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A New Numerical Treatment for the Nonlinear Quadratic Integral Equation in Two-Dimensions

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Abstract
In this paper, a nonlinear quadratic integral equation (**NQIE**) of the second kind in two-dimensions with continuous kernels in two forms of integral operators is considered. Under certain conditions, the existence of a unique solution of **NQIE** is proved by using fixed-point theorem. Then, a suitable numerical method is used to transfer this equation to a system of nonlinear quadratic integral equations (**SNQIEs**) of the second kind. The modified Adomian decomposition method (**MADM**) is used to solve this system and the modified Simpson's rule (**MSR**) is considered to obtain the nonlinear algebraic system (**NAS**). Finally, many applications are treated; the numerical results are computed at different times with different forms of the continuous kernels, and the estimated error, in each case, is calculated.

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Nonlinear Quadratic Integral Equation, Continuous Kernels, Adomian Decomposition Method, Simpson's Rule, Nonlinear Algebraic System.

1. Introduction

There have been dramatic developments in integral equations during the last century. This is due to its linear and nonlinear applications in numerous diverse fields, such as processes engineering, contact and mixed problems, mathematical physics problems, chemistry, biology, theory of elasticity, potential theory and in solving most of boundary value problems of ordinary and partial differential equations, [1]-[8].

Many researchers have been interested in quadratic integral equations. Quadratic integral equations have played an important role and have been applied to improve real-life problems including; modeling the theory of neutron transport, radiative transfer, queuing theory, traffic theory, modeling of radiative transfer, kinetic theory of gases, and many others phenomena, [9]-[15].

Recently, several studies have focused on the effective properties of quadratic integral equations such as; the existence of solutions for several classes, uniqueness, monotonic and positive solutions. The theory of compact operators, the measure of non-compactness, and the Banach contraction fixed-point theorem have all utilized the major mission of the existence theory for integral equations [16]-[21]. Some numerical and analytical methods can be applied to estimate the solutions of the quadratic integral equations. However, **ADM** is the most common method used to obtain numerical solutions for quadratic integral equations, [22]-[27], alongside some other methods, [28] and [29].

Assume **NQIE** as the following formula:

$$\phi(x,t) = f(x,t) + P\phi(x,t) \int_0^t \int_0^1 k(x,y)\, F(t,\tau)\, \gamma\big(y,\tau,\phi(y,\tau)\big)\, dy\, d\tau\,, \qquad (1.1)$$

where the integral operator $P\phi(x,t)$ will take one of the forms:

$$\text{(i)} \quad P\phi(x,t) = \int_0^1 \eta(x,y)\, \psi\,\big(y,t,\phi(y,t)\big)\, dy, \quad x \in [0,1]$$

(ii) $P\phi(x,t)=\int_0^t \xi(t,\tau)\,\chi\big(x,\tau,\phi(x,\tau)\big)\,d\tau, \qquad t\in[0,T]$

In (1.1) the kernels of the NQIE $\eta(x,y)$, $\xi(t,\tau)$, $F(t,\tau)$ and $k(x,y)$ are continuous functions. The functions $f(x,t)$, $\psi\left(y,t,\phi(y,t)\right)$, $\chi\big(x,\tau,\phi(x,\tau)\big)$ and $\gamma\big(y,\tau,\phi(y,\tau)\big)$, are known continuous functions, while $\phi(x,t)$ is unknown, will be determined.

In order to guarantee the existence of a unique solution of (1.1), we assume the following conditions:

(i) The given function $f(x,t)$ with its partial derivatives is continuous with respect to position and time belong to $C([0,1]\times[0,T])$, and its norm is defined by:

$$\|f(x,t)\|_{C([0,1]\times[0,T])} = \max_{x,t}|f(x,t)| \le M.$$

(ii) The kernels of position $k(x,y),\eta(x,y)\ \epsilon\ C([0,1]\times[0,1])$ and satisfy:

$$|k(x,y)|\le c_1\ , \qquad |\,\eta(x,y)|\le c_2 \quad , \qquad (c_1\,,c_2 \text{ are constants}\,).$$

(iii) The kernels of time $F(t,\tau),\xi(t,\tau)$ are continuous with respect to τ for all $0\le\tau\le t\le T$, and satisfy:

$$|F(t,\tau)|\le d_1\ , \qquad |\,\xi(t,\tau)|\le d_2 \quad , \qquad (d_1\,,d_2 \text{ are constants}\,).$$

(iv) For the constants A and A_1, the nonlinear function $\gamma\big(t,x,\phi(x,t)\big)$ satisfies the conditions:

$(a)\ \max_{x,t}\big|\gamma\big(t,x,\phi(x,t)\big)\big| \le\ A\ \|\phi(x,t)\|_{C([0,1]\times[0,T])}$.

$(b)\big|\gamma\big(t,x,\phi_1(x,t)\big)-\gamma\big(t,x,\phi_2(x,t)\big)\big|\le A|\phi_1(x,t)-\phi_2(x,t)|$.

$(c)\ |\gamma(t,x,0)|\le A_1$.

(v) The function $\psi\big(x,t,\phi(x,t)\big)$ satisfies for the constants B and B_1, the following conditions:

$(a)\ \max_{x,t}\big|\psi\big(x,t,\phi(x,t)\big)\big| \le\ B\ \|\phi(x,t)\|_{C([0,1]\times[0,T])}$.

$(b)\big|\psi\big(x,t,\phi_1(x,t)\big)-\psi\big(x,t,\phi_2(x,t)\big)\big|\le B|\phi_1(x,t)-\phi_2(x,t)|$.

$(c)\ |\psi(t,x,0)|\le B_1$.

(vi) The function $\chi\big(x,t,\phi(x,t)\big)$ satisfies for the constants L and L_1, the following conditions:

$(a)\ \max_{x,t}\big|\chi\big(x,t,\phi(x,t)\big)\big| \le\ L\|\phi(x,t)\|_{C([0,1]\times[0,T])}$.

$(b)\big|\chi\big(x,t,\phi_1(x,t)\big)-\chi\big(x,t,\phi_2(x,t)\big)\big|\le L|\phi_1(x,t)-\phi_2(x,t)|$.

$(c)\ |\chi(t,x,0)|\le L_1$.

By the current paper, the **NQIE** of the second kind in two-dimensions with continuous kernels in two forms of integral operator is considered. The existence of a unique solution of **NQIE** is proved. Then, a suitable numerical method is used to transfer this equation to a **SNQIEs** of the second kind. **MADM** and **MSR** are used, to obtain numerically the solution of the **SNQIEs**. In the remainder of this work. Many applications are treated; Maple software is used to obtain the numerical results and the estimated errors. Finally, the conclusion is included a comparison between the numerical solutions of the two methods and their respective errors.

2. The Existence of a Unique Solution of NQIE

By using Banach fixed-point theorem, we will prove the existence of a unique solution of (1.1). Firstly, we must suppose that the integral operator $P\phi(x,t)$ is bounded and continuous; in addition, the operator is contractive.

When the operator $P\phi$ takes the form (i), (1.1) can be adapted in the following form:

$$\phi(x,t)=f(x,t)+\int_0^1\eta(x,y)\,\psi\big(y,t,\phi(y,t)\big)dy\int_0^t\int_0^1 k(x,y)F(t,\tau)\,\gamma\big(y,\tau,\phi(y,\tau)\big)dyd\tau, \quad (2.1)$$

Then, write (2.1) in the integral operator form:

$$\bar{W}\phi(x,t)=f(x,t)+W\phi(x,t) \qquad (2.2)$$

where:

$$W\phi(x,t) = \int_0^1 \eta(x,y)\,\psi\big(y,t,\phi(y,t)\big)dy \int_0^t \int_0^1 k(x,y)F(t,\tau)\,\gamma\big(y,\tau,\phi(y,\tau)\big)dyd\tau. \quad (2.3)$$

Theorem 1

In view of the conditions (i)-(v), (2.1) has a unique solution in the Banach space $C([0,1]\times[0,T])$, under the condition:

$$c_1c_2d_1T[B(Ar_0+A_0)+\mathrm{A}(Br_0+B_0)] < 1\,. \quad (2.4)$$

where $r_0 < 1$ is a constant, satisfies that $|\phi(x,t)| \le r_0$.

Proof

In view of the two formulas (2.2) and (2.3), we have:

$$|\overline{W}\phi(x,t)| \le |f(x,t)| + \int_0^1 |\eta(x,y)|\,\big|\psi\big(y,t,\phi(y,t)\big)\big|dy \int_0^t \int_0^1 |k(x,y)||F(t,\tau)|\,\big|\gamma\big(y,\tau,\phi(y,\tau)\big)\big|dyd\tau,$$

then use the conditions (i)- (iv– a, v– a), to obtain:

$$\|\overline{W}\phi(x,t)\| \le \mathrm{M}+c_1c_2d_1[B\|\phi(x,t)\|\,T\,(Ar_0+A_0)+(Br_0+B_0)\|\phi(x,t)\|\,AT\,]$$
$$\le \mathrm{M}+\sigma\|\phi(x,t)\|\ , \qquad (\sigma = c_1c_2d_1T[B(Ar_0+A_0)+\mathrm{A}(Br_0+B_0)])\,. \quad (2.5)$$

The previous inequality (2.5) shows that, the operator $\overline{W}$ maps the ball S_ρ into itself, where:

$$\rho = \frac{M}{[1-c_1c_2d_1T[B(Ar_0+A_0)+\mathrm{A}(Br_0+B_0)]]}\ , \quad (2.6)$$

since $\rho > 0$ & $M > 0$, therefore we have $\sigma < 1$. Also, (2.5) involves that the operator W is bounded, where:

$$\|W\phi(x,t)\| \le \sigma\,\|\phi(x,t)\|\ . \quad (2.7)$$

Moreover, (2.5) and (2.7) define that the operator $\overline{W}$ is bounded.

Now, consider the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in the space $C([0,1]\times[0,T])$, then from (2.2) and (2.3), we find:

$$|\overline{W}\phi_1(x,t)-\overline{W}\phi_2(x,t)|$$
$$= \int_0^1 |\eta(x,y)|\,\big|\psi\big(y,t,\phi_1(y,t)\big)\big|dy \int_0^t \int_0^1 |k(x,y)||F(t,\tau)|\,\big|\gamma\big(y,\tau,\phi_1(y,\tau)\big)\big|dyd\tau$$
$$- \int_0^1 |\eta(x,y)|\,\big|\psi\big(y,t,\phi_2(y,t)\big)\big|dy \int_0^t \int_0^1 |k(x,y)||F(t,\tau)|\,\big|\gamma\big(y,\tau,\phi_2(y,\tau)\big)\big|dyd\tau$$
$$\le \int_0^1 |\eta(x,y)|\,\big|\psi\big(y,t,\phi_1(y,t)\big)-\psi\big(y,t,\phi_2(y,t)\big)\big|dy$$
$$\times \int_0^t \int_0^1 |k(x,y)||F(t,\tau)|\,\big|\gamma\big(y,\tau,\phi_1(y,\tau)\big)\big|dyd\tau + \int_0^1 |\eta(x,y)|\,\big|\psi\big(y,t,\phi_2(y,t)\big)\big|dy$$
$$\times \int_0^t \int_0^1 |k(x,y)||F(t,\tau)|\,\big|\gamma\big(y,\tau,\phi_1(y,\tau)\big)-\gamma\big(y,\tau,\phi_2(y,\tau)\big)\big|dyd\tau$$

In view of the conditions (ii)- (iv-b,v – b) , we have:

$$\|\overline{W}\phi_1(x,t)-\overline{W}\phi_2(x,t)\| \le$$
$$c_1c_2d_1 \max_{x,t}\Bigg[\int_0^1 B|\phi_1(y,t)-\phi_2(y,t)|dy \int_0^t \int_0^1 \big|\gamma\big(y,\tau,\phi_1(y,\tau)\big)-\gamma(y,\tau,0)+\gamma(y,\tau,0)\big|dyd\tau$$
$$+\int_0^1 \big|\psi\big(y,t,\phi_2(y,t)\big)-\psi(y,t,0)+\psi(y,t,0)\big|dy \int_0^t \int_0^1 \mathrm{A}|\phi_1(y,\tau)-\phi_2(y,\tau)|dyd\tau\Bigg]$$

$$\|\overline{W}\phi_1(x,t)-\overline{W}\phi_2(x,t)\| \le$$
$$c_1c_2d_1[B\|\phi_1(x,t)-\phi_2(x,t)\|\,T\,(Ar_0+A_0)+(Br_0+B_0)\|\phi_1(x,t)-\phi_2(x,t)\|\,AT\,]$$

$$\|\bar{W}\phi_1(x,t) - \bar{W}\phi_2(x,t)\| \le \sigma\, \|\phi_1(x,t) - \phi_2(x,t)\|. \tag{2.8}$$

From (2.8), we see that the operator $\bar{W}$ is continuous in the space $C([0,1]\times[0,T])$. Moreover, $\bar{W}$ is a contraction operator under the condition $\sigma < 1$. So, from Banach fixed point theorem, $\bar{W}$ has a unique fixed point which is, of course, the unique solution of (2.1).

Now, consider the NQIE, when the operator $P\phi$ takes the form (ii):

$$\phi(x,t) = f(x,t) + \int_0^t \xi(t,\tau)\,\chi\big(x,\tau,\phi(x,\tau)\big)\,d\tau \int_0^t\int_0^1 k(x,y)F(t,\tau)\,\gamma\big(y,\tau,\phi(y,\tau)\big)dyd\tau, \tag{2.9}$$

Theorem 2

In view of the conditions (i)-(iv) and (vi), (2.9) has a unique solution in the Banach space $C([0,1]\times[0,T])$, under the following condition:

$$c_1 d_1 d_2 T^2[L(Ar_0 + A_0) + \mathrm{A}(Lr_0 + L_0)] < 1\,. \tag{2.10}$$

The proof can be easily obtained by the same way of Theorem 1.

3. Nonlinear System of Quadratic Integral Equations (NSQIEs)

In this section, a numerical method is used in (2.1) to obtain a **NSQIEs**, [7] and [8].

If we divide the interval $[0,T]$ into l subintervals, by means of the points: $0 = t_0 < t_1 < \cdots < t_l = T$, where $t = t_r$, $\tau = t_s$, $r,s = 0,1,2,\dots,l$, then use the quadrature formula, the time integral term of (2.1) becomes:

$$\int_0^{t_r}\int_0^1 k(x,y)F(t_r,\tau)\gamma\big(y,\tau,\phi(y,\tau)\big)dyd\tau = \sum_{s=0}^{r} u_s\, F_{r,s}\int_0^1 k(x,y)\gamma_s\big(y,\phi_s(y)\big)dy\ , \tag{3.1}$$

now, use (3.1) in (2.1), to get:

$$\phi_r(x) = f_r(x) + \int_0^1 \eta(x,y)\,\psi_r\big(y,\phi_r(y)\big)dy \sum_{s=0}^{r} u_s\, F_{r,s}\int_0^1 k(x,y)\gamma_s\big(y,\phi_s(y)\big)dy. \tag{3.2}$$

In the same technique, (2.9) will be take the form:

$$\phi_r(x) = f_r(x) + \sum_{j=0}^{r} v_j\, \xi_{r,j}\, \chi_j\big(x,\phi_j(x)\big) \sum_{s=0}^{r} u_s\, F_{r,s}\int_0^1 k(x,y)\gamma_s\big(y,\phi_s(y)\big)dy, \tag{3.3}$$

where h is the step-size of integration and u_s, v_j are the weights,

$$u_s = \begin{cases} h/2 & ;\quad s=0\ ,s=r \\ h & ;\quad 0<s<r \end{cases}\ ,\qquad v_j = \begin{cases} h/2 & ;\quad j=0\ ,j=r \\ h & ;\quad 0<j<r \end{cases}\ . \tag{3.4}$$

We used the following notations

$$\phi(x,t_r) = \phi_r(x)\ ,\quad F(t_r,t_s) = F_{r,s}\ ,\quad \xi(t_r,t_j) = \xi_{r,j}\ ,\quad \chi(y,t_j,\ \phi(y,t_j)) = \chi_j\big(y,\phi_j(y)\big)$$

$$\gamma(x,t_s,\phi(x,t_s)) = \gamma_s\big(x,\phi_s(x)\big),\ \ \psi(y,t_r,\phi(y,t_r)) = \psi_r\big(y,\phi_r(y)\big), f(x,t_r) = f_r(x), \tag{3.5}$$

each of the formulas (3.2) and (3.3) represents **NSQIEs** of the second kind.

Definition 1

The error of using quadratic method in (3.2) can be determine by:

$$E_s = \left|\int_0^t\int_0^1 |k(x,y)||F(t,\tau)|\,\big|\gamma\big(y,\tau,\phi(y,s)\big)\big|dyd\tau - \sum_{s=0}^{r} u_s\, F_{r,s}\int_0^1 k(x,y)\gamma_s\big(y,\phi_s(y)\big)dy\right|, \tag{3.6}$$

while, the error in (3.3), can be determined by

$$E_s = \left| \int_0^t \xi(t,\tau)\,\chi\big(x,\tau,\phi(x,\tau)\big)\,d\tau \int_0^t \int_0^1 k(x,y)F(t,\tau)\,\gamma\big(y,\tau,\phi(y,\tau)\big)dyd\tau \right.$$
$$\left. - \sum_{j=0}^{r} v_j\,\xi_{r,j}\,\chi_j\left(x,\phi_j(x)\right) \sum_{s=0}^{r} u_s\,F_{r,s} \int_0^1 k(x,y)\gamma_s\big(y,\phi_s(y)\big)dy \right|. \qquad (3.7)$$

4. The Modified Adomian Decomposition Method

ADM has been proven as a powerful and reliable scheme for solving a wide variety of linear and nonlinear problems; like integral equations, boundary value problems, ordinary or partial differential equations, algebraic equations, and so on, [16] and [17]. The **ADM** includes generating the solution in the form of a series which terms are determined by a recurrence relationship using the Adomian polynomials. One can admit that it is practically difficult to find the exact sum of an Adomian series. Indeed, we only able to calculate a finite terms of the series. On the other hand, the Adomian series is quickly convergent and a truncation error can be easily calculated. Fixed-point theorem was used in many researches for proving **ADM** convergence, the convergence was ensured with weak hypothesis; It was discussed and proved by different methods in [18]-[21].

Assume that the functions $\phi_r(x)$ in the system (3.2) can be expressed as an infinite series:

$$\phi_r(x) = \sum_{n=0}^{\infty} \phi_{r,n}(x), \qquad (4.1)$$

alongside, the nonlinear terms $\gamma_s\big(x,\phi_s(x)\big)$, $\psi_r\big(x,\phi_r(x)\big)$ of (3.2) can be supposed in the form:

$$\gamma_s\big(x,\phi_s(x)\big) = \sum_{n=0}^{\infty} A_{s,n} \quad , \quad \psi_r\big(x,\phi_r(x)\big) = \sum_{n=0}^{\infty} \bar{A}_{r,n} \quad , \qquad (4.2)$$

where the Adomian polynomials $A_{s,n}$, $\bar{A}_{r,n}$ can be determined by:

$$A_{s,n} = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\,\gamma_s\left(\sum_{i=0}^{n} \lambda^i\,\phi_{s,i}\right)\right)_{\lambda=0} \quad , \quad \bar{A}_{r,n} = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\,\psi_r\left(\sum_{i=0}^{n} \lambda^i\,\phi_{r,i}(x)\right)\right)_{\lambda=0} . \qquad (4.3)$$

Another formula of Adomian polynomials, is given by:

$$A_{s,n} = \gamma_s\big(S_{s,n}\big) - \sum_{i=0}^{n-1} A_{s,i} \quad , \quad \bar{A}_{r,n} = \psi_r\big(S_{r,n}\big) - \sum_{i=0}^{n-1} \bar{A}_{r,i} \quad , \qquad (4.4)$$

where, the partial sum $S_{s,n}$ is:

$$S_{r,n} = \sum_{i=0}^{n} \phi_{r,i}(x) \quad , \qquad (4.5)$$

after applying the **ADM** on (3.2), the Adomian decomposition method introduces the recurrence relation:

$$\phi_{r,0}(x) = f_r(x) \,;$$
$$\phi_{r,i}(x) = \int_0^1 \eta(x,y)\,\bar{A}_{r,i-1}(y)dy \sum_{s=0}^{r} u_s\,F_{r,s} \int_0^1 k(x,y)A_{s,i-1}(y)dy\,, (i \geq 1)\,. \qquad (4.6)$$

For **MADM**, the modification of the free term is written in the form:

$$f_r(x) = \sum_{n=0}^{\infty} f_{r,n}(x) \,, \qquad (4.7)$$

in view of (4.7), the modification of the solution can be modified to:

$$\phi_{r,0}(x) = f_{r,0}(x) \,;$$

$$\phi_{r,i}(x) = f_{r,i}(x) + \int_0^1 \eta(x,y)\, \bar{A}_{r,i-1}(y)dy \sum_{s=0}^{r} u_s\, F_{r,s} \int_0^1 k(x,y) A_{s,i-1}(y)dy\,, (i \geq 1). \quad (4.8)$$

The application of (4.8) is more useful when the free term is an exponential or periodic function, the obtained series converges to an exact solution of **QNIE** (2.1).

By the same technique, the solution of **NSQIEs** (3.3) will be:

$$\phi_{r,0}(x) = f_{r,0}(x)\,;$$

$$\phi_{r,i}(x) = f_{r,i}(x) + \sum_{j=0}^{r} v_j\, \xi_{r,j}\, \bar{\bar{A}}_{j,i-1}(x) \sum_{s=0}^{r} u_s\, F_{r,s} \int_0^1 k(x,y) A_{s,i-1}(y)dy\,, (i \geq 1). \quad (4.9)$$

where:

$$\chi_j\left(x, \phi_j(x)\right) = \sum_{n=0}^{\infty} \bar{\bar{A}}_{j,n} \quad . \quad (4.10)$$

The series in (4.9) converges to an exact solution of **QNIE** (2.9).

5. Modified Simpson's Quadrature Rule

In this section, we approximate the **NSQIEs** of (3.2), by using Modified Simpson's quadrature rule formula, [31] and [32], to obtain:

$$\phi_{r,p} = f_{r,p} + \frac{h}{3}\left[4\sum_{i=1}^{N/2} \eta_{p,2i-1}\, \psi_{r,2i-1} + 2\sum_{i=1}^{(N/2)-1} \eta_{p,2i}\, \psi_{r,2i} + \eta_{p,0}\, \psi_{r,0} + \eta_{p,N}\, \psi_{r,N}\right]$$
$$\times \sum_{s=0}^{r} u_s\, F_{r,s}\ \frac{h}{3}\left[4\sum_{j=1}^{N/2} k_{p,2j-1}\, \gamma_{s,2j-1} + 2\sum_{j=1}^{(N/2)-1} k_{p,2j}\, \gamma_{s,2j} + k_{p,0}\, \gamma_{s,0} + k_{p,N}\, \gamma_{s,N}\right], \quad (5.1)$$

which can be adapted in the form:

$$\phi_{r,p} = f_{r,p} + \frac{h}{9}\sum_{i=1}^{N/2} \left[\eta_{p,2i-2}\, \psi_{r,2i-2} + 4\, \eta_{p,2i-1}\, \psi_{r,2i-1} + \eta_{p,2i}\, \psi_{r,2i}\right]$$
$$\times \sum_{s=0}^{r} u_s\, F_{r,s} \sum_{j=1}^{N/2} \left[k_{p,2j-2}\gamma_{s,2j-2} + 4\, k_{p,2j-1}\gamma_{s,2j-1} + k_{p,2j}\, \gamma_{s,2j}\right], \quad (5.2)$$

by using Modified Simpson's quadrature rule formula (3.3) can adapted in the following **NAS**

$$\phi_{r,p}(x) = f_{r,p}(x) +$$
$$\frac{h}{3}\sum_{j=0}^{r} v_j\, \xi_{r,j}\, \chi\left(x, \phi_j(x)\right) \sum_{s=0}^{r} u_s\, F_{r,s} \sum_{i=1}^{N/2} \left[k_{p,2i-2}\gamma_{s,2i-2} + 4\, k_{p,2i-1}\gamma_{s,2i-1} + k_{p,2i}\, \gamma_{s,2i}\right], (5.3)$$

The solution of the **NAS** system in (5.2), converges to the solution of (2.1) and the solution of (5.3), converges to the solution of (2.9).

Definition 2

The following relation determines the estimate total error $R_{r,N}$ of (5.2) :

$$R_{r,N}=\left|\int_0^1 \eta(x,y)\,\psi\big(y,\phi_r(y)\big)dy\sum_{s=0}^{r} u_s\,F_{r,s}\int_0^1 k(x,y)\gamma\big(y,\phi_s(y)\big)dy\right.$$

$$-\frac{h}{9}\sum_{i=1}^{N/2}\left[\eta_{p,2i-2}\,\psi_{r,2i-2}+4\,\eta_{p,2i-1}\,\psi_{r,2i-1}+\eta_{p,2i}\,\psi_{r,2i}\right]$$

$$\left.\times\sum_{s=0}^{r} u_s\,F_{r,s}\sum_{j=1}^{N/2}\left[k_{p,2j-2}\gamma_{s,2j-2}+4\,k_{p,2j-1}\gamma_{s,2j-1}+k_{p,2j}\,\gamma_{s,2j}\right]\right|, \qquad (5.4)$$

and the estimate total error $R_{r,N}$ of formula (5.3) is:

$$R_{r,N}=\left|\sum_{j=0}^{r} v_j\,\xi_{r,j}\,\chi_j\Big(x,\phi_j(x)\Big)\sum_{s=0}^{r} u_s\,F_{r,s}\int_0^1 k(x,y)\gamma_s\big(y,\phi_s(y)\big)dy\right.$$

$$\left.-\frac{h}{3}\sum_{j=0}^{r} v_j\,\xi_{r,j}\,\chi\Big(x,\phi_j(x)\Big)\sum_{s=0}^{r} u_s\,F_{r,s}\sum_{i=1}^{N/2}\left[k_{p,2i-2}\gamma_{s,2i-2}+4\,k_{p,2i-1}\gamma_{s,2i-1}+k_{p,2i}\,\gamma_{s,2i}\right]\right|. \qquad (5.5)$$

6. Numerical Applications

In this section, we consider the **NQIE** (1.1) in some examples, The results are obtained numerically by Maple 18 software, for $x\in[0,1]$, and $t\in[0,T]$. The next tables give us an exact solution and the numerical solutions, which obtained by using modified Simpson's rule (App. MSR) and modified Adomian decomposition method (App. MAD), beside their corresponding errors (Err. MSR) and (Err. MAD), respectively, at the times $T=0.008, T=0.06$ and $T=0.4$.

Example (1-1): Consider **NQIE** , in the form:

$$\phi(x,t)=f(x,t)+\int_0^1 x^2y\,\cosh\big(\phi(y,t)\big)\,dy\int_0^t\int_0^1(x+y)\,t\,\tau\,\sinh\big(\phi(y,\tau)\big)\,dyd\tau\,. \qquad (6.1)$$

The exact solution is $\phi(x,t)=x\,t$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.0016	0.0016	0	0.0016	3E-12
0.4	0.0032	0.0032	0	0.0032	6E-12
0.6	0.0048	0.0048	1.5700E-10	0.0048	8E-11
0.8	0.0064	0.0064	3.4200E-10	0.0064	3.01E-10
1	0.008	0.008000001	6.4000E-10	0.008000001	7.79E-10

Table 1-1: $T=0.008$, $N=20$ and $l=4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.012	0.012000034	3.386E-08	0.012000011	1.099E-08
0.4	0.024	0.024000176	1.7643E-07	0.024000019	1.893E-08
0.6	0.036	0.036000499	4.9857E-07	0.036000254	2.5434E-07
0.8	0.048	0.048001084	1.08369E-06	0.048000954	9.5392E-07
1	0.06	0.060002028	2.02783E-06	0.060002471	2.47127E-06

Table 1-2: $T=0.06$, $N=20$ and $l=4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.08	0.080069933	6.99328E-05	0.080023838	2.3838E-05
0.4	0.16	0.160364066	3.64066E-04	0.160031011	3.10109E-05
0.6	0.24	0.241030897	1.03090E-03	0.240516324	5.16324E-04
0.8	0.32	0.32225452	2.25452E-03	0.322041602	2.04160E-03
1	0.4	0.404266735	4.26674E-03	0.405550118	5.55012E-03

Table 1-3: $T=0.4$ $N=20$ and $l=4$.

Example (1-2): Consider **NQIE**, in the form:

$$\phi(x,t) = f(x,t) + \int_0^t \exp\left(\frac{t+\tau}{1-\tau}\right) \cosh\big(\phi(x,\tau)\big)\, d\tau \int_0^t \int_0^1 (x+y)\, t\, \tau\, \sinh\big(\phi(y,\tau)\big)\, dy d\tau. \quad (6.2)$$

The exact solution is $\phi(x,t) = x\, t$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.0016	0.0016	0	0.0016	1E-12
0.4	0.0032	0.0032	0	0.0032	1E-12
0.6	0.0048	0.0048	0	0.0048	3E-12
0.8	0.0064	0.0064	0	0.0064	7E-12
1	0.008	0.008	0	0.008	1.1E-11

Table 1-4: $T = 0.008$, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.012	0.0120001	1.003E-07	0.012000031	3.144E-08
0.4	0.024	0.024000124	1.2388E-07	0.024000014	1.365E-08
0.6	0.036	0.036000148	1.4763E-07	0.036000081	8.111E-08
0.8	0.048	0.048000172	1.7157E-07	0.048000171	1.7104E-07
1	0.06	0.060000196	1.9571E-07	0.060000284	2.8353E-07

Table 1-5: $T = 0.06$, $N = 20$ and $l = 4$.

X	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.08	0.082687044	2.6870E-03	0.080802443	8.0244E-04
0.4	0.16	0.163336999	3.3370E-03	0.160364422	3.6442E-04
0.6	0.24	0.244009772	4.0098E-03	0.242184074	2.1841E-03
0.8	0.32	0.324711193	4.7112E-03	0.324748902	4.7489E-03
1	0.4	0.4054477	5.4477E-3	0.408186814	8.1868E-3

Table 1-6: $T = 0.4, N = 20$ and $l = 4$.

Example (2-1): Consider **QNIE**, in the form:

$$\phi(x,t) = f(x,t) + \int_0^1 e^{\frac{x-y}{3}} \phi^2(y,t) dy \int_0^t \int_0^1 (2\,x\,y)\, \frac{t+\tau}{4+t-\tau} \phi^3(y,\tau)\, dy d\tau. \quad (6.3)$$

The exact solution is $\phi(x,t) = t \sin(\pi\, x/2)$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.002472136	0.00247213	4.2067E-16	0.0024721	4.2067E-16
0.4	0.004702282	0.00470228	6.60215E-13	0.0047022	6.60215E-13
0.6	0.006472136	0.00647213	4.19803E-16	0.0064721	4.19803E-16
0.8	0.007608452	0.00760845	1.63877E-12	0.0076084	1.63877E-12
1	0.008	0.008	0	0.008	0

Table 2-1: $T = 0.008$, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.01854102	0.01854102	2.49684E-12	0.0185410	7.50315E-12
0.4	0.035267115	0.03526711	2.45162E-12	0.0352671	2.45162E-12
0.6	0.04854102	0.04854102	2.49685E-12	0.0485410	1.57503E-10
0.8	0.057063391	0.05706339	1.22291E-10	0.0570633	5.72291E-10
1	0.06	0.06	1.7E-10	0.0600000	9.7E-10

Table 2-2: T=0.06, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.123606798	0.12361443	7.63735E-06	0.1236115	4.76935E-06
0.4	0.235114101	0.23513150	1.74003E-05	0.2351166	2.54388E-06
0.6	0.323606798	0.32364175	3.49596E-05	0.3236947	8.79233E-05
0.8	0.380422607	0.38048815	6.55437E-05	0.3807344	3.11875E-04
1	0.4	0.40009839	9.83924E-05	0.4005449	5.44907E-04

Table 2-3: T=0.4, $N = 20$ and $l = 4$.

Example (2-2): Consider **NQIE**, in the form:

$$\phi(x,t) = f(x,t) + \int_0^t \ln|t+\tau-1|\, \phi^2(x,\tau)d\tau \int_0^t \int_0^1 (2\,x\,y)\ \frac{t+\tau}{4+t-\tau} \phi^3(y,\tau)\ dyd\tau. \quad (6.4)$$

The exact solution is $\phi(x,t) = t \sin(\pi\, x/2)$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.002472136	0.00247213	4.2067E-16	0.0024721	4.2067E-16
0.4	0.004702282	0.00470228	6.60215E-13	0.0047022	6.60215E-13
0.6	0.006472136	0.00647213	4.19803E-16	0.0064721	4.19803E-16
0.8	0.007608452	0.00760845	1.63877E-12	0.0076084	1.63877E-12
1	0.008	0.008	0	0.008	0

Table 2-4: T=0.008, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.01854102	0.01854102	2.49684E-12	0.01854102	2.49684E-12
0.4	0.035267115	0.03526711	2.45162E-12	0.035267115	2.45162E-12
0.6	0.04854102	0.04854102	2.49685E-12	0.04854102	2.49685E-12
0.8	0.057063391	0.05706339	1.22908E-11	0.057063391	1.22908E-11
1	0.06	0.06	0	0.06	0

Table 2-5: $T = 0.06$, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.12360679	0.1236010	5.75765E-06	0.12360691	1.2135E-07
0.4	0.23511410	0.2350724	4.16947E-05	0.23511785	3.75848E-06
0.6	0.32360679	0.3234881	1.1865E-04	0.32364225	3.54594E-05
0.8	0.38042260	0.3802036	2.18941E-04	0.38053420	1.11598E-04
1	0.4	0.3996972	3.02752E-04	0.40018124	1.81240E-04

Table 2-6: $T = 0.4$, $N = 20$ and $l = 4$.

Example (3): Consider **NQIE**, in the form:

$$\phi(x,t) = f(x,t) + \int_0^1 \frac{x\,y}{3} \exp(\phi(y,t)-1)\ dy \int_0^t \int_0^1 e^{\tau-t}\ \frac{y-x}{y+x+5}\ \phi^2(y,\tau)\ dyd\tau. \quad (6.5)$$

The exact solution is $\phi(x,t) = \mathrm{t} \ln|((x+1))|$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.00145857	0.001458572	3.51637E-13	0.001458572	1.3352E-11
0.4	0.00269178	0.002691778	3.02969E-14	0.002691778	8.9697E-12
0.6	0.00376003	0.003760029	3.41151E-14	0.003760029	8.1966E-11
0.8	0.00470229	0.004702293	2.16953E-13	0.004702293	3.6522E-10
1	0.00554518	0.005545177	1.3048E-10	0.005545176	9.8948E-10

Table 3-1: $T = 0.008, N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.01093929	0.010939306	1.27924E-08	0.010939288	5.8276E-09
0.4	0.02018833	0.02018835	1.53627E-08	0.02018833	3.8473E-09
0.6	0.02820022	0.028200224	6.30526E-09	0.028200183	3.4635E-08
0.8	0.0352672	0.035267183	1.67741E-08	0.035267046	1.5417E-07
1	0	0	0	0	0

Table 3-2: T=0.06, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.07292862	0.072932711	4.0878E-06	0.072926609	2.0133E-06
0.4	0.13458889	0.134593809	4.91405E-06	0.134587412	1.4824E-06
0.6	0.18800145	0.188003421	1.9695E-06	0.187991338	1.0114E-05
0.8	0.23511467	0.235108908	5.75846E-06	0.235068565	4.6101E-05
1	0.27725887	0.277239358	1.95143E-05	0.277128458	1.3041E-04

Table 3-3: T=0.4, $N = 20$ and $l = 4$.

Example (3-2): Consider **NQIE**, in the form:

$$\phi(x,t) = f(x,t) + \int_0^1 \frac{\sinh(\tau-2)}{\cosh t} \exp(\phi(x,\tau)-1)\, d\tau \int_0^t \int_0^1 e^{\tau-t} \frac{y-x}{y+x+5}\, \phi^2(y,\tau) dy d\tau . \quad (6.6)$$

The exact solution is $\phi(x,t) = \text{t} \ln |((x+1))|$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.00145857	0.001458572	0	0.001458572	1.3E-11
0.4	0.00269178	0.002691778	0	0.002691778	9E-12
0.6	0.00376003	0.003760029	0	0.003760029	7E-12
0.8	0.00470229	0.004702293	0	0.004702293	4.1E-11
1	0.00554518	0.005545177	0	0.005545178	9.2E-11

Table 3-4: $T = 0.008$, $N = 20$ and $l = 4$.

X	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.01093929	0.010939252	4.12576E-08	0.010939334	4.08024E-08
0.4	0.02018833	0.02018831	2.45773E-08	0.020188363	2.90327E-08
0.6	0.02820022	0.028200209	8.82474E-09	0.028200241	2.32953E-08
0.8	0.0352672	0.035267206	6.05587E-09	0.035267328	1.28246E-07
1	0.04158883	0.041588851	2.01164E-08	0.04158912	2.89476E-07

Table 3-5: $T = 0.06$, $N = 20$ and $l = 4$.

x	Exact	App. MSR	Err. MSR	App. MAD	Err. MAD
0	0	0	0	0	0
0.2	0.07292862	0.072863881	6.47422E-05	0.072990944	6.2321E-05
0.4	0.13458889	0.134549379	3.95161E-05	0.134634597	4.5702E-05
0.6	0.18800145	0.187986963	1.44884E-05	0.188040004	3.85522E-05
0.8	0.23511467	0.235124801	1.01352E-05	0.235333908	2.19242E-04
1	0.27725887	0.277293086	3.42141E-05	0.277770896	5.12023E-04

Table 3-6: T=0.4, $N = 20$ and $l = 4$.

7. Conclusions

By the current research, we considered a **NQIE** with continuous kernels in two forms of the integral operator. The theorem of Banach fixed-point has been used to prove the existence of a unique solution of **NQIE**. **SNIEs** of the second kind has been obtained by quadratic numerical method. Moreover, the **MADM**, as a best numerical method in treating nonlinear problems, has been used to solve this system, and the **MSR** has been applied on the system to obtain a **NAS**.

Finally, Some applications were treated to obtain numerical results.

Many different cases can be established when a **NQIE** takes special types. In future works, we can suppose and solve a linear and a nonlinear two-dimensional quadratic integral equation with a singular kernel in position or time.

The previous numerical results of Tables (1-1) to (3-6), have shown:

1. The convergence of the approximate solutions of **MADM** and **MSR** to the exact solution.
2. The results provide further confirmation of the effectiveness of **MADM** and **MSR** for obtaining the approximate solutions for linear and nonlinear problems.
3. The time was obviously effect on the values of the error.
4. The values of the error was clearly Rising when x was approaching one
5. The error was vanished when $x = 0$.
6. Adomian method was more accurate at the points next to the origin's point of the position, on the contrary, Simpson was more accurate near the other end when $x = 1$.

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