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Analysis of an Age-Structured Epidemic Model with Differential Infectivity

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Abstract

The major challenge in studying the behavior of a viral infection is the variation that occurs in the level of infection. In this paper, a multi-infected group age-structured epidemic model has been considered. The existence and uniqueness of the nonnegative solution in this model has been proved. Threshold results determining the existence of endemic states have been established under various conditions. The local stability of the steady states have been discussed in this article.

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Key Word and Phrases

Age-Structure, Abstract Cauchy Problem, C_0 -semigroup, Positive Operator, Endemic States, Threshold Value, Equilibrium and Stability Analyses.

1. Introduction

Viral disease is one of the most common disease at present. The number of diseases transmitted by viruses are growing at a rapid rate. Several research works across the globe is aimed at investigating the probable reasons behind the disease. In a certain study [1], the temporal progression of the clinical, radiological and virological changes in community outbreak of severe acute respiratory syndrome (SARS) was examined. The study results claimed that worsening in the patient condition is not related to uncontrolled viral replication but may be related to immunopathological damage caused by the virus. Viral level often determine the ability of transmission for some diseases such as malaria and fever, where the infectivity depends on the number of parasites or viral loads in infected hosts or vectors [2],[3]. There is a huge volume of research on the viral action of Human Immunodeficiency Virus. A differential infectivity model was proposed in [4],[5],[6]. Furthermore, [5] subdivided the infected population into n subgroups, $SI_1I_2\cdots I_nR$. Upon infection, an individual enters subgroup j with probability p_j and stays in that group until it is inactive in transmission, where $\sum_{j=1}^{n} p_j = 1$. Highly active antiretroviral therapy (HAART) is currently one of the most important component for treating HIV-1 infection. A study on the efficacy of HAART in suppression of HIV-1 has been examined in [7]. It was observed that the rate of virological failure of HAART was highly documented among the SWISS cohort group where the study was conducted, but the chances of clinical progression was low even in patients with viral rebound.

While ODE models are often used when the population structures (age, sex, etc.) are neglected, there are many cases in which incorporating one or more of these structures into the model may provide additional and important information which may be helpful in the understanding of the disease dynamics. The incorporation of age-dependent demographical and/or epidemiological parameters usually leads to a system of first-order partial differential equations with nonlocal boundary conditions. In articles like [8], the author has considered the mixing strategies and emphasized the role played by proportionate mixing through an age structured model. The authors also developed expression in terms of preference function for general solution of the framework. In this paper, we study a more general age-structured $SI_1I_2\cdots I_nR$ model that includes multiple infected-groups of human populations. We need to specify a general assumption that ensures the uniqueness of the positive equilibrium as well as the local stability result, which follows subsequently.

The paper has been organized as follows. In Section 2, we describe the multi-infected-group model and the reduced system under the assumption that the total population has reached its stable age distribution. In Section 3, we find the C_0-semigroup which has been generated by the system of linear age-structured model. Furthermore, we discussed the existence and uniqueness of nonnegative solution. Our main theorems on the existence of steady states are given in Section 4. The main results about the stability analysis of the equilibrium solutions have been presented in Section 5.

2. The Model

Let us define the state of stress at a point in the stationary frame S^0, by the following stress tensor: (Fig.1) We subdivide a closed population into $n+2$ compartments containing susceptible, n infective and recovered individuals, which means that susceptible individuals become the infected individuals with differential infectivity, and become the recovered individuals with permanent immunity. Corresponding to the differential infectivity, the infectious individuals are divided into n classes, I_1, I_2, $\cdots$, I_n. We assume that the population is in a stationary demographic state. Let $N(a), 0\le a\le r_m$ (r_m denotes the highest age attained by the individuals in the population) be the density with respect to age of the total number of individuals, under our assumptions, $N(a)$ satisfies:

$$N(a)=\mu^* N\exp(-\int_0^a \mu(\sigma)d\sigma). \tag{2.1}$$

where $\mu(a)$ denotes the instantaneous death rate at age a of the population, the constant N is the total size of the population and μ^* is the crude death rate, we assume that $\mu(a)$ is nonnegative, locally integrable on $[0,r_m)$, and satisfies:

$$\int_0^{r_m} \mu(\sigma)d\sigma=+\infty.$$

The crude death rate is determined such that:

$$\mu^*\int_0^{r_m} f(a)da=1,$$

where $f(a)=\exp(-\int_0^a \mu(\sigma)d\sigma)$ is the survival function which is the proportion of individuals who survive to age a. Then we have the relation:

$$N(a)=\mu^* Nf(a). \tag{2.2}$$

Next let $S(a,t)$, $I_j(a,t)$, $(j=1,2,\cdots,n)$ and $R(a,t)$ be the densities of respectively the susceptible , infected population in the jth class and immune population at time t of age a. Hence, we have:

$$N(a)=S(a,t)+\sum_{j=1}^{n} I_j(a,t)+R(a,t). \tag{2.3}$$

Let γ_j be the recovery rate in the class I_j and p_j be the probability that an individual enter the class I_j ($\sum_{j=1}^{n} p_j = 1$). Let $\beta_j(a,b)$ be the age-dependent transmission coefficient of class I_j, that is, the probability that a susceptible person of age a meets an infectious person of age b in the class I_j and becomes infective, per unit of time. Define the force of infectious of class I_j by $\lambda_j(a,t)$ given as:

$$\lambda_j(a,t) = \int_0^{r_m} \beta_j(a,\sigma) I_j(\sigma,t) d\sigma. \tag{2.4}$$

Moreover we assume that the death rate of the population is not affected by the presence of the disease and hence depend only on time. Under the above assumption, the spread of the disease can be described by the system of partial differential equations:

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) S(a,t) = -(\mu(a) + \lambda(a,t)) S(a,t), \tag{2.5$_1$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) I_1(a,t) = p_1 \lambda(a,t) S(a,t) - (\mu(a) + \gamma_1) I_1(a,t), \tag{2.5$_2$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) I_2(a,t) = p_2 \lambda(a,t) S(a,t) - (\mu(a) + \gamma_2) I_2(a,t), \tag{2.5$_3$}$$

$$\vdots$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) I_n(a,t) = p_n \lambda(a,t) S(a,t) - (\mu(a) + \gamma_n) I_n(a,t), \tag{2.5$_{n+1}$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) R(a,t) = \sum_{j=1}^{n} \gamma_j I_j(a,t) - \mu(a) R(a,t), \tag{2.5$_{n+2}$}$$

where:

$$\lambda(a,t) = \sum_{j=1}^{n} \lambda_j(a,t),$$

with boundary conditions:

$$S(0,t) = \mu^* N, \quad I_j(0,t) = 0 (j = 1, \cdots, n), \quad R(0,t) = 0. \tag{2.6}$$

Consider the fraction of susceptible, class I_j and immune population at age a and time t:

$$s(a,t) = \frac{S(a,t)}{N(a)}, \quad i_j(a,t) = \frac{I_j(a,t)}{N(a)} (j = 1, \cdots, n), \quad r(a,t) = \frac{R(a,t)}{N(a)}.$$

Then the system (2.5_1) - (2.5_{n+2}) can be written to a simpler form:

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) s(a,t) = -\lambda(a,t) s(a,t), \tag{2.7$_1$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) i_1(a,t) = p_1 \lambda(a,t) s(a,t) - \gamma_1 i_1(a,t), \tag{2.7$_2$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) i_2(a,t) = p_2 \lambda(a,t) s(a,t) - \gamma_2 i_2(a,t), \tag{2.7$_3$}$$

$$\vdots$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) i_n(a,t) = p_n \lambda(a,t) s(a,t) - \gamma_n i_n(a,t), \tag{2.7$_{n+1}$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a}) r(a,t) = \sum_{j=1}^{n} \gamma_j i_j(a,t), \tag{2.7$_{n+2}$}$$

$$s(0,t) = 1, i_j(0,t) = 0, (j = 1, \cdots, n), \quad r(0,t) = 0, \tag{2.8}$$

where:

$$\lambda(a,t) = \int_0^{r_m} (\sum_{j=1}^{n} \beta_j(a,\xi) i_j(\xi,t)) N(\xi) d\xi, \quad N(a) = \mu^* N f(a), \tag{2.9}$$

$$s(a,t) + \sum_{j=1}^{n} i_j(a,t) + r(a,t) = 1. \tag{2.10}$$

In the following, we mainly consider the system (2.7_1) - (2.7_{n+2}) with the initial conditions:

$$s(a,0) = s^0(a), \quad i_j(a,0) = i_j^0(a) (j = 1, \cdots, n), \quad r(a,0) = r^0(a). \tag{2.11}$$

3. Existence and Uniqueness of Nonnegative Solution

In this section we shall show that the initial-boundary value problem (2.7_1) - (2.7_{n+2}), (2.8), (2.11) has a unique solution. First we note that it suffices to consider the system in terms of only $s(a,t)$, $i_j(a,t)$ $(j=1,\cdots,n)$ since, once these functions are known, $r(a,t)$ can be obtained by $r(a,t)=1-s(a,t)-\sum_{j=1}^{n} i_j(a,t)$.

First we introduce a new variable $\hat{s}$ by $\hat{s}(a,t)=s(a,t)-1$. Then we obtain the new system for $\hat{s}(a,t)$, and $i_j(a,t)$ $(j=1,\cdots,n)$:

$$(\tfrac{\partial}{\partial t}+\tfrac{\partial}{\partial a})\hat{s}(a,t)=-\lambda(a,t)(\hat{s}(a,t)+1), \qquad (3.1_1)$$

$$(\tfrac{\partial}{\partial t}+\tfrac{\partial}{\partial a})i_1(a,t)=p_1\lambda(a,t)(\hat{s}(a,t)+1)-\gamma_1 i_1(a,t), \qquad (3.1_2)$$

$$(\tfrac{\partial}{\partial t}+\tfrac{\partial}{\partial a})i_2(a,t)=p_2\lambda(a,t)(\hat{s}(a,t)+1)-\gamma_2 i_2(a,t), \qquad (3.1_3)$$

$$\vdots$$

$$(\tfrac{\partial}{\partial t}+\tfrac{\partial}{\partial a})i_n(a,t)=p_n\lambda(a,t)(\hat{s}(a,t)+1)-\gamma_n i_n(a,t), \qquad (3.1_{n+1})$$

$$\hat{s}(0,t)=0, \quad i_j(0,t)=0 (j=1,\cdots,n).$$

Let us consider the initial-boundary value problem described by the system (3.1_1) - (3.1_{n+1}) as an abstract Cauchy problem on the Banach space $\mathbf{X}=L^1(0,r_m)\times L^1(0,r_m)\times\cdots\times L^1(0,r_m)$ with norm $\|x\|=\sum_{j=1}^{n+1}\|x\|_1$ for $x(a)=(x_1(a),x_2(a),\cdots,x_{n+1}(a))^T\in\mathbf{X}$, where $\|\cdot\|_1$ is the ordinary norm of $L^1(0,r_m)$. Let A be a linear operator defined by:

$$(Ax)(a)=\begin{pmatrix} -\dfrac{dx_1(a)}{da} \\ -\dfrac{dx_2(a)}{da}-\gamma_1 x_2(a) \\ \vdots \\ -\dfrac{dx_{n+1}(a)}{da}-\gamma_n x_{n+1}(a) \end{pmatrix}, \qquad (3.2)$$

$$x(a)=(x_1(a),x_2(a),\cdots,x_{n+1}(a))^T\in D(A),$$

where p^T is the transpose of the vector p and the domain $D(A)$ is given as:

$$D(A)=\{x\in\mathbf{X}\mid x_i \text{ is absolutely continuous on } [0,r_m),\ x(0)=(0,0,0)^T\}.$$

Suppose that $\beta_j(a,b)\in L^\infty((0,r_m)\times(0,r_m))$. We define a nonlinear operator $F:\mathbf{X}\to\mathbf{X}$ by:

$$(Fx)(a)=\begin{pmatrix} -(\sum_{j=1}^{n} Q_j x_{j+1})(a)(1+x_1(a)) \\ p_1(\sum_{j=1}^{n} Q_j x_{j+1})(a)(1+x_1(a)) \\ \vdots \\ p_n(\sum_{j=1}^{n} Q_j x_{j+1})(a)(1+x_1(a)) \end{pmatrix}, \quad x\in\mathbf{X}, \qquad (3.3)$$

where Q_j is a bound linear operator on $L^1(0,r_m)$, $(j=1,\cdots,n)$ given by:

$$(Q_j f)(a)=\int_0^{r_m}\beta_j(a,\sigma)N(\sigma)f(\sigma)d\sigma. \tag{3.4}$$

Note that $Q_j f\in L^\infty(0,r_m)$ for $f\in L^1(0,r_m)$ and hence the nonlinear operator F is defined on the whole space $\mathbf{X}$. Let $u(t)=(\hat{s}(\cdot,t),i_1(\cdot,t),\cdots,i_n(\cdot,t))^T\in\mathbf{X}$. Then we can rewrite the initial-boundary value problem (3.1_1)-(3.1_{n+1}) as the abstract semilinear initial value problem in $\mathbf{X}$:

$$\frac{d}{dt}u(t)=Au(t)+F(u(t)),\quad u(0)=u_0\in\mathbf{X}, \tag{3.5}$$

where $u^0(a)=(\hat{s}^0(a),i_1^0(a),\cdots,i_n^0(a))^T$, $\hat{s}^0(a)=s^0(a)-1$. It is easily seen that the operator A is the infinitesimal generator of C_0 - semigroup $T(t)$, $t\geq 0$ and F is continuously Frechet differentiable on $\mathbf{X}$. Then for each $u^0\in\mathbf{X}$, there exists a maximal interval of existence $[0,t_0)$, and a unique continuous mild solution $t\to u(t,u^0)$ from $[0,t_0)$ to $\mathbf{X}$ such that:

$$u(t,u^0)=T(t)u^0+\int_0^t T(t-\tau)F(u(\tau,u^0))d\tau, \tag{3.6}$$

for all $t\in[0,t_0)$ and either $t_0=+\infty$ or $t_0<+\infty$ and $\lim_{t\to t_0^-}\|u(t,u_0)\|=\infty$. Moreover, if $u_0\in D(A)$, then $u(t,u_0)\in D(A)$ for $0\leq t<t_0$ and the function $t\to u(t,u_0)$ is continuously differentiable and satisfies (3.5) on $[0,t_0)$ (see [9], P_{194}, Proposition 4.16).

Since $S(a,t)=\mu^* Nf(a)(1+\hat{s}(a,t))$, from above discussion we obtain that the solution $(S(a,t),I_1(a,t),\cdots,I_n(a,t),R(a,t))^T$, $t\in(0,t_0)$ is continuously differentiable and satisfies (2.5)-(2.6), where either $t_0=+\infty$ or $t_0<+\infty$ and

$$\lim_{t\to t_0}(\|S(a,t)\|+\|I_1(a,t)\|+\cdots+\|I_n(a,t)\|+\|R(a,t)\|)=+\infty.$$

From $\|N(a)\|=\|N(a,t)\|=N$, we easily obtain $t_0=+\infty$. Thus we have the following result.

Theorem 3.1
The abstract Cauchy problem (3.5) has a unique global classical solution on $\mathbf{X}$ with respect to initial data $u_0\in D(A)$.

Therefore, it follows immediately that the initial-boundary value problem (2.5)-(2.6) has a unique global classical solution with respect to the initial data.

4. Existence of Steady States

Let $u^*=(s^*(a),i_1^*(a),\cdots,i_n^*(a))$ be the steady state solution of the set of equations (2.7_1)-(2.7_{n+2}). It is easy to verify the following:

$$s^*(a)=\exp(-\int_0^a\lambda^*(\sigma)d\sigma), \tag{4.1$_1$}$$

$$i_1^*(a)=p_1\int_0^a\exp(-\gamma_1(a-\sigma))\lambda^*(\sigma)\exp(-\int_0^\sigma\lambda^*(\eta)d\eta)d\sigma, \tag{4.1$_2$}$$

$$i_2^*(a)=p_2\int_0^a\exp(-\gamma_2(a-\sigma))\lambda^*(\sigma)\exp(-\int_0^\sigma\lambda^*(\eta)d\eta)d\sigma, \tag{4.1$_3$}$$

$$\vdots$$

$$i_n^*(a)=p_n\int_0^a\exp(-\gamma_n(a-\sigma))\lambda^*(\sigma)\exp(-\int_0^\sigma\lambda^*(\eta)d\eta)d\sigma, \tag{4.1$_{n+1}$}$$

where

$$\lambda^*(a) = \int_0^{r_m} (\sum_{j=1}^n \beta_j(a,\xi) i_j^*(\xi)) N(\xi) d\xi, \qquad (4.1_{n+2})$$

Substituting $(4.1_2) - (4.1_{n+1})$ into (4.1_{n+2}) and changing the order of integration, we obtain an equation for $\lambda^*(a)$:.

$$\lambda^*(a) = \int_0^{r_m} \varphi(a,\sigma)\lambda^*(\sigma)\exp(-\int_0^{\sigma} \lambda^*(\eta)d\eta)d\sigma, \qquad (4.2)$$

where:

$$\varphi(a,\sigma) = \sum_{j=1}^{n} p_j \varphi_j(a,\sigma), \ \ \varphi_j(a,\sigma) = \int_{\sigma}^{r_m} \beta_j(a,\xi) N(\xi)\exp(-\gamma_j(\xi-\sigma))d\xi. \qquad (4.3)$$

From (4.1_{n+2}), it follows that:

$$|\lambda^*(a)| \le n\mu^* N \|\beta\|_\infty \|i^*\|_1,$$

where:

$$\|\beta\|_\infty = max\{\|\beta_1\|_\infty, \cdots, \|\beta_n\|_\infty\}, \quad \|i^*\|_1 = max\{\|i_1^*\|_1, \cdots, \|i_n^*\|_1\},$$

in which $\|\cdot\|_\infty, \|\cdot\|_1$ denote a L^∞ -norm and a L^1 -norm, respectively. Then it follows that $\lambda^* \in L_+^\infty(0,r_m)$ since $i_j^* \in L_+^1(0,r_m)(j=1,2,\cdots,n)$. It is clear that one solution of (4.2) is $\lambda^*(a) \equiv 0$, which corresponds to the equilibrium state with no disease. In order to investigate a nontrivial solution for (4.2), we define a nonlinear operator $\Phi(x)$ in the Banach space $X = L^1(0,r_m)$ with the positive cone $X_+ = \{\psi \in X, \psi \ge 0, a.e.\}$ by :

$$(\Phi x)(a) = \int_0^{r_m} \varphi(a,\sigma)x(\sigma)\exp(-\int_0^{\sigma} x(\eta)d\eta)d\sigma, \quad x \in X. \qquad (4.4)$$

Since the range of Φ is included in $L^\infty(0,r_m)$, the solutions of (4.2) correspond to fixed points of the operator Φ. Observe that the operator Φ has a positive linear majorant T defined by:

$$(Tx)(a) = \int_0^{r_m} \varphi(a,\sigma)x(\sigma)d\sigma, \quad x \in X. \qquad (4.5)$$

We summarize the Perron-Frobenius theory for the positive operators in an ordered Banach space. Let X be a real or complex Banach space and let X^* be its dual, i.e., the space of all linear functionals on X. The value of $F \in X^*$ at $\psi \in X$ is denoted by $\langle F, \psi \rangle$. A close subset X_+ is called a cone if the following holds(see [19], P_{889}).

Definition 4.1

A positive operator $T \in B(X)$ is called semi-nonsupporting if and only if for every pair $\psi \in X_+$, $\{0\}, F \in X_+^*$, $\{0\}$, there exists a positive integer $p = p(\psi, F)$ such that $\langle F, T^p\psi \rangle > 0$. A positive operator $T \in B(X)$ is called nonsupporting if and only if for every pair $\psi \in X_+$, $\{0\}, F \in X_+^*$, $\{0\}$, there exists an integer $p = p(\psi, F)$ such that $\langle F, T^n\psi \rangle > 0$ for all $n \ge p$.

The reader may refer to [10],[11] for the proof of the following theorem:

Theorem 4.2

Let the cone X_+ be total, $T \in B(E)$ be semi-nonsupporting with respect to X_+ and let $r(T)$ be a pole of the resolvent $R(\lambda, T)$. Then the following hold:

(1) $r(T) \in P_\sigma(T)$, $\{0\}$, $r(T)$ is a simple pole of the resolvent.

(2) The eigenspace corresponding to $r(T)$ is one-dimensional and the corresponding eigenvector $\psi \in X_+$ is a nonsupporting point. The relation $T\varphi = \mu\varphi$ with $\varphi \in X_+$ implies that $\varphi = c\psi$ for some constant c.
(3) The eigenspace of T^* corresponding to $r(T)$ is also a one-dimensional subspace of X^* spanned by a strictly positive functional $F \in X^*$.
(4) Assume that X is a Banach lattice. If $T \in B(X)$ is nonsupporting, then the peripheral spectrum of T consists only of $r(T)$, i.e., $|\lambda| < r(T)$ for $\lambda \in \sigma(T)$, $\{r(T)\}$.

The following comparison theorem is due to [12].

Theorem 4.3
Suppose that X is a Banach lattice. Let S and T be positive operator in $B(X)$.
(1) If $S \le T$, then $r(S) \le r(T)$.
(2) If S and T are semi-nonsupporting operators, then $S \le T, S \ne T$ implies that $r(S) < r(T)$.
With the above explanations, we are in the perfect shape to investigate the nature of the majorant operator T defined by (4.5). We initiate the process with the following assumption:

Assumption 4.4
(1)

$$\beta_j(a,\xi) \in L_+^\infty((0,r_m)\times(0,r_m)).$$

(2)

$$\lim_{h\to 0}\int_0^{r_m} |\beta_j(a+h,\xi)-\beta_j(a,\xi)|\,da = 0 \text{ uniformly for } \xi \in R, (j=1,2,\cdots,n). \tag{4.6}$$

where β_j is extended by $\beta_j(a,\xi)=0$ for $a,\xi \in (-\infty,0)\bigcup(r_m,+\infty)$.
(3) There exist $l \in Z, (1 \le l \le n)$ and a number κ with $r_m > \kappa > 0$ and $\varepsilon > 0$ such that

$$\beta_l(a,\xi) \ge \varepsilon \text{ for almost all } (a,\xi) \in (0,r_m)\times(r_m-\kappa, r_m). \tag{4.7}$$

Then we can prove that :

Lemma 4.5
Under Assumption 4.4, the operator $T: X \to X$ is nonsupporting and compact.

Proof
Define the positive linear functional $F \in X_+^*$ by:

$$\langle F,\psi\rangle = \int_0^{r_m} g(\sigma)\psi(\sigma)d\sigma, \quad \psi \in X, \tag{4.8}$$

where $g(\sigma)$ is given by:

$$g(\sigma) = \int_\sigma^{r_m} p_l s(\xi) N(\xi)\exp(-\gamma_l(\xi-\sigma))d\xi, \tag{4.9}$$

where the function $s(\xi)$ is defined as $s(\xi)=0, \xi \in (0, r_m-\kappa), s(\xi)=\varepsilon, \xi \in [r_m-\kappa, r_m)$. Hence $\beta_l(a,\xi) \ge s(\xi)$ for almost all $(a,\xi) \in (0,r_m)\times(0,r_m)$. Since $g(\sigma) > 0$ for all $\sigma \in [0,r_m)$, the functional F is strictly positive and :

$$\langle F,x\rangle e \le Tx, \quad e = 1 \in X_+, \quad x \in X_+.$$

Then for any integer n, we have:

$$T^{n+1}x \ge \langle F,x\rangle\langle F,e\rangle^n e.$$

Therefore we obtain $\langle Y, T^n x\rangle > 0$, $n \ge 1$ for every pair $x \in X_+$, $\{0\}$, $Y \in X_+^*$, $\{0\}$, that is,

T is nonsupporting. Next observe that :

$$\int_0^{r_m} |\varphi(a+h,\sigma)-\varphi(a,\sigma)|\,da$$
$$=\int_0^{r_m} \Big|\int_\sigma^{r_m} \sum_{j=1}^{n} \beta_j(a+h,\xi)p_jN(\xi)\exp(-\gamma_j(\xi-\sigma))d\xi$$
$$-\int_\sigma^{r_m} \sum_{j=1}^{n} \beta_j(a,\xi)p_jN(\xi)\exp(-\gamma_j(\xi-\sigma))d\xi\Big|\,da$$
$$=\int_0^{r_m} \Big|\int_\sigma^{r_m} \sum_{j=1}^{n} [\beta_j(a+h,\xi)-\beta_j(a,\xi)]p_jN(\xi)\exp(-\gamma_j(\xi-\sigma))d\xi\Big|\,da$$
$$\le \mu^* N r_m \int_0^{r_m}\int_0^{r_m} \Big|\sum_{j=1}^{n} \beta_j(a+h,\xi)-\beta_j(a,\xi)\Big|\,d\xi da. \tag{4.10}$$

In order to prove the compactness of T, we identify the Banach space X with the subspace of $L^1(R)$ such that $X=\{\psi\in L^1(R)\,|\,\psi(a)=0 \text{ for } a\in(-\infty,0)\cup(r_m,\infty)\}$. Then we can interpret T as an operator on $L^1(R)$ such that X is its invariant subspace, so it is sufficient to show that the operator T is compact in $L^1(R)$. Let K be a bounded subset of $L^1(R)$. Then it follows immediately that $T(K)$ is also a bounded subset. Observe that :

$$\int_R |(Tx)(a+h)-(Tx)(a)|\,da \le \int_R\int_R |\varphi(a+h,\sigma)-\varphi(a,\sigma)||x(\sigma)|\,d\sigma da$$
$$\le \|x\| \sup_{0\le\sigma\le r_m} \int_R |\varphi(a+h,\sigma)-\varphi(a,\sigma)|\,da.$$

Together with the condition (4.6) and (4.10) it follows that $T(K)$ is an equicontinuous family in L^1-norm. Moreover it follows from $T(K)\subset X$ that:

$$\int_{|\sigma|\ge r_m} |(Tx)(\sigma)|\,d\sigma=0,\quad x\in K.$$

Thus we can apply the compactness criterion by Frechet-Kolmogorov ([13], P_{275}), that is, $T(K)$ is relatively compact in $L^1(R)$. Thus T is a compact operator. This completes the proof.

From Theorem 4.2, it follows that the spectral radius $r(T)$ of operator T is the only positive eigenvalue with a positive eigenvector and also an eigenvalue of the dual operator T^* with a strictly positive eigenfunctional.

Now we can prove the following:

Theorem 4.6 (Threshold results).
Let $r(T)$ be the spectral radius of the operator T defined by (4.5). Then the following holds:
(1) If $r(T)\le 1$, the only non-negative solution x of the equation $x=\Phi(x)$ is the trivial solution $x\equiv 0$.
(2) If $r(T)>1$, the equation $x=\Phi(x)$ has at least one non-zero positive solution.
Subsequently, in order to investigate the uniqueness problem for nontrivial positive fixed points of the operator Φ, we introduce the concept of concave operator (see [14]).

Lemma 4.7 [14].
Suppose that the operator $A: X_+\to X_+$ is monotone and concave. If for any $x\in X_+$ satisfying $\alpha_1 u_0\le x\le \beta_1 u_0$ $(\alpha_1=\alpha_1(x)>0, \beta_1=\beta_1(x)>0)$ and $0<t<1$, there exists $\eta=\eta(x,t)>0$ such that:

$$A(tx) \geq tAx + \eta u_0, \tag{4.11}$$

then A has at most one positive fixed point.

Here, we can prove the following:

Theorem 4.8
If for all $(a,\sigma) \in [0,r_m) \times [0,r_m)$, the inequality

$$\beta_j(a,\sigma)N(\sigma) - \gamma_j \varphi_j(a,\sigma) \geq 0 (j = 1,2,\cdots,n), \tag{4.12}$$

holds, and $r(T) > 1$, then Φ has only one positive fixed point.

Proof
From Lemma 4.7 and Theorem 4.6 (and Definition 4.7 in [19]), it is sufficient to show that under condition (4.12), the operator Φ is a monotone concave operator satisfying the condition (4.10). From (4.4) and (4.3) it follows that:

$$\begin{aligned}
\Phi(x)(a) &= \int_0^{r_m} \varphi(a,\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)d\sigma \\
&= \int_0^{r_m} \varphi(a,\sigma)[-\frac{d}{d\sigma}(\exp(-\int_0^\sigma x(\eta)d\eta)]d\sigma \\
&= -\varphi(a,\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)|_{\sigma=0}^{\sigma=r_m} + \int_0^{r_m} \exp(-\int_0^\sigma x(\eta)d\eta)\frac{d}{d\sigma}\varphi(a,\sigma)d\sigma \\
&= \varphi(a,0) - \varphi(a,r_m)\exp(-\int_0^{r_m} x(\eta)d\eta) + \int_0^{r_m} \exp(-\int_0^\sigma x(\eta)d\eta) \\
&\quad \cdot[-\sum_{j=1}^n p_j\beta_j(a,\sigma)N(\sigma) + \sum_{j=1}^n \int_\sigma^{r_m} p_j\gamma_j\beta_j(a,\xi)N(\xi)\exp(-\gamma_j(\xi-\sigma))d\xi]d\sigma \\
&= \varphi(a,0) + \int_0^{r_m} \exp(-\int_0^\sigma x(\eta)d\eta)[-\sum_{j=1}^n p_j\beta_j(a,\sigma)N(\sigma) + \sum_{j=1}^n p_j\gamma_j\varphi_j(a,\sigma)]d\sigma \\
&= \varphi(a,0) - \int_0^{r_m} \exp(-\int_0^\sigma x(\eta)d\eta)\sum_{j=1}^n p_j[\beta_j(a,\sigma)N(\sigma) - \gamma_j\varphi_j(a,\sigma)]d\sigma,
\end{aligned}$$

from which together with condition (4.12) we know that Φ is a monotonic operator. Next from (4.4) and (4.3) we observe that:

$$\alpha(x)u^0 \leq \Phi(x)(a) \leq \beta(x)u^0$$

where $u^0 \equiv 1$ and:

$$\alpha(x) = \int_0^{r_m} g(\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)d\sigma,$$

$$\beta(x) = M\int_0^{r_m} h(\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)d\sigma.$$

Here :

$$M = max\{ess\sup\beta_1(a,b), ess\sup\beta_2(a,b),\cdots, ess\sup\beta_n(a,b)\} < +\infty,$$

$g(\sigma)$ is given by (4.9) and $h(\sigma)$ is defined by:

$$h(\sigma) = n\int_\sigma^{r_m} N(\xi)d\xi.$$

It follows that $\alpha(x) > 0$ and $\beta(x) > 0$ for $x \in X_+$, $\{0\}$. Moreover we obtain:

$$\begin{aligned}
\Phi(tx)(a) - t\Phi(x)(a) &= t\int_0^{r_m} \varphi(a,\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)[\exp((1-t)\int_0^\sigma x(\eta)d\eta) - 1]d\sigma \\
&\geq \int_0^{r_m} g(\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)[\exp((1-t)\int_0^\sigma x(\eta)d\eta) - 1]d\sigma,
\end{aligned}$$

from which we conclude that Φ is a concave operator and the condition (4.11) is satisfied by assuming $u^0 = 1$ and :

$$\eta(x,t) = t\int_0^{r_m} g(\sigma)x(\sigma)\exp(-\int_0^{\sigma} x(\eta)d\eta)[\exp((1-t)\int_0^{\sigma} x(\eta)d\eta) - 1]d\sigma.$$

This completes the proof of Theorem 4.8.

We need to look for relationships that could guarantee condition (4.12) We say that if $\beta(a,\xi)N(\xi)$ is continuous and non-increasing as a function of $\xi \in (0, r_m)$, then (4.12) holds. In the following we will show :

$$\begin{aligned}
&\beta_j(a,\sigma)N(\sigma) - \gamma_j\varphi_j(a,\sigma) \\
&= \beta_j(a,\sigma)N(\sigma) - \gamma_j\int_{\sigma}^{r_m} \beta_j(a,\xi)N(\xi)\exp(-\gamma_j(\xi-\sigma))d\xi \\
&\geq \beta_j(a,\sigma)N(\sigma)[1 - \gamma_j\int_{\sigma}^{r_m} \exp(-\gamma_1(\xi-\sigma))d\xi] \\
&= \beta_j(a,\sigma)N(\sigma)[1 - \gamma_j \frac{1}{-\gamma_j}\int_{\sigma}^{r_m} \frac{d}{d\xi}\exp(-\gamma_j(\xi-\sigma))d\xi] \\
&= \beta_j(a,\sigma)N(\sigma)[1 + \exp(-\gamma_j(\xi-\sigma))|_{\sigma=0}^{\sigma=r_m}] \\
&= \beta_j(a,\sigma)N(\sigma)\exp(-\gamma_j(r_m-\sigma)) \\
&\geq 0.
\end{aligned}$$

In particular, condition (4.12) holds if β_j is independent of age of infective σ, because $N(\sigma)$ is a decreasing function.

The assumption that $\beta_j(a,\xi)N(\xi)$ is non-increasing on ξ implies that the number of age a infected by younger individuals is always greater than the number of those infected by older individuals. This assumption may not be realistic for some diseases. Here we use the assumption to explain the validity of condition (4.12).

5. Stability Analysis for Equilibrium Solutions

In order to investigate the local stability of the equilibrium solutions:

$$(s^*(a), i_1^*(a), i_2^*(a), \cdots, i_n^*(a))^T$$

of $(2.1_1)-(2.7_{n+1})$, we first rewrite $(2.1_1)-(2.7_{n+1})$ into equations for small perturbations. Let

$$s(a,t) = s^*(a) + \eta(a,t),\ i_1(a,t) = i_1^*(a) + \delta_1(a,t), \cdots, i_n(a,t) = i_n^*(a) + \delta_n(a,t).$$

From $(2.1_1)-(2.7_{n+1})$, we have:

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a})\eta(a,t) = -\lambda(a,t)[\eta(a,t) + s^*(a)] - \lambda^*(a)\eta(a,t), \tag{5.1$_1$}$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a})\delta_1(a,t) = p_1[\lambda(a,t)(\eta(a,t) + s^*(a)) + \lambda^*(a)\eta(a,t)] - \gamma_1\delta_1(a,t), \tag{5.1$_2$}$$

$$\vdots$$

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a})\delta_n(a,t) = p_n[\lambda(a,t)(\eta(a,t) + s^*(a)) + \lambda^*(a)\eta(a,t)] - \gamma_n\delta_n(a,t), \tag{5.1$_{n+1}$}$$

where:

$$\lambda(a,t) = \int_0^{r_m} \sum_{j=1}^{n} (\beta_j(a,\sigma)\delta_j(\sigma,t))N(\sigma)d\sigma,$$

$$\eta(0,t) = 0, \quad \delta_j(0,t) = 0 (j = 1, \cdots, n).$$

Therefore we can formulate $(5.1_1)-(5.1_{n+1})$ as an abstract semilinear problem on the Banach space X .

$$\frac{dx(t)}{dt} = Ax(t) + G(x(t)), \quad x(t) = (\eta(t), \delta_1(t), \cdots, \delta_n(t))^T \in X, \tag{5.2}$$

where the generator A is defined by :

$$(Ax)(a) = (-\frac{dx_1(a)}{da}, -\frac{dx_2(a)}{da} - \gamma_1 x_2(a), \cdots, -\frac{dx_{n+1}(a)}{da} - \gamma_n x_{n+1}(a))^T \tag{5.3}$$

with the domain:

$$D(A) = \{x \in X \mid x_j \text{ is absolutely continuous on } [0, r_m),\ j = 1, 2, \cdots, n, x(0) = 0\}.$$

The nonlinear term G is defined as :

$$G(x) = \begin{pmatrix} -(\sum_{j=1}^{n} Q_j x_{j+1})(x_1 + s^*) - \lambda^* x_1 \\ p_1[(\sum_{j=1}^{n} Q_j x_{j+1})(x_1 + s^*) + \lambda^* x_1] \\ \vdots \\ p_n[(\sum_{j=1}^{n} Q_j x_{j+1})(x_1 + s^*) + \lambda^* x_1] \end{pmatrix}. \tag{5.4}$$

for $x(a) = (x_1(a), x_2(a), \cdots, x_n(a))^T \in X$, where the operator Q_j is defined by (3.4). The linearized equation around $x = 0$ is given by :

$$\frac{d}{dt} x(t) = (A + C)x(t), \tag{5.5}$$

where the bounded linear operator C is the Frechet derivative of $G(x)$ at $x = 0$ and given by:

$$Cx = \begin{pmatrix} -(\sum_{j=1}^{n} Q_j x_{j+1})s^* - \lambda^* x_1 \\ p_1[(\sum_{j=1}^{n} Q_j x_{j+1})(x_1 + s^*) + \lambda^* x_1] \\ \vdots \\ p_n[(\sum_{j=1}^{n} Q_j x_{j+1})(x_1 + s^*) + \lambda^* x_1] \end{pmatrix}.$$

Now let us consider the resolvent equation for $A + C$:

$$(\lambda I - A - C)u = \psi, \quad u \in D(A), \quad \psi \in X, \quad \lambda \in C. \tag{5.6}$$

Then we have:

$$\frac{du_1(a)}{da} + (\lambda + \lambda^*(a))u_1(a) = \psi_1(a) - (\sum_{j=1}^{n} Q_j u_{j+1})(a)s^*(a), \tag{5.7$_1$}$$

$$\frac{du_2(a)}{da} + (\lambda + \gamma_1)u_2(a) = \psi_2(a) + p_1[(\sum_{j=1}^{n} Q_j u_{j+1})(a)s^*(a) + \lambda^*(a)u_1(a)], \tag{5.7$_2$}$$

$$\vdots$$

$$\frac{du_{n+1}(a)}{da} + (\lambda + \gamma_n)u_{n+1}(a) = \psi_{n+1}(a) + p_n[(\sum_{j=1}^{n} Q_j u_{j+1})(a)s^*(a) + \lambda^*(a)u_1(a)], \tag{5.7$_{n+1}$}$$

From (5.7$_1$), we obtain :

$$u_1(a) = \varphi_1(a) + P_\lambda(a), \tag{5.8}$$

where:

$$\varphi_1(a)=\exp(-\lambda a)\Pi(a)\int_0^a[\psi_1(\sigma)\exp(\lambda\sigma)\Pi^{-1}(\sigma)d\sigma;$$

$$P_\lambda(a)=-\int_0^a(\sum_{j=1}^n Q_j u_{j+1})(\sigma)\exp(-\lambda(a-\sigma))\Pi(a)d\sigma,$$

in which $\Pi(a)$ is defined by:

$$\Pi(a)=\exp(-\int_0^a\lambda^*(\eta)d\eta)=s^*(a).$$

By (5.7_2) - (5.7_{n+1}), we get :

$$u_2(a)=\varphi_2(a)+P_{\lambda 1}(a),$$
$$\vdots$$
$$u_{n+1}(a)=\varphi_{n+1}(a)+P_{\lambda n}(a),$$

where:

$$\varphi_2(a)=\int_0^a\exp(-(\lambda+\gamma_1)(a-\sigma))\varphi_2(\sigma)d\sigma,$$
$$\vdots$$
$$\varphi_{n+1}(a)=\int_0^a\exp(-(\lambda+\gamma_n)(a-\sigma))\varphi_{n+1}(\sigma)d\sigma,$$
$$P_{\lambda 1}(a)=p_1\int_0^a\exp(-(\lambda+\gamma_1)(a-\sigma))[(\sum_{j=1}^n Q_j u_{j+1})(\sigma)\Pi(\sigma)+\lambda^*(\sigma)u_1(\sigma)]d\sigma,$$
$$\vdots$$
$$P_{\lambda n}(a)=p_n\int_0^a\exp(-(\lambda+\gamma_n)(a-\sigma))[(\sum_{j=1}^n Q_j u_{j+1})(\sigma)\Pi(\sigma)+\lambda^*(\sigma)u_1(\sigma)]d\sigma.$$

We define $\theta(\sigma)$ as follows:

$$\theta(\sigma)=\sum_{j=1}^n(Q_j u_{j+1})(\sigma). \tag{5.9}$$

Substituting (5.8) and those expressions into (5.9) we obtain :

$$\theta(\sigma)=\sum_{j=1}^n J_j(\sigma)+E_\lambda(\sigma)+\sum_{j=1}^n L_j(\sigma). \tag{5.10}$$

where:

$$J_j(\sigma)=\int_0^{r_m}\beta_j(\sigma,a)N(a)\int_0^a\exp(-(\lambda+\gamma_j)(a-\xi))\psi_{j+1}(\xi)d\xi da,$$
$$L_j(\sigma)=\int_0^{r_m}p_j\beta_j(\sigma,a)N(a)\int_0^a\exp(-(\lambda+\gamma_j)(a-\xi))$$
$$\cdot\lambda^*(\xi)\exp(-\lambda\xi)\Pi(\xi)\int_0^\xi\varphi_1(\eta)\exp(\lambda\eta)\Pi^{-1}(\eta)d\eta d\xi da,$$

$$E_\lambda(\sigma)=\sum_{j=1}^n E_{\lambda j}(\sigma),$$
$$E_{\lambda j}(\sigma)=\int_0^{r_m}p_j\beta_j(\sigma,a)N(a)\int_0^a\exp(-(\lambda+\gamma_j)(a-\xi))\theta(\xi)\Pi(\xi)d\xi da$$
$$-\int_0^{r_m}p_j\beta_j(\sigma,a)N(a)\int_0^a\exp(-(\lambda+\gamma_j)(a-\xi))\lambda^*(\xi)\Pi(\xi)$$
$$\cdot\int_0^\xi\theta(\eta)\exp(-\lambda(\xi-\eta))d\eta d\xi da,$$
$$j=1,2,\cdots,n.$$

Let us define :

$$\theta_{\lambda j}(\sigma,\xi)=\int_{\xi}^{r_m}\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)(a-\xi))da,$$
$$\varphi_{\lambda j}(\sigma,\xi)=\int_{\xi}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)da,$$
$$\varphi_{\lambda}(\sigma,\xi)=\sum_{j=1}^{n}\int_{\xi}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)da,$$
$$j=1,2,\cdots,n.$$

Then we can rewrite the above representations for $J_j(\sigma), L_j(\sigma)$, and $E_{\lambda j}(\sigma)$ as:

$$\begin{aligned}
J_j(\sigma)&=\int_0^{r_m}\int_{\xi}^{r_m}\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)(a-\xi))da\psi_{j+1}(\xi)d\xi\\
&=\int_0^{r_m}\theta_{\lambda j}(\sigma,\xi)\psi_{j+1}(\xi)d\xi,\\
L_j(\sigma)&=\int_0^{r_m}\int_{\xi}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)da\\
&\quad\cdot\exp(\gamma_j\xi)\lambda^*(\xi)\Pi(\xi)\int_0^{\xi}\varphi_1(\eta)\exp(\lambda\eta)\Pi^{-1}(\eta)d\eta d\xi\\
&=\int_0^{r_m}\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\lambda^*(\xi)\Pi(\xi)\int_0^{\xi}\varphi_1(\eta)\exp(\lambda\eta)\Pi^{-1}(\eta)d\eta d\xi\\
E_{\lambda j}(\sigma)&=\int_0^{r_m}\int_{\xi}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)da\exp((\lambda+\gamma_j)\xi)\theta(\xi)\Pi(\xi)d\xi\\
&\quad-\int_0^{r_m}\int_{\xi}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)da\exp(\gamma_j\xi)\lambda^*(\xi)\Pi(\xi)\\
&\quad\cdot\int_0^{\xi}\theta(\eta)\exp(\lambda\eta)d\eta d\xi\\
&=\int_0^{r_m}\varphi_{\lambda j}(\sigma,\xi)\exp((\lambda+\gamma_1)\xi)\theta(\xi)\Pi(\xi)d\xi\\
&\quad-\int_0^{r_m}\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\lambda^*(\xi)\Pi(\xi)\int_0^{\xi}\theta(\eta)\exp(\lambda\eta)d\eta d\xi.
\end{aligned}$$

If we define linear operators on the Banach space $L^1(0,r_m)$ by:

$$\begin{aligned}
&(S_{\lambda j}\psi)(\sigma)=\int_0^{r_m}\theta_{\lambda j}(\sigma,\xi)\psi(\xi)d\xi,\\
&(U_{\lambda j}\psi)(\sigma)=\int_0^{r_m}\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\lambda^*(\xi)\Pi(\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta d\xi,\\
&U_{\lambda}=\sum_{j=1}^{n}U_{\lambda j};\\
&(T_{\lambda j}\psi)(\sigma)=\int_0^{r_m}\varphi_{\lambda j}(\sigma,\xi)\exp((\lambda+\gamma_j)\xi)\Pi(\xi)\psi(\xi)d\xi,\\
&T_{\lambda}=\sum_{j=1}^{n}T_{\lambda j};\\
&(V_{\lambda}\psi)(\sigma)=(T_{\lambda}\psi)(a)-(U_{\lambda}\psi)(a).
\end{aligned}\tag{5.11}$$

Then the following expression holds:

$$(V_{\lambda}\psi)(\sigma)=\int_0^{r_m}\chi_{\lambda}(\sigma,\eta)\psi(\eta)d\eta,\tag{5.12}$$

where:

$$\chi_{\lambda}(\sigma,\eta)=\sum_{j=1}^{n}\chi_{\lambda j}(\sigma,\eta),\tag{5.13}$$

and:

$$\chi_{\lambda j}(\sigma,\eta)=\int_{\eta}^{r_m} p_j\beta_j(\sigma,\xi)N(\xi)\exp(-\lambda(\xi-\eta))$$
$$\cdot[\Pi(\xi)-\exp(-\gamma_j\xi)\int_{\eta}^{\xi}\gamma_j\Pi(a)\exp(\gamma_j a)da]d\xi. \quad (5.14)$$

It is not difficult to verify the above expression if we note that :

$$(U_{\lambda j}\psi)(\sigma)=\int_0^{r_m}\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\lambda^*(\xi)\Pi(\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta d\xi$$
$$=\int_0^{r_m}(-\frac{d\Pi(\xi)}{d\xi})[\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta]d\xi$$
$$=-[\Pi(\xi)\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta]|_0^{r_m}$$
$$+\int_0^{r_m}\Pi(\xi)\frac{\partial}{\partial\xi}[\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta]d\xi$$
$$=\int_0^{r_m}[\Pi(\xi)\frac{\partial\varphi_{\lambda j}}{\partial\xi}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta$$
$$+\gamma_j\Pi(\xi)\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta]d\xi+(T_{\lambda j}\psi)(\sigma)$$
$$=(T_{\lambda j}\psi)(\sigma)-\int_0^{r_m}\Pi(\xi)p_j\beta_j(\sigma,\xi)N(\xi)\exp(-(\lambda+\gamma_j)\xi)\exp(\gamma_j\xi)$$
$$\cdot\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta d\xi+\int_0^{r_m}\gamma_j\Pi(\xi)\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta d\xi .$$

From:

$$\int_0^{r_m}\gamma_j\Pi(\xi)\varphi_{\lambda j}(\sigma,\xi)\exp(\gamma_j\xi)\int_0^{\xi}\psi(\eta)\exp(\lambda\eta)d\eta d\xi$$
$$=\int_0^{r_m}\int_{\eta}^{r_m}\gamma_j\Pi(\xi)\exp(\gamma_j\xi)\varphi_{\lambda j}(\sigma,\xi)d\xi\psi(\eta)\exp(\lambda\eta)d\eta$$
$$=\int_0^{r_m}\int_{\eta}^{r_m}\gamma_j\Pi(\xi)\exp(\gamma_j\xi)\int_{\xi}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)dad\xi\psi(\eta)\exp(\lambda\eta)d\eta$$
$$=\int_0^{r_m}\int_{\eta}^{r_m}p_j\beta_j(\sigma,a)N(a)\exp(-(\lambda+\gamma_j)a)\int_{\eta}^{a}\gamma_j\Pi(\xi)\exp(\gamma_j\xi)d\xi da\psi(\eta)\exp(\lambda\eta)d\eta$$
$$=\int_0^{r_m}\int_{\eta}^{r_m}p_j\beta_j(\sigma,\xi)N(\xi)\exp(-(\lambda+\gamma_j)\xi)\int_{\eta}^{\xi}\gamma_j\Pi(a)\exp(\gamma_j a)dad\xi\psi(\eta)\exp(\lambda\eta)d\eta,$$

and:

$$(T_{\lambda j}\psi)(\sigma)-(U_{\lambda j}\psi)(\sigma)=\int_0^{r_m}\chi_{\lambda j}(\sigma,\eta)d\eta,$$

we can reach (5.14). From above definitions and (5.10), it follows that :

$$\theta(\sigma)=\sum_{j=1}^{n}(S_{\lambda j}\psi_{j+1})(\sigma)+(U_{\lambda}\psi_1\Pi^{-1})(\sigma)+(T_{\lambda}\theta)(\sigma)-(U_{\lambda}\theta)(\sigma).$$

Hence, we have:

$$\theta(\sigma)=(I-T_{\lambda}+U_{\lambda})^{-1}[\sum_{j=1}^{n}(S_{\lambda j}\psi_{j+1})(\sigma)+(U_{\lambda}\psi_1\Pi^{-1})(\sigma)]$$
$$=(I-V_{\lambda})^{-1}[\sum_{j=1}^{n}(S_{\lambda j}\psi_{j+1})(\sigma)+(U_{\lambda}\psi_1\Pi^{-1})(\sigma)]. \quad (5.15)$$

From (5.8), (5.9), (5.15) and the expressions during (5.8) and (5.9), we can conclude that:

Lemma 5.1

The perturbed operator $A+C$ has a compact resolvent and :

$$\sigma(A+C)=P_\sigma(A+C)=\{\lambda\in C\,|\,1\in P_\sigma(V_\lambda)\},\tag{5.16}$$

where $\sigma(A)$ and $P_\sigma(A)$ denote the spectrum of A and the point spectrum of A respectively.

Proof

From (5.8) and (5.15), we obtain the expression for u_1 :

$$\begin{aligned}u_1(a)&=\int_0^a \exp(-\lambda a)\Pi(a)\exp(\lambda\sigma)\Pi^{-1}(\sigma)\psi_1(\sigma)d\sigma\\&-\int_0^a \exp(-\lambda a)\Pi(a)\exp(\lambda\sigma)\Pi^{-1}(\sigma)\Pi(\sigma)(\sum_{j=1}^{n}(Q_j u_{j+1}))(\sigma)d\sigma\\&=(H\psi_1)(a)-W(\psi_1,\psi_2,\cdots,\psi_{n+1})(a),\end{aligned}$$

where the operators H and W are defined by :

$$(H\psi_1)(a)=\int_0^a G(a,\sigma)\psi_1(a)d\sigma,$$

$$W(\psi_1,\psi_2,\cdots,\psi_{n+1})(a)=\int_0^a G(a,\sigma)\Pi(\sigma)(I-V_\lambda)^{-1}[\sum_{j=1}^{n}(S_{\lambda j}\psi_{j+1})(\sigma)\ +(U_\lambda\psi_1\Pi^{-1})(\sigma)]d\sigma,$$

in which:

$$G(a,\sigma)=\exp(-\lambda a)\exp(\lambda\sigma)\Pi(a)\Pi^{-1}(\sigma).$$

Since H is a Volterra operator with a continuous kernel, it is a compact operator on $L^1(0,r_m)$. On the other hand, by the same manner as the proof of Lemma 4.5, we can prove that T_λ and U_λ are compact for all $\lambda\in C$. Let $\Lambda=\{\lambda\in C\,|\,1\in\sigma(V_\lambda)\}$. Then it follows that when $\lambda\in C$, Λ the operator W is a compact operator from $\mathbf{X}$ to $L^1(0,r_m)$. By the same way, we can prove that $u_2(a)$ $\cdots$, $u_{n+1}(a)$ can be represented by compact operators from $\mathbf{X}$ to $L^1(0,r_m)$. Consequently, we know that $A+C$ has a compact resolvent. So we get that $\sigma(A+C)=P_\sigma(A+C)$ (see [13, P_{187}]). From above argument, it follows that C , $\Lambda\subset\rho(A+C)$ ($\rho(A+C)$ denotes the resolvent set of $A+C$), that is, $\Lambda\supset\sigma(A+C)=P_\sigma(A+C)$. Since V_λ is a compact operator, we know that $\sigma(V_\lambda)$, $\{0\}=P_\sigma(V_\lambda)$, $\{0\}$ and if $\lambda\in\Lambda$, there exists an eigenfunction ψ_λ such that $V_\lambda\psi_\lambda=\psi_\lambda$. Then it is easily seen that if we define the following functions

$$\begin{aligned}u_1(a)&=-\exp(-\lambda a)\Pi(a)\int_0^a \exp(\lambda\sigma)\psi_\lambda(\sigma)d\sigma,\\u_2(a)&=p_1\int_0^a \exp(-(\lambda+\gamma_1)(a-\sigma))[\psi_\lambda(\sigma)\Pi(\sigma)+\lambda^*(\sigma)u_1(\sigma)]d\sigma,\\&\vdots\\u_{n+1}(a)&=p_n\int_0^a \exp(-(\lambda+\gamma_n)(a-\sigma))[\psi_\lambda(\sigma)\Pi(\sigma)+\lambda^*(\sigma)u_1(\sigma)]d\sigma,\end{aligned}$$

$(u_1(a),u_2(a),\cdots,u_{n+1}(a))^T$ gives an eigenvector of $A+C$ corresponding to the eigenvalue λ. Then $\Lambda\subset P_\sigma(A+C)$ and we conclude that (5.16) holds.

Lemma 5.2

Let $(t),t\geq 0$ be the C_0-semigroup generated by the perturbed operator $A+C$. Then $T(t),t\geq 0$ is eventually norm continuous and:

$$\omega_0(A+C)=s(A+C)=\sup\{Re\mu\,|\,\mu\in\sigma(A+C)\},\tag{5.17}$$

where $\omega_0(A+C)$ denotes the growth bound of the semigroup $T(t),t\geq 0$, and $s(A+C)$ is the spectral bound of the generator $A+C$.

Proof
We define bounded operators C_1 and C_2 by :

$$C_1 u = (-\lambda^* u_1, p_1\lambda^* u_1, \cdots, p_n\lambda^* u_1)^T,$$
$$C_2 u = (-s^*\sum_{j=1}^{n}(Q_j u_{j+1}), P_1 s^*\sum_{j=1}^{n}(Q_j u_{j+1}), \cdots, P_n s^*\sum_{j=1}^{n}(Q_j u_{j+1}))^T, \quad u \in \mathbf{X}.$$

Then $C = C_1 + C_2$ and $A + C_1$ generates a C_0-semigroup $S(t), t \geq 0$. Since $S(t)$ is a nilpotent semigroup, so it is eventually norm continuous. Using Assumption 4.4 and similar proof to Lemma 4.5, we can prove that C_2 is a compact operator in $\mathbf{X}$. Therefore, from Theorem 1.30 in ([16], P_{44}), $T(t)$ is also eventually norm continuous. Since the spectral mapping theorem holds for the eventually norm continuous semigroup ([9], P_{87}), we obtain (5.17).

If $\omega_0(A+C) < 0$, the equilibrium $x = 0$ of system (5.2) is locally exponentially asymptotically stable in the sense that there exist $\varepsilon > 0, M \geq 1$ and $\gamma < 0$ such that if $x \in \mathbf{X}$ and $\|x\| \leq \varepsilon$, then the solution $x(t,x)$ of (5.2) exists globally and $\|x(t,x)\| \leq M\exp(\gamma t)\|x\|$ for all $t \geq 0$. This is the main part of the principle of linearized stability (see [9]). Therefore in order to study the stability of the equilibrium states, we have to know the structure of the set of singular points $\Lambda = \{\lambda \in C \mid 1 \in P_\sigma(V_\lambda)\}$. Since $\|V_\lambda\| \to 0$ if $Re\lambda \to +\infty$, $I - V_\lambda$ is invertible for large values of $Re\lambda$. By the theorem of Steinberg [17], the function $\lambda \to (I - V_\lambda)^{-1}$ is meromorphic in the complex domain, and hence the set Λ is a discrete set whose elements are poles of $(I - V_\lambda)^{-1}$ of finite order.

Now we shall make use of positive operator theory once more. Our main purpose here is to determine the dominant singular point, i.e., the element of the set Λ with the largest real part. From (5.16) and (5.17), the dominant singular point gives the growth bound of the semigroup $T(t)$ generated by $A+C$. First we will show that:

Lemma 5.3
Suppose that the following conditions hold:

$$\frac{1}{p_j} i_j^*(r_m) < \exp(-\gamma_j r_m), \quad (j = 1, 2, \cdots, n). \tag{5.18}$$

Then the operator $V_\lambda, \lambda \in R$ is nonsupporting with respect to X_+ and the following holds:

$$\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} r(V_\lambda) = 0. \tag{5.19}$$

Proof
Since $\Pi(\xi) - \exp(-\gamma_j \xi)\int_\eta^\xi \gamma_j \Pi(a) e^{\gamma_j a}$ is an increasing function of η, $(j = 1, 2, \cdots, n)$, we have:

$$
\begin{aligned}
&\Pi(\xi)-\exp(-\gamma_j\xi)\int_\eta^\xi \gamma_j\Pi(a)e^{\gamma_j a}da\\
&\geq \Pi(\xi)-\exp(-\gamma_j\xi)\int_0^\xi \gamma_j\Pi(a)e^{\gamma_j a}da\\
&=\Pi(\xi)-\exp(-\gamma_j\xi)\int_0^\xi \Pi(a)\frac{e^{\gamma_j a}}{da}da\\
&=\Pi(\xi)-\exp(-\gamma_j\xi)[\Pi(a)e^{(\gamma_j a)}|_0^\xi-\int_0^\xi e^{\gamma_j a}\frac{d\Pi(a)}{da}da]\\
&=\Pi(\xi)-\exp(-\gamma_j\xi)[\Pi(\xi)e^{(\gamma_j\xi)}-1+\int_0^\xi e^{\gamma_j a}\lambda^*(a)\Pi(a)da]\\
&=\exp(-\gamma_j\xi)-\int_0^\xi \exp(-\gamma_j(\xi-a))\lambda^*(a)\Pi(a)da\\
&=\exp(-\gamma_j\xi)-\frac{1}{p_j}i_j^*(\xi)\\
&\geq \exp(-\gamma_j r_m)-\frac{1}{p_j}i_j^*(r_m).
\end{aligned} \tag{5.20}
$$

In the following, we will show that Assumption 5.4 guarantees that the operator V_λ is strictly positive. In fact, from (5.13) and 5.14 we can see that :

$$
\begin{aligned}
\chi_{\lambda j}(\sigma,\eta)=\int_\eta^{r_m} & p_j\beta_j(\sigma,\xi)N(\xi)\exp(-\lambda(\xi-\eta))\\
&\cdot[\Pi(\xi)-\exp(-\gamma_j\xi)\int_\eta^\xi \gamma_j\Pi(a)\exp(\gamma_j a)da]d\xi\\
&\geq \int_\eta^{r_m} p_j\beta_j(\sigma,\xi)N(\xi)\exp(-\lambda(\xi-\eta))[\exp(-\gamma_j r_m)-\frac{1}{p_j}i_j^*(r_m)]d\xi.
\end{aligned} \tag{5.21}
$$

and :

$$
\begin{aligned}
&\chi_\lambda(\sigma,\eta)=\sum_{j=1}^n \chi_{\lambda j}(\sigma,\eta)\\
&\geq \sum_{j=1}^n\int_\eta^{r_m} p_j\beta_j(\sigma,\xi)N(\xi)\exp(-\lambda(\xi-\eta))[\exp(-\gamma_j r_m)-\frac{1}{p_j}i_j^*(r_m)]d\xi\\
&=\sum_{j=1}^n G_j(r_m)\int_\eta^{r_m} p_j\beta_j(\sigma,\xi)N(\xi)\pi_\lambda(\xi,\eta)d\xi,
\end{aligned} \tag{5.22}
$$

where:

$$
G_j(r_m)=\exp(-\gamma_j r_m)-\frac{1}{p_j}i_j^*(r_m),\quad \pi_\lambda(\xi,\eta)=\exp(-\lambda(\xi-\eta)).
$$

If we define $G_0(r_m)=min\{G_1(r_m),G_2(r_m),\cdots,G_n(r_m)\}$, we can obtain :

$$
\chi_\lambda(a,\rho)\geq G_0(r_m)\varphi_\lambda(\sigma,\eta). \tag{5.23}
$$

From the above discussion we know that if conditions 5.18 hold, then the operator $V_\lambda,\lambda\in R$ is positive.

Therefore, in order to show the nonsupporting property of $V_\lambda,\lambda\in R$, it suffices to prove that the integral operator $\tilde{T}_\lambda$ defined by :

$$
(\tilde{T}_\lambda\psi)(\sigma)=\int_0^{r_m}\varphi_\lambda(\sigma,\eta)\psi(\eta)d\eta,\quad \psi\in X, \tag{5.24}
$$

is nonsupporting. It is easy to verify the inequality :

$$\tilde{T}_\lambda \psi \geq \langle f_\lambda, \psi \rangle e, \quad e = 1 \in X_+, \quad \psi \in X_+, \tag{5.25}$$

where the linear function f_λ is defined by:

$$\langle f_\lambda, \psi \rangle = \int_0^{r_m} [\int_\eta^{r_m} s(x)N(x)\pi_\lambda(x,\eta)dx]\psi(\eta)d\eta.$$

Then it follows that for all integers n,

$$\tilde{T}_\lambda^{n+1}\psi \geq \langle f_\lambda, \psi \rangle \langle f_\lambda, e \rangle^n e.$$

Since f_λ is strictly positive and the constant function $e = 1$ is a quasi-interior point of $L^1(0, r_m)$, it follows that $\left\langle F, \tilde{T}_\lambda^n \psi \right\rangle > 0$ for every pair $\psi \in X_+$, $\{0\}, F \in X_+^*$, $\{0\}$. Then $\tilde{T}_\lambda, \lambda \in R$ is nonsupporting. Next we show (5.19).

From (5.23) and (5.25), we obtain:

$$V_\lambda \psi \geq G_0(r_m)\tilde{T}_\lambda \psi \geq G_0(r_m)\langle f_\lambda, \psi \rangle e, \quad \lambda \in R, \quad \psi \in X_+.$$

Taking duality pairing with the eigenfunctional F_λ of V_λ that corresponds to $r(V_\lambda)$, one has:

$$r(V_\lambda)\langle F_\lambda, \psi \rangle \geq G_0(r_m)\langle F_\lambda, e \rangle \langle f_\lambda, \psi \rangle.$$

If we let $\psi = e$, we arrive at the inequality:

$$r(V_\lambda) \geq G_0(r_m)\langle f_\lambda, e \rangle,$$

where:

$$\begin{aligned} \langle f_\lambda, e \rangle &= \int_0^{r_m} \int_\eta^{r_m} s(x)N(x)\pi_\lambda(x,\eta)dxd\eta \\ &= \int_0^{r_m} \int_0^{x} s(x)N(x)\pi_\lambda(x,\eta)d\eta dx \\ &= \int_0^{r_m} s(x)N(x)\int_0^{x} \pi_\lambda(x,\eta)d\eta dx. \end{aligned}$$

and:

$$\begin{aligned} \langle f_\lambda, e \rangle &\geq \varepsilon \int_{r_m-\kappa}^{r_m} N(x)\int_0^x \exp(-\lambda(x-\eta))d\eta dx \\ &= \varepsilon \int_{r_m-\kappa}^{r_m} N(x)\frac{1}{\lambda}[1-\exp(-\lambda x)]dx. \end{aligned}$$

Since $N(x) > 0$ for $x \in [r_m - \kappa, r_m)$, we know that:

$$\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty.$$

On the other hand, we obtain:

$$V_\lambda \psi \leq T_\lambda \psi \leq \tilde{T}_\lambda \psi \leq \langle g_\lambda, \psi \rangle e, \quad \lambda \in R, \quad \psi \in X_+,$$

where the positive functional g_λ is defined by:

$$\langle g_\lambda, \psi \rangle = np_0 M \int_0^{r_m} [\int_\eta^{r_m} N(x)\pi_\lambda(x,\eta)dx]\psi(\eta)d\eta,$$

where:

$$M = max\{ess\sup \beta_1(\sigma,\xi), \cdots, ess\sup \beta_n(\sigma,\xi)\}, \qquad p_0 = max\{p_1, p_2, \cdots, p_n\}.$$

Then we obtain the estimate:

$$r(V_\lambda) \leq \langle g_\lambda, e \rangle = np_0 M \int_0^{r_m} N(x)\frac{1}{\lambda}[1-\exp(-\lambda x)]dx.$$

From which we can conclude that:

$$\lim_{\lambda \to +\infty} r(V_\lambda) = 0.$$

This completes the proof.

From condition (5.18) and the expression (5.22), the kernel $\chi_\lambda(a,\rho)$ is strictly decreasing as a function of $\lambda \in R$. Using Proposition 4.3, we know that the function $\lambda \to r(V_\lambda)$ is strictly decreasing for $\lambda \in R$. Moreover, if there exists $\lambda \in R$ such that $r(V_\lambda)=1$, then $\lambda \in \Lambda$, because $r(V_\lambda) \in P_\sigma(V_\lambda)$. From the monotonicity of $r(V_\lambda)$ and (5.19), it is easy to see that the following holds:

Lemma 5.5
Under condition (5.18), there exists a unique $\lambda_0 \in R \cap \Lambda$ such that $r(V_{\lambda_0})=1$, and $\lambda_0 > 0$ if $r(V_0)>1$; $\lambda_0 = 0$ if $r(V_0)=1$; $\lambda_0 < 0$ if $r(V_0)<1$.

Next, by using the similar argument as Theorem 6.13 in [18] we can prove that λ_0 is the dominant singular point:

Lemma 5.6
Suppose that condition (5.18) holds. If there exists a $\lambda \in \Lambda, \lambda \neq \lambda_0$, then $Re\lambda < \lambda_0$.

Proof
Suppose that $\lambda \in \Lambda$ and $V_\lambda \psi = \psi$, then $|V_\lambda \psi| = |\psi|$, where $|\psi|(a) = |\psi(a)|$. From the expression (5.22), it follows that $V_{Re\lambda}|\psi| \geq |\psi|$. Taking duality pairing with $F_{Re\lambda} \in X_+^*$ on both sides, we have $r(V_{Re\lambda})\langle F_{Re\lambda}, |\psi|\rangle \geq \langle F_{Re\lambda}, |\psi|\rangle$, from which we conclude that $r(V_{Re\lambda}) \geq 1$, because $F_{Re\lambda}$ is strictly positive. Since $r(V_\lambda), \lambda \in R$ is a decreasing function, we obtain that $Re\lambda \leq \lambda_0$. If $Re\lambda = \lambda_0$, then $V_{\lambda_0}|\psi| = |\psi|$. In fact, if we suppose that $V_{\lambda_0}|\psi| > |\psi|$, taking duality pairing with the eigenfunction F_0 corresponding to $r(V_{\lambda_0})=1$ on both sides yields $\langle F_0, |\psi|\rangle > \langle F_0, |\psi|\rangle$ which is a contradiction. Then we can write that $|\psi| = c\psi_0$ for some constant c which we may assume to be one, where ψ_0 is the eigenfunction corresponding to $r(V_{\lambda_0})=1$. Hence, $\psi(a) = \psi_0(a)\exp(i\nu(a))$ for some real-valued function $\nu(a)$. If we substitute this relation into $V_{\lambda_0}\psi_0 = |V_\lambda \psi|$ we obtain:

$$\sum_{j=1}^{n}\int_0^{r_m}\int_\eta^{r_m} p_j[\Pi(\xi) - \exp(-\gamma_j\xi)\int_\eta^\xi \gamma_j \Pi(a)\exp(\gamma_j a)da] \cdot \beta_j(\sigma,\xi)N(\xi)\exp(-\lambda(\xi-\eta))d\xi\psi_0(\eta)d\eta$$

$$= \left|\sum_{j=1}^{n}\int_0^{r_m}\int_\eta^{r_m} p_j[\Pi(\xi) - \exp(-\gamma_j\xi)\int_\eta^\xi \gamma_j \Pi(a)\exp(\gamma_j a)da]\right| \cdot \left|\beta_j(\sigma,\xi)N(\xi)e^{-(\lambda_0+iI_m\lambda)(\xi-\eta)}e^{i\nu(\eta)}d\xi\psi_0(\eta)d\eta\right|.$$

From Lemma 6.12 in [18], it follows that $-Im\lambda(\xi\eta) + \nu(\eta) = \theta$ for some constant θ. Using the relation $V_\lambda\psi = \psi$ we obtain that $e^{i\theta}V_{\lambda_0}\psi_0 = \psi_0 e^{i\nu(\eta)}$, So $\theta = \nu(\eta)$, which implies that $Im\lambda = 0$. This completes the proof of Lemma 5.6.

Theorem 5.7
Under condition (5.18), the equilibrium state

$$(s^*(a), i_1^*(a), i_2^*(a), \cdots, i_n^*(a))^T$$

for (2.7_1) - (2.7_{n+1}) is locally asymptotically stable if $r(V_0)<1$ and locally unstable if $r(V_0)>1$.

Proof
From Lemma 5.5 and 5.6, we conclude that $\sup\{Re\lambda : 1 \in P_\sigma(V_\lambda)\} = \lambda_0$. Hence it follows that $s(A+C) = \sup\{Re\lambda : 1 \in P_\sigma(V_\lambda)\} < 0$ if $r(V_0) < 1$, and $s(A+C) > 0$ if $r(V_0) > 1$. This completes the proof.

Now we can state the local stability results for our epidemic model:

Theorem 5.8 (Local stability results)
Let $r(T)$ be the spectral radius of the operator T defined by (4.5). Then the followings hold:
(1) If $r(T) < 1$, the trivial equilibrium point of (2.7_1) - (2.7_{n+1}) is locally asymptotically stable.
(2) If $r(T) > 1$, the trivial equilibrium point of (2.7_1) - (2.7_{n+1}) is unstable.
(3) If $r(T) > 1$ and condition (5.18) holds for an endemic steady state, it is locally asymptotically stable.

Proof
The proof is similar with ([19], Theorem 5.8).

We have not determined what kind of conditions could guarantee (5.18). Since it would be difficult to answer the question if we consider it under most general conditions.

6. Conclusions

Dividing the infected class into several compartment, provides an opportunity to look into the effect of the infection under the variation of the viral load. It also provides a platform to discuss the nature of the disease progression as the viral load varies among different classes. The age-structured behavior of the model adds in explaining the distribution of the disease in separate classes by age. This paper presents the model and also focusses on the various threshhold parameters which guides the existence of the endemic solutions. From the practical point of view, these results can be utlized in controlling the progression of the disease with the change in the viral load. This is a general model, which discusses the effect of disease in general. This can be utilized in discussing the effect of specific diseases by suitable choice of parameters. Moreover this paper assumes permanent immunity in the individuals once they recover. This nature can be relaxed to increase the scope of discussion by modifying the model to include diseases which also posses temporary immunity in recoved individual.

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