frac{d}{dt} E^N(t) dt \right| \le T^{1/2} \left(\left\| G_{T,1}^N \right\|_T + \left\| G_{T,2}^N \right\|_T \right) \\ &\le 4T^{1/2} \left(cT^{r-1} N^{3/2-r} \, | u |_{r,T} + cMT^r N^{-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right) + c_e \left\| E^N \right\|_T \right) \\ &\le cT^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right). \end{aligned}$$

Consequently, by Lemma 1 we have:

$$| u(T) - u^N(T) | \le | u(T) - \Pi_N u(T) | + | E^N(T) | \le c_\beta T^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right).$$

It is the desired result.

Theorem 2. (H^1 -convergence) Under the same assumption as in Theorem 1, we have

$$| u - u^N |_{1,T} \le c_\beta T^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right), \tag{3.19}$$

where c_β is a positive constant depending on M and β.

Proof. From (3.10) we know:

$$\left\| \frac{d}{dt} E^N \right\|_T \le \sqrt{2} \left(\left\| G_{T,1}^N \right\|_T + \left\| G_{T,2}^N \right\|_T \right).$$

Combining estimates (3.14)(3.15) with inequality above, we have:

$$\left\| \frac{d}{dt} E^N \right\|_T \le cT^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right).$$

Consequently, by Lemma 1 one has:

$$| u - u^N |_{1,T} \le | u - \Pi_N u |_{1,T} + | E^N |_{1,T} \le c_\beta T^r N^{3/2-r} \left(| u |_{r,T} + \|u\|_T + \|u\|_{L^\infty(0,T)} \right).$$

It is the desired result.

3.2 The Nonlinear Case

We use the same notation $u^N(t)$ which is the numerical solution to (2.5) in this section. We also need to compare the numerical solution $u^N(t)$ with the interpolation $\Pi_N u(t)$ of solution to (1.2). Denote that:

$$G_{T,3}^N(t) = \Pi_N\left(\int_0^t K(t,\tau)F(u(\tau)) - \tilde{\Pi}_N^k K(t,\tau)\Pi_N F(u^N(\tau))d\tau\right). \tag{3.20}$$

Then it follows from (1.2) that for $1 \le k \le N$,

$$\frac{d}{dt}\Pi_N u(t_{T,k}^N) + \int_0^{t_{T,k}^N} K(t_{T,k}^N,\tau)F(u(\tau))d\tau = g(t_{T,k}^N) - G_{T,1}^N(t_{T,k}^N) \tag{3.21}$$

Further, let $E^N(t) = u^N(t) - \Pi_N u(t).$ Subtracting (2.5) from (3.21) yields:

$$\frac{d}{dt}E^N(t_{T,k}^N) = G_{T,1}^N(t_{T,k}^N) + G_{T,3}^N(t_{T,k}^N), \quad 1 \le k \le N, \qquad E^N(0) = 0. \tag{3.22}$$

Next we estimate $\|G_{T,3}^N\|_T$. Firstly, we have:

$$\begin{aligned}\left|G_{T,3}^N(t)\right| \le & \left|\Pi_N \int_0^t \left[K(t,\tau) - \tilde{\Pi}_N^k K(t,\tau)\right]F(u(\tau))d\tau\right| \\ & + \left|\Pi_N \int_0^t \tilde{\Pi}_N^k K(t,\tau)\left[F(u(\tau)) - \Pi_N F(u(\tau))\right]d\tau\right| \\ & + \left|\Pi_N \int_0^t \tilde{\Pi}_N^k K(t,\tau)\Pi_N\left[F(u(\tau)) - F(\Pi_N u(\tau))\right]d\tau\right| \\ & + \left|\Pi_N \int_0^t \tilde{\Pi}_N^k K(t,\tau)\Pi_N\left[F(\Pi_N u(\tau)) - F(u^N(\tau))\right]d\tau\right| \\ =: & |\Pi_N J_{11}| + |\Pi_N J_{12}| + |\Pi_N J_{13}| + |\Pi_N J_{14}|.\end{aligned}$$

Now we need assumption on nonlinear term $F(\cdot)$. Assume that:

$$F(z) \in C^r(\mathbf{R})(r \ge 2).$$

With aid of the abovementioned assumptions, we can deduce:

$$\begin{aligned}\|J_{11}\|_T \le & \left\|K - \tilde{\Pi}_N^k K\right\|_{L^2(D)}\|F(u)\|_T \\ \le & ct^r N^{-r}\left\|\frac{\partial^r}{\partial\tau^r}K\right\|_{L^2(D)}\|F(u)\|_T \le cT^r N^{-r}\left\|\frac{\partial^r}{\partial\tau^r}K\right\|_{L^2(D)}\|F(u)\|_T, \\ |J_{11}|_{1,T} \le & \left\|\left[K(t,t) - \tilde{\Pi}_N^k K(t,t)\right]F(u(t))\right\|_T \\ & + \left\|\int_0^t\left[\frac{\partial}{\partial t}\left(K(t,\tau) - \tilde{\Pi}_N^k K(t,\tau)\right)\right]F(u(\tau))d\tau\right\|_T \\ \le & ct^r N^{-r}\left\|\frac{\partial^r}{\partial\tau^r}K\right\|_{L^2(D)}\|F(u)\|_{L^\infty(0,T)} + ct^r N^{-r}\left\|\frac{\partial^{r+1}}{\partial\tau^r\partial t}K\right\|_{L^2(D)}\|F(u)\|_T \\ \le & cT^r N^{-r}\left(\left\|\frac{\partial^r}{\partial\tau^r}K\right\|_{L^2(D)}\|F(u)\|_{L^\infty(0,T)} + \left\|\frac{\partial^{r+1}}{\partial\tau^r\partial t}K\right\|_{L^2(D)}\|F(u)\|_T\right), \\ \|J_{12}\|_T \le & \left\|\tilde{\Pi}_N^k K\right\|_{L^2(D)}\|F(u) - \Pi_N F(u)\|_T \\ \le & cT^r N^{-r}\left(\|K\|_{L^2(D)} + TN^{-1}\left\|\frac{\partial}{\partial\tau}K\right\|_{L^2(D)}\right)|F(u)|_{r,T},\end{aligned}$$

$$\begin{aligned}
\left|J_{12}\right|_{1,T} &\le \left\|\tilde{\Pi}_N^k K(t,t)\left[F(u(t))-\Pi_N F(u(t))\right]\right\|_T \\
&\quad + \left\|\int_0^t \frac{\partial}{\partial t}\tilde{\Pi}_N^k K(t,\tau)\left[F(u(\tau))-\Pi_N F(u(\tau))\right]d\tau\right\|_T \\
&\le \left\|\tilde{\Pi}_N^k K\right\|_{L^\infty(D)}\left\|F(u)-\Pi_N F(u)\right\|_T + \left\|\frac{\partial}{\partial t}\tilde{\Pi}_N^k K\right\|_{L^2(D)}\left\|F(u)-\Pi_N F(u)\right\|_T \\
&\le cT^r N^{-r}\left(N^{1/2}\|K\|_{L^\infty(D)} + \left\|\frac{\partial}{\partial \tau}K\right\|_{L^2(D)} + TN^{-1}\left\|\frac{\partial^2}{\partial\tau\partial t}K\right\|_{L^2(D)}\right)|F(u)|_{r,T},
\end{aligned}$$

Let:

$$u_M = \max_{0\le s\le T}\left\{|u(s)+TN^{-1}|u'(s)|\right\},$$

$$C_F(z_1,z_2) = \max_{|z|\le|z_1|+|z_2|}|F'(z)| + \left(|z_1|+|z_2|\right)\max_{|z|\le|z_1|+|z_2|}|F''(z)|. \tag{3.23}$$

and C_0 be a positive constant such that:

$$\max_{0\le t\le T}|u-\Pi_N u|\le C_0. \tag{3.24}$$

Denote $\tilde{e}(t)=\Pi_N u-u$, then:

$$\begin{aligned}
&\left\|F(u)-F(\Pi_N u)\right\|_T + TN^{-1}|F(u)-F(\Pi_N u)|_{1,T} \\
&\le \left\|\int_0^1 F'(u+\theta\tilde{e})\tilde{e}\,d\theta\right\|_T + TN^{-1}\left\|\int_0^1\left(F''(u+\theta\tilde{e})\frac{d}{dt}(u+\theta\tilde{e})\tilde{e} + F'(u+\theta\tilde{e})\left(\frac{d}{dt}\tilde{e}\right)\right)d\theta\right\|_T \\
&\le C_F(u_M,C_0)\|\tilde{e}\|_T + c|\tilde{e}|_{1,T}\le cT^{r-1}N^{3/2-r}|u|_{r,T}.
\end{aligned}$$

Then we have:

$$\begin{aligned}
\left\|J_{13}\right\|_T &\le \left\|\tilde{\Pi}_N^k K\right\|_{L^2(D)}\left\|\Pi_N\left[F(u)-F(\Pi_N u)\right]\right\|_T \\
&\le cT^{r-1}N^{3/2-r}\left(\|K\|_{L^2(D)} + TN^{-1}\left\|\frac{\partial}{\partial\tau}K\right\|_{L^2(D)}\right)|u|_{r,T},
\end{aligned}$$

$$\begin{aligned}
\left|J_{13}\right|_{1,T} &\le \left\|\tilde{\Pi}_N^k K(t,t)\Pi_N\left[F(u(t))-F(\Pi_N u(t))\right]\right\|_T \\
&\quad + \left\|\int_0^t \frac{\partial}{\partial t}\tilde{\Pi}_N^k K(t,\tau)\Pi_N\left[F(u(\tau))-F(\Pi_N u(\tau))\right]d\tau\right\|_T \\
&\le \left(\left\|\tilde{\Pi}_N^k K\right\|_{L^\infty(D)} + \left\|\frac{\partial}{\partial t}\tilde{\Pi}_N^k K\right\|_{L^2(D)}\right)\left\|\Pi_N\left[F(u)-F(\Pi_N u)\right]\right\|_T \\
&\le cT^{r-1}N^{3/2-r}\left(N^{1/2}\|K\|_{L^\infty(D)} + \left\|\frac{\partial}{\partial t}K\right\|_{L^2(D)} + TN^{-1}\left\|\frac{\partial^2}{\partial\tau\partial t}K\right\|_{L^2(D)}\right)|u|_{r,T}.
\end{aligned}$$

In order to estimate $\left\|\Pi_N J_{14}\right\|_T$, we assume that:

$$|F(z_1)-F(z_2)|\le\gamma|z_1-z_2|,\quad \forall z_1,z_2\in\mathbf{R}. \tag{3.25}$$

Because of $E^N(0)=0$,

$$|E^N(t)|^2 = \int_0^t \frac{d}{dy}\left(E^N(y)\right)^2 dy = 2\int_0^t E^N(y)\frac{d}{dy}E^N(y)dy,$$

$$\left\|E^N\right\|_T \le 2T|E^N|_{1,T}. \tag{3.26}$$

From (3.25) and (3.26), we get:

$$\left\| F(\Pi_N u) - F(u^N) \right\|_T \le \gamma \left\| E^N \right\|_T \le 2\gamma T \mid E^N \mid_{1,T} . \tag{3.27}$$

Let:

$$u_L = \max_{0\le s\le T} \left\{ \mid \Pi_N u(s) \mid + \left| \frac{d}{ds} \Pi_N u(s) \right| \right\} \tag{3.28}$$

and if:

$$\max_{0\le t\le T} \mid E^N \mid \le C_1, \tag{3.29}$$

we can obtain:

$$\begin{aligned} &\mid F(\Pi_N u) - F(u^N) \mid_{1,T} \\ &= \left\| \int_0^1 F''(\Pi_N u + \theta E^N) \left(\frac{d}{dt} (\Pi_N u + \theta E^N) \right) E^N + F'(\Pi_N u + \theta E^N) \left(\frac{d}{dt} (E^N) \right) d\theta \right\| \\ &\le C_F(u_L, C_1)(2T+1) \mid E^N \mid_{1,T} . \end{aligned}$$

This means:

$$\begin{aligned} \left\| J_{14} \right\|_T &\le \left\| \tilde{\Pi}_N^k K \right\|_{L^2(D)} \left\| \Pi_N \left[F(\Pi_N u) - F(u^N) \right] \right\|_T \\ &\le c\left(2\gamma T + TN^{-1}\right) \left(\left\| K \right\|_{L^2(D)} + TN^{-1} \left\| \frac{\partial}{\partial \tau} K \right\|_{L^2(D)} \right) \mid E^N \mid_{r,T} . \end{aligned}$$

$$\begin{aligned} \left| J_{14} \right|_{1,T} &\le \left\| \tilde{\Pi}_N^k K(t,t) \Pi_N \left[F(\Pi_N u(t)) - F(u^N(t)) \right] \right\|_T \\ &\quad + \left\| \int_0^t \frac{\partial}{\partial t} \tilde{\Pi}_N^k K(t,\tau) \Pi_N \left[F(\Pi_N u(\tau)) - F(u^N(\tau)) \right] d\tau \right\|_T \\ &\le \left(\left\| \tilde{\Pi}_N^k K \right\|_{L^\infty(D)} + \left\| \frac{\partial}{\partial t} \tilde{\Pi}_N^k K \right\|_{L^2(D)} \right) \left\| \Pi_N \left[F(\Pi_N u) - F(u^N) \right] \right\|_T \\ &\le c\left(2\gamma T + TN^{-1}\right) \left(N^{1/2} \left\| K \right\|_{L^\infty(D)} + \left\| \frac{\partial}{\partial t} K \right\|_{L^2(D)} + TN^{-1} \left\| \frac{\partial^2}{\partial \tau \partial t} K \right\|_{L^2(D)} \right) \mid E^N \mid_{r,T} . \end{aligned}$$

Now we have:

$$\begin{aligned} \left\| G_{T,3}^N \right\|_T &\le cMT^{r-1} N^{3/2-r} \left(\left\| F(u) \right\|_T + \left\| F(u) \right\|_{L^\infty(0,T)} + \mid F(u) \mid_{r,T} + \mid u \mid_{r,T} \right) \\ &\quad + c_e (2\gamma T + TN^{-1}) \mid E^N \mid_{1,T} . \end{aligned} \tag{3.30}$$

Theorem 3. Under assumption of Theorem 1 of $K(\cdot,\cdot)$, let nonlinear term $F(\cdot)$ such that:

$$\mid F(z_1) - F(z_2) \mid \le \gamma \mid z_1 - z_2 \mid, \quad \forall z_1, z_2 \in \mathbf{R}, \quad F \in C^r(\mathbf{R})$$

with $2c_e T(2\gamma + N^{-1}) \le \beta < 1$. And $u \in H^r(0,T) \cap L^\infty(0,T) (2 \le r \le N+1)$ is the solution to (2.2). Then we have:

$$\mid u - u^N \mid_{1,T} \le c_\beta T^r N^{3/2-r} \left(\mid F(u) \mid_{r,T} + \left\| F(u) \right\|_T + \left\| F(u) \right\|_{L^\infty(0,T)} + \mid u \mid_{r,T} \right), \tag{3.31}$$

and:

$$\left\| u - u^N \right\|_T \le c_\beta T^r N^{3/2-r} \left(\mid F(u) \mid_{r,T} + \left\| F(u) \right\|_T + \left\| F(u) \right\|_{L^\infty(0,T)} + \mid u \mid_{r,T} \right), \tag{3.32}$$

where c_β is a positive constant depending on M, γ and β.

Proof. From (3.22) we know:

$$\mid E^N \mid_{1,T} \le \sqrt{2} \left(\left\| G_{T,1}^N \right\|_T + \left\| G_{T,3}^N \right\|_T \right).$$

Combining estimates of $\left\| G_{T,1}^N \right\|_T$ and $\left\| G_{T,3}^N \right\|_T$ leads:

$$|E^N|_{1,T} \le c_\beta T^r N^{3/2-r}\left(|u|_{r,T} + |F(u)|_{r,T} + \|F(u)\|_T + \|F(u)\|_{L^\infty(0,T)}\right).$$

Then we have:

$$\begin{aligned}|u-u^N|_{1,T} &\le |u-\Pi_N u|_{1,T} + |E^N|_{1,T}\\ &\le c_\beta T^{r+1} N^{3/2-r}\left(|u|_{r,T} + |F(u)|_{r,T} + \|F(u)\|_T + \|F(u)\|_{L^\infty(0,T)}\right).\end{aligned}$$

Furthermore, by using (3.26) we can obtain:

$$\|E^N\|_T \le c_\beta T^{r+1} N^{3/2-r}\left(|u|_{r,T} + |F(u)|_{r,T} + \|F(u)\|_T + \|F(u)\|_{L^\infty(0,T)}\right).$$

Also we have:

$$\begin{aligned}\|u-u^N\|_T &\le \|u-\Pi_N u\|_T + \|E^N\|_T\\ &\le c_\beta T^r N^{3/2-r}\left(|u|_{r,T} + |F(u)|_{r,T} + \|F(u)\|_T + \|F(u)\|_{L^\infty(0,T)}\right).\end{aligned}$$

The proof is completed.

Remark 2. Note that the assumption (3.29) which is about boundness of E^N can be satisfied since convergence results in Theorem 3.

4. Multi-Step Version of the Collocation Method

In actual computation, it is not convenient to resolve the discrete system (2.15) with very large mode N. On the other hand, for ensuring the convergence of scheme (2.5), the length of T is limited. To remedy these deficiencies, we now provide the multi-step version of Legendre-Gauss-Radau collocation method, which simplifies computation, saves work, and still keeps the spectral accuracy.

Let M be a positive integer, and $N_m, 1 \le m \le M$ be positive integers. We divide the interval $[0,T]$ as:

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m < \ldots < t_M = T.$$

We set $\tau_m = t_m - t_{m-1}$. Replacing T and N by τ_1 and N_1 in (2.5-2.6), we obtain the local numerical solution $u_1^{N_1} \in P_{N_1}(t_0, t_1)$, with the initial value $u_1^{N_1}(0) = g(t_{T,0}^{N_1})$.

Next, we evaluate the local numerical solutions $u_m^{N_m} \in P_{N_m}(t_{m-1}, t_m)(m = 2,3,\ldots,M)$ by using the following process,

$$\begin{aligned}\frac{d}{dt}u_m^{N_m}(t_{\tau_m,k}^{N_m}) &= g(t_{\tau_m,k}^{N_m}) - \int_{t_{m-1}}^{t_{\tau_m,k}^{N_m}} \tilde{\Pi}_N^{\tau_k} K(t_{\tau_m,k}^{N_m},\tau)\Pi_N F(u_m^{N_m}(\tau))d\tau\\ &\quad - \sum_{s=1}^{m-1}\int_{t_{s-1}}^{t_s} \tilde{\Pi}_N^{\tau_s} K(t_{\tau_m,k}^{N_m},\tau)\Pi_N F(u_s^{N_s}(\tau))d\tau, \qquad 1 \le k \le N_m, \end{aligned} \tag{4.1}$$

$$u_m^{N_m}(t_{\tau_m,0}^{N_m}) = u_{m-1}^{N_{m-1}}(t_{m-1}), \qquad 2 \le m \le M. \tag{4.2}$$

Obviously, the local numerical solution $u_m^{N_m}(t)$ is a proper approximation to the local exact solution $u_m(t) = u|_{[t_{m-1},t_m]}$, with the approximate initial value $u_m^{N_m}(t_{m-1}) = u_{m-1}^{N_{m-1}}(t_{m-1})$. Specifically, we denote $L_{\tau_k,l}(t)$ the shifted Legendre polynomial of order l on $[t_{k-1}, t_k]$, namely:

$$L_{\tau_k,l}(t) = L_l\left(\frac{2}{\tau_k}(t - t_{k-1}) - 1\right).$$

The corresponding shifted Legendre-Gauss-Radau collocation points and weights are:

$t_{\tau_k,j}^{N_k} = \frac{\tau_k}{2}(\xi_j + 1) + t_{k-1}$ and $\omega_{\tau_k,j}^{N_k} = \frac{\tau_k}{2}\rho_j$, $j = 0,1,\ldots,N_m$, respectively. Let:

$$\mathbf{u}^{\tau_k} = (u_k^{N_k}(t_{\tau_k,1}^{N_k}), u_k^{N_k}(t_{\tau_k,2}^{N_k}),\dots, u_k^{N_k}(t_{\tau_k,N_k}^{N_k}))^T,$$
$$\mathbf{g}^{\tau_k} = (g(t_{\tau_k,1}^{N_k}), g(t_{\tau_k,2}^{N_k}),\dots, g(t_{\tau_k,N_k}^{N_k}))^T,$$
$$\mathbf{F}(\mathbf{u}^{\tau_k}) = (F(u_k^{N_k}(t_{\tau_k,1}^{N_k})), F(u_k^{N_k}(t_{\tau_k,2}^{N_k})),\dots, F(u_k^{N_k}(t_{\tau_k,N_k}^{N_k})))^T,$$
$$\mathbf{A}^{\tau_k} = (a_{jl})_{N_k\times(N_k+1)} = \left(L_{\tau_k,l}(t_{\tau_k,j}^{N_k})\right)_{1\le j\le N_k;0\le l\le N_k},$$
$$\mathbf{B}^{\tau_m} = (b_{kl})_{N_m\times(N_m+1)} = \left(\int_{t_{m-1}}^{t_{\tau_m,k}^{N_m}} \widetilde{\Pi}_N^{\tau_m} K(t_{\tau_m,k}^{N_m},\tau)L_{\tau_m,l}(\tau)d\tau\right)_{1\le k\le N_m;0\le l\le N_m},$$
$$\mathbf{C}^{\tau_k} = (c_{lj})_{(N_k+1)\times N_k} = \left(\frac{2l+1}{\tau_k}L_{\tau_k,l}(t_{\tau_k,j}^{N_k})\omega_{\tau_k,j}^{N_k}\right)_{1\le j\le N_k;0\le l\le N_k},$$
$$\mathbf{h}^{\tau_k} = (h_l)_{(N_k+1)\times 1} = \left(\frac{2l+1}{\tau_k}L_{\tau_k,l}(t_{\tau_k,0}^{N_k})\omega_{\tau_k,0}^{N_k}\right)_{0\le l\le N_k},$$
$$\mathbf{b}^{\tau_k} = (b_l)_{1\times(N_k+1)} = \left(L_{\tau_k,l}(t_k)\right)_{0\le l\le N_k},$$
$$\mathbf{D}^{\tau_k} = (d_{jl})_{(N_k+1)\times(N_k+1)} = \left(\frac{2l+1}{\tau_k}L_{\tau_k,l}(t_{\tau_k,j}^{N_k})\omega_{\tau_k,j}^{N_k}\right)_{0\le j\le N_k;0\le l\le N_k},$$
$$\mathbf{K}^{\tau_n} = (k_{kj})_{N_m\times(N_n+1)} = \left(K(t_{\tau_m,k}^{N_m}, t_{\tau_n,j}^{N_n})\omega_{\tau_n,j}^{N_n}\right)_{1\le k\le N_k;0\le j\le N_n},$$
$$\hat{\mathbf{F}}(\mathbf{u}^{\tau_k}) = (f_j)_{(N_k+1)\times 1} = \left(F(u_k^{N_k}(t_{\tau_k,j}^{N_k}))\right)_{0\le j\le N_k},$$
$$\mathbf{v}^{\tau_k} = (v_j)_{(N_k+1)\times 1} = \left(u_k^{N_k}(t_{\tau_k,j}^{N_k})\right)_{0\le j\le N_k}.$$

For solving the local numerical solution $u_1^{N_1} \in P_{N_1}(t_0,t_1)$, we need to solve the system as :

$$\mathbf{A}^{\tau_1}\mathbf{C}^{\tau_1}\mathbf{u}^{\tau_1} + \mathbf{B}^{\tau_1}\mathbf{C}^{\tau_1}\mathbf{F}(\mathbf{u}^{\tau_1}) = \mathbf{g}^{\tau_1} - \mathbf{A}^{\tau_1}\mathbf{h}^{\tau_1}U_0 - \mathbf{B}^{\tau_1}\mathbf{h}^{\tau_1}F(U_0)$$

and:

$$u_1^{N_1}(t_1) = \mathbf{b}^{\tau_1}\mathbf{D}^{\tau_1}\mathbf{v}^{\tau_1}.$$

Next, step by step, the local numerical solution $u_m^{N_m} \in P_{N_m}(t_{m-1},t_m)(m = 2,3,\dots,M)$ can be obtain by solving:

$$\mathbf{A}^{\tau_m}\mathbf{C}^{\tau_m}\mathbf{u}^{\tau_m} + \mathbf{B}^{\tau_m}\mathbf{C}^{\tau_m}\mathbf{F}(\mathbf{u}^{\tau_m}) = \mathbf{g}^{\tau_m} - \mathbf{A}^{\tau_m}\mathbf{h}^{\tau_m}u_m^{N_m}(t_{m-1}) - \mathbf{B}^{\tau_m}\mathbf{h}^{\tau_m}F(u_m^{N_m}(t_{m-1})) - \sum_{k=1}^{m-1}\left(\mathbf{K}^{\tau_k}\hat{\mathbf{F}}(\mathbf{u}^{\tau_k})\right),$$

$$u_m^{N_m}(t_m) = \mathbf{b}^{\tau_m}\mathbf{D}^{\tau_m}\mathbf{v}^{\tau_m}$$

Noting that $t_{\tau_k,0}^{N_k} = t_{k-1}$, the process above can be down without any gap.

Obviously, the local numerical solution $u_m^{N_m}(t)$ is a proper approximation to the local exact solution $U_m(t)$, with the approximate initial value $u_m^{N_m}(t_{m-1}) = u_{m-1}^{N_{m-1}}(t_{m-1})$.

5. Numerical Experiments

By this section, we will present some numerical experiments to confirm the effectiveness and robustness of the scheme (2.5) or (2.15).

The first issue in performing the proposed method is how to solve the nonlinear system (2.15). Since the matrix $\mathbf{B}$ in the right-hand side of the system (2.15) depends on kernel function $K(t,\tau)$, we can not claim any useful result on it. Anyway, a simple iterate method is: for $k = 0,1,2,...$

$$\mathbf{u}^{k+1} = (\mathbf{AC})^{-1}\left(\mathbf{g} - \mathbf{BCF}(\mathbf{u}^k) - \mathbf{Ah}U_0 - \mathbf{Bh}F(U_0)\right) \qquad (5.1)$$

If the sequence $\mathbf{u}^{k+1}$ converge, then we can obtain a good approximation solution of the system (2.15).

Example 1 Our first example is concerned with linear problem in one-dimension. Consider Volterra integro-differential equation (1.1) with:

$$g(t) = 2\exp(2t) + \frac{\exp(t^2 + 2t) - 1}{t+2}, \quad K(t,\tau) = \exp(t\tau), \quad F(u) = u.$$

The exact solution is $u(t) = \exp(2t)$.

Because the exact solution is known, we can compute the error between numerical solution and exact solution. The errors in L^∞ and L^2 are defined by:

$$\mathrm{Err\,inf} = \max_{0\le k\le N}\left|u^N(t_{T,k}^N) - u(t_{T,k}^N)\right|, \quad \mathrm{Err2} = \sqrt{\sum_{k=0}^{N}\left|u^N(t_{T,k}^N) - u(t_{T,k}^N)\right|^2 \omega_{T,k}^N},$$

where $u(t)$ and $u^N(t)$ are exact and numerical solution, respectively. Another pointwise error at $t = T$ is $\mathrm{Err}(T) = \left|u^N(T) - u(T)\right|$.

In Table 1, we present the errors with different N. The results show "spectral accuracy".

Table 1 Errors with different N for Example 1.

N	4	6	8	10	12
Errinf	7.5336e-03	5.3924e-05	2.1392e-07	5.3993e-10	1.5978e-12
Err2	4.8957e-03	3.4040e-05	1.3309e-07	3.3289e-10	6.0824e-13
Err(1)	2.2326e-03	1.6024e-05	6.3583e-08	1.5996e-10	2.2560e-13

Example 2 Our second example is concerned with nonlinear problem in one-dimension. Consider Volterra integro-differential equation (1.2) with:

$$g(t) = \cos t + \frac{1-\cos t}{2} + \frac{\sin 2t - 2t}{8} + \frac{\cos t - \exp(-2t) - 2\sin t}{10} + \frac{2 - \exp(-2t) - \sin 2t - \cos 2t}{16},$$

$$K(t,\tau) = -\frac{1 - \exp(-2(t-\tau))}{2}, \; F(u) = u - u^2.$$

The exact solution is $u(t) = \sin t$.

For solving the nonlinear system (2.15), simple iterate method (5.1) is employed with tolerate $\varepsilon = 10^{-16}$ (that is, $\left\|\mathbf{u}^{k+1} - \mathbf{u}^k\right\|_2 \le \varepsilon$).

In Table 2, we present the errors with different N. Also the iteration number of simple iterate method (5.1) is given. It is show that a few iterations can achieve good precision. The results show "spectral accuracy".

Table 2 Errors with different N for Example 2.

N	4	6	8	10	12
Iterations	8	7	7	7	6
Errinf	8.1718e-05	1.4291e-07	1.3981e-10	1.2712e-13	1.5177e-13
Err2	6.2979e-05	1.0790e-07	1.0451e-10	6.6495e-14	2.7142e-14
Err(1)	6.1917e-05	1.0772e-07	1.0515e-10	5.9286e-14	5.3291e-15

Example 3 Consider Volterra integro-differential equation (1.2) with:

$$g(t)=\frac{\exp(t)-1}{2}-\frac{\exp(t)-\cos(6\pi t)+6\pi\sin(6\pi t)}{2(1+36\pi^2)}$$
$$+\exp(t)\sin(3\pi t)+3\pi\exp(t)\cos(3\pi t),$$
$$K(t,\tau)=\exp(t-3\tau),\quad F(u)=u^2.$$

The exact solution is $u(t)=\exp(t)\sin(3\pi t)$.

The errors in L^∞ and L^2 are defined by the same as in example 1. For solving the nonlinear system (2.15), simple iterate method (5.1) is employed with tolerate $\varepsilon=10^{-16}$ (that is, $\left\|\mathbf{u}^{k+1}-\mathbf{u}^k\right\|_2\le\varepsilon$). In Table 3, we present the errors and iterate numbers with different N. The results also show "spectral accuracy".

Table 3 Errors with different N for Example 3

N	8	12	16	20	24	28
Iterations	-	11	10	10	9	10
Errinf	8.4862e-02	3.9052e-04	1.3631e-06	2.9890e-09	2.3864e-12	3.2441e-13
Err2	5.3712e-02	2.4405e-04	8.4872e-07	1.8684e-09	1.5316e-12	6.7479e-14
Err(1)	3.7932e-02	1.7916e-04	6.2291e-07	1.3753e-09	1.1381e-12	1.9874e-14

Example 4 Consider the multi-step method (4.1) to solve linear Volterra integro-differential equation (1.1) with:

$$g(t)=-\sin t+\frac{t}{2}\left(\frac{1-\cos 2t}{2}\right),\qquad K(t,s)=t\sin s,\qquad F(u)=u.$$

The exact solution is $u(t)=\cos t$. We consider that the computational interval $[0,T](T=10)$ is divided equally as $t_i=i\tau, i=0,1,...,M=T/\tau$. For simplicity, numbers N_k of collocation points in subinterval $[t_{k-1},t_k]$ are same. The numerical results are listed in Tables 4-7.

Table 4 Errors with different N for Example 4 with $\tau=10$.

N	8	12	16	20	24	28
Err(10)	5.4443e-01	6.3693e-03	2.5312e-06	3.6964e-09	2.0853e-10	7.8837e-10

Table 5 Errors with different N for Example 4 with $\tau=5$.

N	10	12	14	16	18	20
Err(10)	2.1937e-03	3.3463e-05	2.7518e-07	1.2771e-09	2.0768e-10	2.5006e-10

Table 6 Errors with different N for Example 4 with $\tau=2.5$.

N	6	8	10	12	14	16
Err(10)	8.5062e-02	6.0013e-04	2.3388e-06	6.3053e-09	1.0163e-10	1.7211e-10

Table 7 Errors with different N for Example 4 with $\tau=2$.

N	6	8	10	12	14	16
Err(10)	2.3879e-02	8.9386e-05	2.2541e-07	3.6889e-10	1.5094e-11	9.2655e-11

6. Concluding Remarks

In this paper, we proposed a Legendre-Gauss-Radau collocation method for solving Volterra-type integro-differential equation of second kind. This method is easy to be implemented for nonlinear problems. In particular, benefiting from the rapid convergence of the Legendre-Gauss-Radau interpolation, this method possesses spectral accuracy. It means that for any fixed mode N, the smoother the exact solutions become, the more accurate the numerical results will be.

We also provided a multi-step version of Legendre-Gauss-Radau collocation method.We could use this process with moderate mode N to evaluate the numerical solution, step by step. This simplifies actual computation and saves work essentially. Furthermore, for any fixed mode N, the numerical solutions have high convergence rate. Meanwhile, for any fixed step size in time, the numerical errors decay very rapidly as the smoothness of solutions and the mode N increase.

The numerical results demonstrated the spectral accuracy of proposed methods and coincided well with the theoretical analysis. It was also indicated that our new methods provide more accurate numerical results. On the other hand, in the derivation of the existing collocation method, one could use the Lagrange interpolation which is unstable for large N. Whereas we used the Gauss-type interpolation as in [18], [19], [20], which makes our methods much more stable for large N. This is also confirmed by the numerical results.

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