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An Augmentation Preconditioner for Asymmetric Saddle Point Problems with Singular (1,1) Blocks

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Abstract
In this paper, an augmentation preconditioner for asymmetric saddle point problems with singular (1,1) blocks is introduced on the base of the recent article by He and Huang [Two augmentation preconditioners for nonsymmetric and indefinite saddle point linear systems with singular (1, 1) blocks, Comput. Math. Appl., 62 (2011) 87-92]. We study the spectral characteristics of the preconditioned matrix in detail. Theoretical analysis shows that all the eigenvalues of the preconditioned matrix are strongly clustered. Numerical experiments are given to demonstrate the efficiency of the presented preconditioner.

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Key Word and Phrases
Augmentation, Preconditioner, Saddle Point Systems, Krylov Subspace Method, Eigenvalue.

1. Introduction

Consider the following general nonsingular saddle point problems:

$$Kx \equiv \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1}$$

where $A \in R^{nxn}$ and $B, C \in R^{mxn}$ have full rank. Without loss of generality, we assume that K is nonsingular and A is singular with nullity. The form of (1) frequently occurs in a large number of applications, such as the (linearized) Navier-Stokes equations [1,15,18], the time-harmonic Maxwell equations [2-4,16,17], the linear programming (LP) problem and the quadratic programming (QP) problem [5-6]. In recent years, a great deal of effort has been made to solve saddle point problems. Most of the work has aimed at developing effective preconditioning techniques [7].

Recently, He and Huang [8] introduced the following preconditioners:

$$P_1 = \begin{bmatrix} A + B^T W^{-1} C & B^T \\ C & -W \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} A + B^T W^{-1} C & B^T \\ C & -C(A + B^T W^{-1} C)^{-1} B^T \end{bmatrix},$$

where P_1, P_2, W and $A + B^T W^{-1} C$ are invertible. It was shown that if A is singular with nullity s, then the eigenvalue distribution of the preconditioned matrix $P_1^{-1} K$ is as follows: 1 with algebraic multiplicity *n-m*, $\frac{1\pm i}{2}$ $(i^2 = -1)$ with algebraic multiplicity *2s* and the remaining eigenvalues satisfying $\frac{1}{2}\pm\sqrt{\frac{1-2d}{1+2d}}$ with algebraic multiplicity *2*(*m*-*s*), where d is the generalized eigenvalues of $dAx = B^T W^{-1} Cx$. The eigenvalue distribution of $P_1^{-1}K$ and $P_2^{-1}K$ have a similar conclusion [8]. Obviously, this is favorable to Krylov subspace methods, which rely on the matrix-vector products and the number of distinct eigenvalues and eigenvectors of the preconditioned matrix [9,10,14]. It is well-known fact that the preconditioning technique attempts to make the spectral property better to improve the rate of convergence of Krylov subspace mehtods [11].

In the light of the preconditioning idea, this paper is devoted to giving a new augmentation preconditioner for saddle point linear systems (1), that is to say:

$$T = \begin{bmatrix} A + B^T W^{-1} C & (1+\sqrt{2})B^T \\ (1-\sqrt{2})C & -2W \end{bmatrix},$$

where T, W and $A + B^T W^{-1} C$ are invertible. Obviously, the preconditioner T is different from the preconditioners P_1 and P_2. It is shown that, in contrast to augmentation preconditioners P_1 and P_2, all the eigenvalues of the proposed new preconditioned matrix are more strongly clustered. Numerical experiments show that the new preconditioner is slightly more efficient than the preconditioner P_1.

Forming the preconditioners P_1, P_2 and T may be computationally expensive (in particular setting up the Schur complement and solving for it) and in practice cheaper alternatives must be sought. Nevertheless, understanding the spectral properties of P_1, P_2 and T is useful since it can illustrate the behavior of preconditioners based on approximations of their components. Therefore, our focus throughout this paper is on the analysis. Finally, numerical experiments are reported to confirm the presented results and illustrate the efficiency of the presented preconditioner.

2. Main Results

To conveniently discuss, the following two lemmas are required.

Lemma 2.1 [12]

If the asymmetric coefficient matrix

$$K = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$$

is nonsingular, then

$$rank(B) = rank(C) = m,\ \mathbb{N}(A)\cap\mathbb{N}(C) = \{0\}\ \ and\ \ \mathbb{N}(A^T)\cap\mathbb{N}(B) = \{0\},$$

where N(·) denotes the null space of matrix.

Lemma 2.2 [12]

If the asymmetric coefficient matrix

$$K=\begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$$

is nonsingular, then the rank of the matrix A is at least n-m, and hence its nullity is at most m.

The following theorem provides the eigenvalue distribution of the preconditioned matrix $T^{-1}K$.

Theorem 2.1

Suppose that K is nonsingular and that its (1,1) block A is singular with nullity s (s<<m), then λ=1 is an eigenvalue of $T^{-1}K$ with geometric multiplicity n-m+$2s$. The remaining $2(m$-$s)$ eigenvalues satisfy the relation:

$$\lambda=\frac{\delta}{2+\delta},$$

where δ are the generalized eigenvalues defined by

$$\delta Av = B^T W^{-1} Cv \quad \textbf{(2)}$$

Let $\{z_i\}_{i=1}^{n-m}$ be a basis of N(C) and $\{z_i\}_{i=1}^{s}$ a basis of N(A). Then a set of linear independent eigenvectors corresponding to λ=1 can be found: the n-m vectors $[z_i^T, 0^T]^T$ and the $2s$ vectors $[x_i^T, -\frac{\sqrt{2}}{2}(W^{-1}Cx_i)^T]^T$.

Proof.

Let λ be an eigenvalue of $T^{-1}K$ with eigenvector $[v^T, q^T]^T$. Then:

$$\begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}\begin{bmatrix} v \\ q \end{bmatrix}=\lambda\begin{bmatrix} A+B^TW^{-1}C & (1+\sqrt{2})B^T \\ (1-\sqrt{2})C & -2W \end{bmatrix}\begin{bmatrix} v \\ q \end{bmatrix},$$

which can be rewritten into:

$$Av+B^Tq=\lambda(A+B^TW^{-1}C)v+(1+\sqrt{2})\lambda B^Tq, \quad (3)$$

$$Cv=(1-\sqrt{2})\lambda Cv-2\lambda Wq. \quad (4)$$

Since K is nonsingular, it is not difficult to obtain that $\lambda \neq 0$ and $v \neq 0$. By (4), we get:

$$q=\frac{(1-\sqrt{2})\lambda-1}{2\lambda}W^{-1}Cv.$$

Substituting it into (3) yields:

$$2(\lambda^2-\lambda)Av+(\lambda-1)^2B^TW^{-1}Cv=0. \quad (5)$$

If $v \in N(C)$, then (5) implies that:

$$2(\lambda^2-\lambda)Av=0.$$

It follows that $\lambda = 1$ and let $\{z_i\}_{i=1}^{n-m}$ be a basis of N(C), then n-m vectors $[z_i^T, 0^T]^T$ are linearly independent eigenvectors associated with $\lambda = 1$.

If $v \in N(A)$ then from (5) we obtain:

$$(\lambda-1)^2B^TW^{-1}Cv=0,$$

from which it follows that $\lambda = 1$ and $2s$ vectors $[x_i^T, -\frac{\sqrt{2}}{2}(W^{-1}Cx_i)^T]^T$ are linearly independent eigenvectors associated with $\lambda = 1$.

Assume that $\lambda \neq 1$. Combining (2) with (5) yields:

$$2(\lambda^2-\lambda)=-\delta(\lambda-1)^2$$

It is easy to see that the rest $2(m$-$s)$ eigenvalues satisfy

$$\lambda=\frac{\delta}{2+\delta}. \quad (6)$$

Remark 2.1

(6) gives an explicit formula in terms of the generalized eigenvalues of (5) and becomes tightly clustered as $\delta \to \infty$. Since λ is a strictly increasing function of δ on $(0,\infty)$, it is easy to see that the remaining eigenvalues $\lambda \to 1$ as $\delta \to \infty$. The simplest choice is that $W^{-1} = \gamma I$ $(\gamma > 0)$ [16], which leads to the rest $2(m\text{-}s)$ eigenvalues satisfying:

$$\lambda = \frac{\gamma\delta}{2+\gamma\delta},$$

where δ are the generalized eigenvalue defined by $\delta Ax = B^T Cx$. Obviously, the parameter γ should be chosen to be large such that the eigenvalues are strongly clustered, but not too large such that the (2,2) block of T is too near singular. In actual implementation, for simplicity, the choice of the matrix W is often a scalar matrix.

From Lemma 2.2 , it is easy to get that the nullity of A is at most m. Theorem 2.1 shows that the higher it is, the more strongly the eigenvalues are clustered. Combining Lemma 2.2 with Theorem 2.1, the following theorem is given.

Theorem 2.2

Suppose that K is nonsingular and that its (1,1) block A is singular with nullity m. Then $\lambda = 1$ is an eigenvalue of $T^{-1}K$ with geometric multiplicity $n+m$.

The important consequence of Theorem 2.2 is that the preconditioned matrix $T^{-1}K$ have minimal polynomials of degree at most 1. Therefore, Krylov subspace methods like GMRES applied to the preconditioned linear systems with coefficient matrix $T^{-1}K$ converge in one step if roundoff errors are ignored.

3. Numerical Experiments

In this section, some numerical examples are reported to demonstrate the performance of two preconditioners T and P_1. In our numerical experiments, all the computations are done with MATLAB 7.0.

Example 1. Consider the following Oseen problem:

$$\begin{cases} -v\Delta u + \omega \cdot \mathrm{grad} u + \mathrm{grad} p = f, \ in\ \Omega, \\ -\mathrm{div} u = 0, \ in\ \Omega, \end{cases} \tag{7}$$

with suitable boundary condition on $\partial\Omega$, where w is given such that $div(\omega) = 0$, v is the viscosity. In our experiments, two values of the viscosity parameter are used for the Oseen equation: $v = 1$ and $v = 0.01$. The test problems are leaky two-dimensional lid-driven cavity problem in square $(0 \le x \le 1.0 \le y \le 1)$. Using IFISS [13] to discretize (7), we take a finite element subdivision based on uniform grid of square element. The mixed finite element used is the bilinear constant-velocity-pressure Q_1 - P_0 pair with stabilization (the stabilization parameter is chosen to 1/4). We get the (1,1) block A of the coefficient matrix corresponding to the discretization of the conservative term is positive real, i.e., $A + A^T$ is symmetric positive definite. Since the matrix B produced by this package is rank deficient, we drop the first two rows of B to get a full rank matrix. Corresponding to B, we also drop the first two rows and columns of the (2, 2) block. The matrix $\hat{K}$ arises from the discretization of the Oseen equations (7) and is of the form as follows:

$$\hat{K} = \begin{bmatrix} F_1 & & B_u^T \\ & F_2 & B_v^T \\ B_u & B_v & 0 \end{bmatrix}.$$

To introduce the form (1), the matrices $[B_u, B_v]$ and $[B_u, B_v]^T$ are replaced by a random matrix

$\hat{C}$ with the same sparsity as $[B_u, B_v]$ and a random matrix $\hat{B}^T$ with the same sparsity as $[B_u, B_v]^T$, respectively. $\hat{C}(1:m,1:m)$ and $\hat{B}(1:m,1:m)$ are replaced by $C_1 = \hat{C}(1:m,1:m) - \frac{3}{2}I_m$ and $B_1 = \hat{B}(1:m,1:m) - \frac{3}{2}I_m$, respectively, such that C_1 and B_1 are nonsingular.

Define $C_2 = \hat{C}(1:m, m+1:n)$ and $B_2 = \hat{B}(1:m, m+1:n)$, then we have $C = [C_1, C_2]$ and $B = [B_1, B_2]$ with $B_1, C_1 \in R^{mxm}$ and $B_2, C_2 \in R^{mx(n-m)}$. The above strategy leads to the following saddle point-type matrix:

$$K = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$$

with

$$rank(C) = rank(B) = m.$$

From Lemma 2.2, noting that the nullity of A is at most m, we construct the following saddle point-type matrices:

$$K_i = \begin{bmatrix} A_i & B^T \\ C & 0 \end{bmatrix}, \ i = 1, 2,$$

where A_i is constructed from A by filling its first $i \times m/2$ rows and columns with zero entries. In this case, the real matrix A_i is positive semidefinite matrix and its nullity is $i \times m/2$. We take two meshes *h*: 1/16, 1/32.

In our numerical computations, the matrix W in the augmentation block precoditioners is taken as $W = I_m$. The incomplete LU factorization of $A_i + B^T C$ $(i = 1, 2)$ with drop tolerance $\tau = 0.0001$ is used. Figure 1 plots the eigenvalues of K, $P^{-1}K$ and $T^{-1}K$ for $\nu = 0.01$ and *h*=1/16, where left in Figure 1 corresponds to nullity *m* and right in Figure 1 corresponds to nullity m/2.

From Figure 1, it is easy to see that the preconditioners T and P_1 indeed make the spectrum of the coefficient matrix K better. It can clearly see that the higher the nullity of the (1,1) block is, the stronger the eigenvalues of the preconditioned matrices are clustered. The preconditioned matrix $T^{-1}K$ has only one clustering points: 1. The preconditioned matrix $P^{-1}K$ has three clustering points: 1 and (1±i)/2. The clustering degree of the preconditioned matrix $T^{-1}K$ is superior to that of the preconditioned matrix $P^{-1}K$.

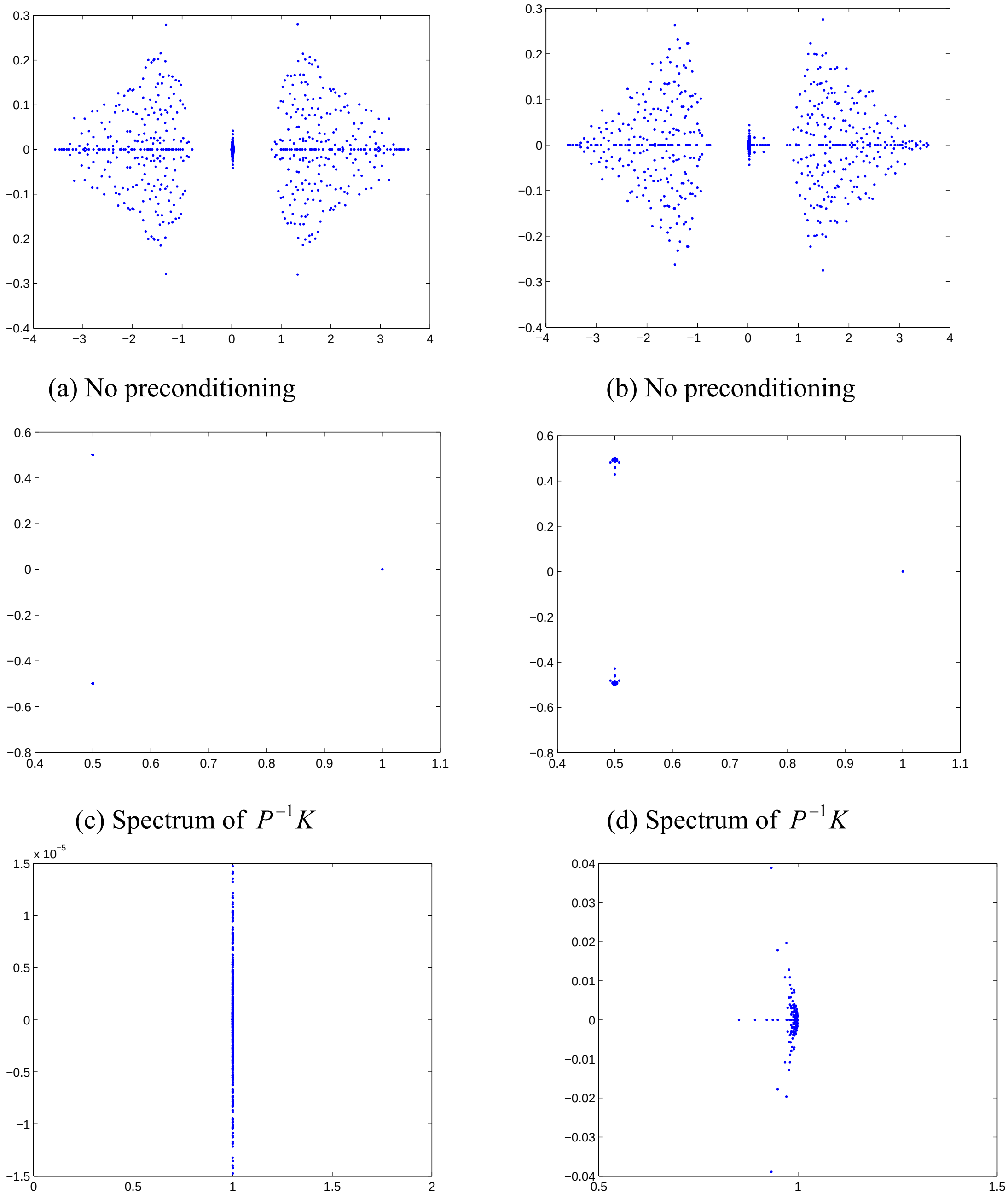
In the sequence, we will use the preconditioned GMRES(*l*) to solve the corresponding saddle point linear systems (1), where the right-hand side *f* is taken such that the solution is all ones. The purpose of these experiments is just to investigate the influence of the eigenvalue distribution on the convergence behavior of GMRES(*l*). In general, the choice of the restart parameter *l* (*l*<< *n*) is no general rule, which mostly depends on a matter of experience in practice. In our numerical experiments, for the sake of simplicity, we take *l*=20.

All tests are started from the zero vector and the stopping criterion is chosen as follows $\|r^{(k)}\|_2 / \|r^{(0)}\|_2 \le 10^{-6}$. In Tables 1 and 2, we present some results to illustrate the convergence behaviors of GMRES(20) preconditioned by T and P_1. The purpose of these experiments is just to investigate the influence of the eigenvalue distribution on the convergence behavior of the GMRES(20) method. "IT" denotes the number of iteration. "CPU(s)" denotes the time (in seconds) required to solve a problem.

Table 1 Iteration number and CPU(s) of GMRES(20) with nullity *m*.

	T ($v=1$)		P_1 ($v=1$)		T ($v=0.01$)		P_1 ($v=0.01$)	
	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)
16´ 16	5	0.1875	7	0.2031	8	0.1813	10	0.2813
32´ 32	21	13.2656	22	14.7344	21	20.7969	31	33.1406

From Tables 1-2, the preconditioners T and P_1 are quite competitive in terms of convergence rate, robustness and efficiency. Further, it is easy to see that the preconditioner T outperforms the preconditioner P_1 from the iteration numbers and CPU's time. Compared with the preconditioner P_1, the preconditioner T may be preferentially considered under certain conditions.



(a) No preconditioning (b) No preconditioning

(c) Spectrum of $P^{-1}K$ (d) Spectrum of $P^{-1}K$

(e) Spectrum of $T^{-1}K$ (f) Spectrum of $T^{-1}K$

Fig. 1 Spectra of the matri K, $P_1^{-1}K$ and $T^{-1}K$.

Table 2 Iteration number and CPU(s) of GMRES(20) with nullity $m/2$.

	$T\ (v=1)$		$P_1\ (v=1)$		$T\ (v=0.01)$		$P_1\ (v=0.01)$	
	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)
16´ 16	18	0.6719	19	0.7656	9	0.3594	13	0.4531
32´ 32	23	26.2969	23	26.9219	15	25.8906	19	31.9375

Example 2. A matrix from the UF Sparse Matrix Collection.

The test matrix is Garon/Garon1, coming from UF Sparse Matrix Collection, which is an ill-conditioned matrix arising from computation fluid dynamics problem. The characteristics of the test matrix are listed in Table 3: see [19] for details.

Table 3 Characteristics of the test matrix from the UF Sparse Matrix Collection.

Matrix name	n	m	$nnz(A)$	$nnz(B)$	$nnz((C))$
Garon/Garon1	2775	400	58949	12889	12885

Table 4 Iteration number and CPU(s) of GMRES(20).

T		P_1	
IT	CPU(s)	IT	CPU(s)
10	2.2190	20	4.3900

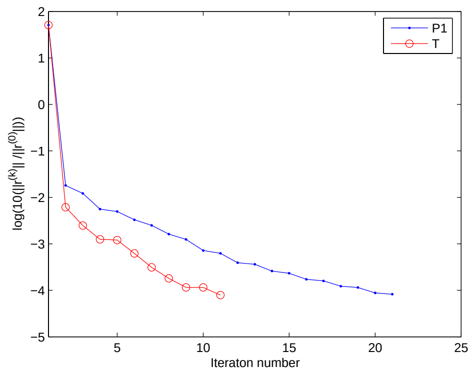


Fig. 2 Iteration number of GMRES(20)

The numerical results from using the GMRES(20) method with two preconditioners T and P_1 to solve the systems of linear equations with the coefficient matrix of Garon/Garon1 are given in Table 4, and Figure 2 corresponds to Table 4.

From Table 4 and Figure 2, it is not difficult to find that the preconditioner T outperforms the preconditioner P_1. In other words, compared with the preconditioner P_1, the preconditioner T is more feasible and competitive.

4. Conclusions

By the current research, we have presented a new augmentation preconditioner for asymmetric saddle point problems with singular (1,1) blocks. The spectral characteristics of the preconditioned matrix have been discussed in detail. Theoretical analysis shows that all the eigenvalues of the preconditioned matrix are more strongly clustered than that of the preconditioned matrix in [8]. Numerical experiments are given to demonstrate the efficiency of the presented preconditioner.

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