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A Modified Search Direction Iteration Method for the Absolute Value Equations

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Abstract
In this paper, based on minimization technique and preconditioning technique, a modified search direction iteration method is presented to solve the absolute value equations $Ax-|x|=b$. The convergence properties of the proposed method are given. Theoretical analysis shows that the proposed method is not only suitable for the real symmetric matrix A, but also suitable for the real asymmetric indefinite matrix A. Numerical experiments are reported to demonstrate the effectiveness of the proposed method.

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Key Word and Phrases
Absolute Value Equation, Positive Definite Matrix, Minimization Technique, Preconditioning Technique.

1. Introduction

We consider the absolute value equations (AVE)

$$Ax-|x|=b, \tag{1.1}$$

where $A\in R^{nxn}$, $b\in R^n$ and $|x|$ denotes all components of the vector $x\in R^n$ by absolute value. A general form of the AVE (1.1) is first introduced in [1] as follows:

$$Ax-B|x|=b, \tag{1.2}$$

which has been investigated in a more general context in [2]. As is known in [5], the general linear complementarity problem (LCP) [3], [4] which subsumes many mathematical programming problems can be formulated as the AVE (1.1). This implies that the AVE (1.1) is NP-hard in its general form [2].

In recent years, to efficiently solve the AVE (1.1), some numerical methods have been developed, such as the smoothing Newton method [6], the generalized Newton method [7], [9], [10], the sign accord method [11]. On other forms of the iteration method, one can see [12]-[14] for more details.

Recently, based on the minimization technique, Noor et al. [15] proposed the following search direction iterative scheme for solving the AVE (1.1):

$$x_k = x_{k-1} + \alpha_k v_k, \tag{1.3}$$

with $\alpha_k = -\frac{\langle Ax_{k-1}-|x_{k-1}|-b, v_k\rangle}{\langle Cv_k, v_k\rangle}$, $C=A-D$, $D=diag(sign(x))$, $v_k(k=1,2,\ldots)$ being the search direction, and discussed some convergence properties of the search direction iterative scheme (1.3). Numerical experiments are reported to illustrate that this search direction iterative scheme (1.3) is feasible.

By investigating the search direction iterative scheme (1.3), it is not difficult to find that the search direction iterative scheme (1.3) is only suitable for the real symmetric matrix A and the symmetric positive definite matrix C. When the involved matrix A in (1.1) is a real asymmetric indefinite matrix A, the search direction iterative scheme (1.3) may be invalid. In this case, it needs to establish the modified iterative scheme for solving the AVE (1.1). To overcome this disadvantage of the iterative scheme (1.3) and improve the convergence of the iterative scheme (1.3), based on the preconditioning technique, a modified search direction iteration method is presented to solve the AVE (1.1). Compared with the search direction iterative scheme (1.3), the modified search direction iteration method is not only suitable for the real symmetric matrix A, but also suitable for the real asymmetric indefinite matrix A. The convergence properties of the proposed method are given. Numerical experiments are reported to demonstrate the effectiveness of the proposed method.

For convenience, here briefly explain some terminologies used in the next section. Let R^n be the finite dimension Euclidean space, whose inner product and norm, respectively, are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$. For $x\in R^n$, $sign(x)$ denotes a vector with components equal to 1, 0,-1 depending on whether the corresponding component of x is positive, zero or negative. The diagonal matrix $D(x) = diag(sign(x))$ denotes a diagonal matrix corresponding to $sign(x)$. We consider the matrix A such that $C = P(A-D)$ is positive definite for any arbitrary matrices D and P. The matrix A is said to be symmetric positive definite if it is symmetric and satisfies $x^T Ax > 0$ for all $x\in R^n \setminus \{0\}$. The matrix A is said to be positive definite if its symmetric part $\frac{1}{2}(A^T + A)$ is positive definite.

2. Main Results

In order to accelerate the convergence of the search direction iteration method (1.3) for solving the nonlinear system (1.1), preconditioned methods can be used. Based on this fact, we can consider the preconditioned AVE (1.1):

$$PAx - P\,|\,x\,| = Pb, \tag{2.1}$$

where P, called the preconditioner, is a non-singular matrix. Obviously, the form (2.1) is another general form of the AVE (1.1), which is different from the form (1.2). When $P = I$, the form (2.1) reduces to the AVE (1.1).

Based on (1.1) and (2.1), clearly, the following conclusion is obtained and its proof is omitted.

Theorem 2.1

The solution of the AVE (1.1) is the same as that of the preconditioned AVE (2.1).

Based on Theorem 2.1 and minimization technique, the following result can be obtained, whose proof is similar to the proof of Theorem 2.1 in [15].

Theorem 2.2

If $PC = P(A-D)$ is positive definite matrix for each $x\in R^n$, then $x\in R^n$ is a solution of the preconditioned AVE (2.1) if and only if $x\in R^n$ is a minimum of the function $f(x)$, where:

$$f(x) = \langle PAx, x\rangle - \langle P|x|, x\rangle - 2\langle Pb, x\rangle,\ \ x\in R^n \tag{2.2}$$

Proof. Making the substitution PAx for Ax, $P\,|\,x\,|$ for $|\,x\,|$, and Pb for b in the proof of Theorem 2.1 [15], it is easy to verify that the result in Theorem 2.2 holds.

In [15], the following result was obtained.

Theorem 2.3 [15]

If $C = A-D$ is positive definite matrix for each $x\in R^n$, then $x\in R^n$ is a solution of the AVE (1.1) if and only if $x\in R^n$ is a minimum of the function $f(x)$, where:

$$f(x) = \langle Ax, x\rangle - \langle |x|, x\rangle - 2\langle b, x\rangle,\ \ x\in R^n \tag{2.3}$$

Some remarks on Theorems 2.2 and 2.3 are given.

- Clearly, Theorem 2.2 is a generalization of Theorem 2.3. When $P = I$, then Theorem 2.2

reduces to Theorem 2.3.

- Compared Theorem 2.2 with Theorem 2.3, obviously, the conditions of Theorem 2.2 may be more weaker than that of Theorem 2.3. It is reason that matrix $C = A - D$ in Theorem 2.2 need not be positive definite matrix, but matrix $PC = P(A-D)$ in Theorem 2.2 need be positive definite matrix. That is to say, it needs to choose the preconditioner P make $PC = P(A-D)$ positive definite, which implies that the results of Theorem 2.2 may be suitable for the real asymmetric indefinite matrix A for the AVE (1.1).

Based on Theorem 2.2, a modified search direction iteration method for solving the AVE (1.1) can be established below.

Algorithm 1 (A modified search direction iteration method)
Step 1. Choose an arbitrary vector $x_0 \in R^n$ and an initial search direction $u_1 \neq 0$. Set $k := 1$.
Step 2. If $\|x_k - x_{k-1}\| = 0$, stop.
Step 3. Compute x_k by:

$$x_k = x_{k-1} + \bar{\alpha}_k u_k, \; k = 1, 2, \ldots, \tag{2.4}$$

where

$$\bar{\alpha}_k = -\frac{\langle P(Ax_{k-1} - |x_{k-1}| - b), u_k \rangle}{\langle PCu_k, u_k \rangle}.$$

Step 4. Set $k := k + 1$. Go to step 1.

Remark 2.1 Compared with the modified search direction iteration method (2.4) with the original search direction iteration method (1.3), the former is a generalization of the latter. Not only that, one can choose a proper preconditioner P such that the former is more efficient than the latter (to see the next section). Of course, when one establishes the preconditioner P, it is noted that the preconditioner P should be cheap to construct and apply.

Remark 2.2 It is noted that when $u_k = e_k$, where e_k is the *k*th column of the identity matrix, it is not difficult to find that the modified search direction iteration method (2.4) examines the preconditioned AVE (2.1) one at a time in sequence and previously computed results are used as soon as they are available. Therefore, in this case, the modified search direction iteration method (2.4) can be considered as a preconditioned Gauss-Seidel method for solving the AVE (1.1). In our numerical experiments, we take $u_k = e_k$, some numerical results are reported to illustrate the efficiency of the modified search direction iteration method (2.4) (to see next section).

To obtain the local convergence theory of the modified search direction iteration method (2.4), here it is necessary to define a vector norm $\|x\|_M^2 = \langle Mx, x \rangle$ for any $x \in R^n$, which is called M-norm.

Theorem 2.4
Let $PC = P(A-D)$ be symmetric, $f(x)$ be defined in Theorem 2.2. Then the modified search direction iteration method (2.4) with $u_k = e_k$, where e_k is the *k*th column of the identity matrix, converges linearly to a solution x^* of the preconditioned AVE (2.1) in PC-norm.

Proof. Based on Theorem 2.2, we have:

$$\begin{aligned}
&\left\|x_k - x^*\right\|_{PC}^2 - \left\|x_{k-1} - x^*\right\|_{PC}^2 \\
&= \left\langle PC(x_k - x^*), x_k - x^* \right\rangle - \left\langle PC(x_{k-1} - x^*), x_{k-1} - x^* \right\rangle \\
&= \left\langle PCx_k, x_k \right\rangle - 2\left\langle PCx^*, x_k \right\rangle - [\left\langle PCx_{k-1}, x_{k-1} \right\rangle - 2\left\langle PCx^*, x_{k-1} \right\rangle] \\
&= \left\langle PAx_k - P|x_k|, x_k \right\rangle - 2\left\langle PAx^* - P|x^*|, x_k \right\rangle \\
&\quad - \left[\left\langle PAx_{k-1} - P|x_{k-1}|, x_{k-1} \right\rangle - 2\left\langle PAx^* - P|x^*|, x_{k-1} \right\rangle\right] \\
&= f(x_k) - f(x_{k-1}) \\
&= f(x_{k-1} + \bar{\alpha}_k u_k) - f(x_{k-1}) < 0.
\end{aligned}$$

Then,

$$\left\|x_k - x^*\right\|_{PC}^2 < \left\|x_{k-1} - x^*\right\|_{PC}^2. \tag{2.5}$$

The inequality (2.5) implies that the sequence $\{x_k\}$ is a Fejer sequence in [16]. Therefore, the sequence $\{x_k\}$ converges linearly to a solution of the preconditioned AVE (2.1) in PC-norm.

3. Numerical Experiments

In this section, we give some numerical experiments to demonstrate the performance of the proposed method for solving the AVE (1.1). Here, 'MSR' denotes the modified search direction iteration method (2.4) and 'SR' denotes the original search direction iteration method (1.3). We compare MSR with SR from the point of view of the number of iterations (denoted as IT) and CPU elapsed times (denoted as CPU). All the tests are performed in MATLAB 7.0.

Example 3.1. Let the matrix A be given by Definition 2.1:

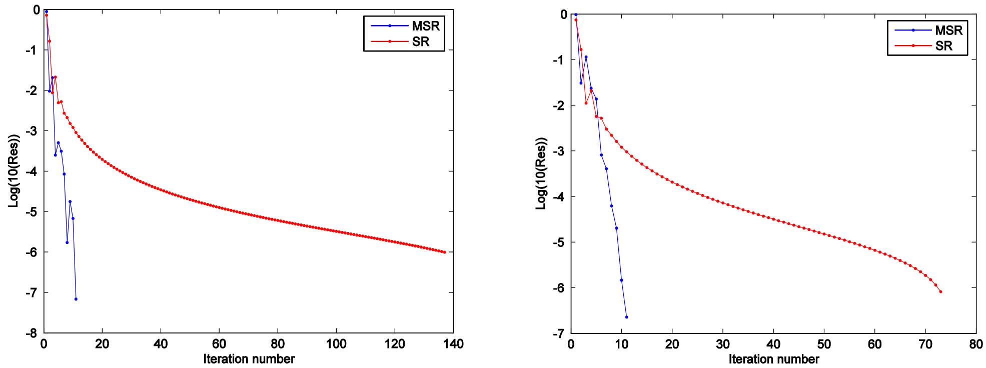
$$a_{ii} = 2n, a_{i,i+1} = a_{i+1,i} = n, a_{i,j} = 0.5, j \neq i, i+1.$$

Let $b = (A - I)e$, where I is the identity matrix and $e = (1,1,\ldots,1)^T$ such that $x = (1,1,\ldots,1)^T$ is the exact solution. The stopping criteria is $\|x_k - x_{k-1}\| < 10^{-6}$ and the initial guess is $x_0 = (x_1, x_2, \ldots, x_n)^T$, $x_i = 0.001i.$

Table 1 IT and CPU for MSR and SR.

	n	32	64	126	256
MSR	IT	11	11	15	8
	CPU(s)	0.0156	0.125	1.648	16.5
SR	IT	73	137	160	149
	CPU(s)	0.0781	0.3594	1.969	18.3125

In the implementation of the MSR method, it is necessary to choose the appropriate preconditioner P for the MSR method. In our numerical computations, we take the preconditioner $P = tridiag(A)$ to yield the least CPU elapsed times and iteration numbers for the MSR method. In this case, Table 1 lists the iteration numbers and CPU elapsed times for the MSR and SR methods. Figure 1 plots the iteration numbers of the MSR and SR methods with $n = 32$ and $n = 64$. Similarly, the figure of the iteration numbers of the MSR and SR methods with $n = 126$ and $n = 256$ is plotted as well and here is omitted.



(a) n=32 (b) n=64

Fig. 1 IT of MSR and SR

From Table 1, the iteration numbers and CPU elapsed times of the MSR method for solving the AVE (1.1) are less than that of the SR method if it is appropriate to choose the preconditioner P. That is to say, compared with the SR method, the MSR method for solving the AVE (1.1) may be given priority under certain conditions.

4. Conclusions

In this paper, a modified search direction iteration method has been presented to solve the absolute value equations. The convergence properties of the proposed method are discussed. Numerical experiments confirm the effectiveness of the proposed method**.**

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