Universal Journal of Integral Equations **3 (2015), 51-60**
www.papersciences.com

On a Discussion of Fredholm – Urysohn Integral Equation with Singular Kernel in Time

M. M. El-Kojok[1] , S. A. Raad[2]
[1]Department of Mathematics, College of Sciences and Arts, Qassim University, Al Qassim, K.S.A.
[2]Department of Mathematics, Faculty Applied Sciences, Umm Al– Qura University, Makka, K.S.A.
[1]maha_mko@hotmail.com, [2]saraad@uqu.edu.sa

Abstract
The existence of a unique solution of Fredholm-Urysohn integral equation (**F-UIE**) of the second kind is discussed and proved. The Fredholm integral term is considered in position with continuous kernel, while the Urysohn integral term in time with singular kernel. A quadratic numerical method is used to obtain a system of Urysohn integral equations (**SUIEs**) of the second kind in time. Moreover, the modified Toeplitz matrix method (**MTMM**), as a numerical method, is used to obtain a nonlinear algebraic system (**NAS**). Many important theorems related to the existence of a unique solution of the **SUIEs,** the **NAS** and the estimate error are considered and proved. Finally, numerical examples, when the kernel of time takes a logarithmic and Carleman forms, are calculated and the estimate error, in each case, is computed.

2010 Mathematics Subject Classification: 65L10, 65R20
Key Word and Phrases
Fredholm - Urysohn Integral Equation (F-UIE), System of Urysohn Integral Equations, Modified Toeplitz Matrix Method, Nonlinear Algebraic System.

1. Introduction

The mixed integral equations have many applications in various sciences [1]-[3]. Many different analytic methods can be used to obtain the solution of the mixed integral equations in the Banach space $L_2[\Omega]\times C[0,T]$, where Ω is the domain of the contact problem and considered in the position, while $t\in[0,T], T<1$, is the time. By the same time, the numerical methods have played an important rule to obtain the numerical solution of the mixed integral equations [4]-[6].

Consider the **F-UIE** of the second kind:

$$\mu\,\phi(x,t)=f(x,t)+\lambda\int_a^b K(x,y)\phi(y,t)\,dy+\lambda\int_0^t F(|t-\tau|)\gamma\big(\tau,x,\phi(x,\tau)\big)d\tau\ . \qquad (1.1)$$

Here, $f(x,t)$ and $\gamma\big(t,x,\phi(x,t)\big)$ are two given functions, while the function $\phi(x,t)$ is unknown in the Banach space $C([a,b]\times[0,T])$. The kernel of position $K(x,y)$ is continuous, while the kernel of time $F(|t-\tau|),\ t,\tau\,\epsilon\,[0,T],\ T<1$, has a singularity. The constant μ defines the kind of the integral equation, while λ is a constant may be complex that has a physical meaning.

2. Existence of a Unique Solution of F-UIE

In this section, the existence of a unique solution of eq. (1.1) will be discussed and proved by virtue of Banach fixed point theorem which can be applied to IEs of the first and second kinds. In this aim, we write eq. (1.1) in the integral operator form:

$$\bar{W}\phi(x,t)=\frac{1}{\mu}f(x,t)+W\phi(x,t)\qquad,\quad \mu\neq0\quad, \qquad (2.1)$$

where,

$$W\phi(x,t)=\frac{\lambda}{\mu}\int_a^b K(x,y)\,\phi(y,t)\,dy+\frac{\lambda}{\mu}\int_0^t F(|t-\tau|)\,\gamma\big(\tau,x,\phi(x,\tau)\big)d\tau. \qquad (2.2)$$

Then, we assume the following conditions:

(i) The kernel of position $K(x,y) \epsilon C([a,b]\times[a,b])$ and satisfies:

$$|K(x,y)| \le c^* \quad , \quad (c^* \text{is a constant}) .$$

(ii) The kernel of time $F(|t-\tau|)$ is a discontinuous function which satisfies:

(a) For each continuous function $\gamma\big(t,x,\phi(x,t)\big)$ and $0 \le t_1 \le t_2 \le t$, the integrals

$$\int_{t_1}^{t_2} F(|t-\tau|)\gamma\big(\tau,x,\phi(x,\tau)\big)\, d\tau \qquad \text{and} \qquad \int_0^t F(|t-\tau|)\gamma\big(\tau,x,\phi(x,\tau)\big)\, d\tau \ ,$$

are continuous in $([a,b]\times[0,T])$.

(b) $F(|t-\tau|)$ is absolutely integrable with respect to τ for all $0 \le t \le T$, thus there exists a constant M, such that:

$$\int_0^t |F(t-\tau)|\, d\tau \le M \qquad .$$

(iii) The given function $f(x,t)$ with its partial derivatives with respect to position and time belong to $C([a,b]\times[0,T])$, and its norm is defined by:

$$\|f(x,t)\|_{C([a,b]\times[0,T])} = \max_{x,t}|f(x,t)| \le L \ , \ (L \text{ is a constant}).$$

(iv) The known function $\gamma\big(t,x,\phi(x,t)\big)$ satisfies for the constants A_1, A_2 and $A \ge \max\{A_1, A_2\}$ the following conditions :

$(a)\ \max_{x,t}\big|\gamma\big(t,x,\phi(x,t)\big)\big| \le A_1\|\phi(x,t)\|_{C([a,b]\times[0,T])}$.

$(b)\ \big|\gamma\big(t,x,\phi_1(x,t)\big) - \gamma\big(t,x,\phi_2(x,t)\big)\big| \le A_2\,|\phi_1(x,t)-\phi_2(x,t)|$.

***Theorem 1*:** Equation (1.1) has a unique solution in the space$C([a,b]\times[0,T])$,under the condition:

$$|\mu| > |\lambda|(c^*(b-a)+MA) . \tag{2.3}$$

To prove this theorem , we must consider the following lemmas .

***Lemma 1* :** The integral operator $\overline{W}$ maps the space $C\ ([a,b]\times[0,T])$ into itself .

***Proof* :** In view of the two formulas (2.1) and (2.2) , and the conditions $(i)-(iv-a)$, we have:

$$\|\overline{W}\,\phi(x,t)\| \le \frac{L}{|\mu|} + \sigma\|\phi(x,t)\| \ , \qquad \left(\sigma = \frac{|\lambda|}{|\mu|}\,(c^*(b-a)+MA)\right). \tag{2.4}$$

The previous inequality (2.4) shows that , the operator $\overline{W}$ maps the ball S_ρ into itself , where:

$$\rho = \frac{L}{[|\mu| - |\lambda|\,(c^*(b-a)+MA)]} \quad . \tag{2.5}$$

Since $\rho > 0$ and $L > 0$, therefore we have $\sigma < 1$. Also, the inequality (2.4) involves the boundedness of the operator W , where:

$$\|W\,\phi(x,t)\| \le \sigma\,\|\phi(x,t)\| \ . \tag{2.6}$$

Moreover , the inequalities (2.4) and (2.6) define the boundedness of the operator $\overline{W}$.

***Lemma 2*:** The integral operator $\overline{W}$ is continuous and contraction in the Banach space $C([a,b]\times[0,T])$.

***Proof* :** For the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in $C([a,b]\times[0,T])$, the formulas (2.1) and (2.2) after using the conditions (i) , (ii) and $(iv-b)$, yield:

$$\|\overline{W}\,\phi_1(x,t) - \overline{W}\,\phi_2(x,t)\| \le \sigma\,\|\phi_1(x,t)-\phi_2(x,t)\| \ . \tag{2.7}$$

From inequality (2.7), we see that the operator $\overline{W}$ is continuous in the Banach space $C([a,b]\times[0,T])$. Moreover , $\overline{W}$ is a contraction operator under the condition $\sigma < 1$.

***Proof of Theorem 1* :** The lemmas (1) and (2) show that, the operator $\overline{W}$ of (2.1) is contractive in the Banach space $C([a,b]\times[0,T])$. So, from Banach space fixed point theorem, $\overline{W}$ has a unique fixed point which is ,of course, the unique solution of eq.(1.1).

3. System of Urysohn Integral Equations

In this section, a quadratic numerical method is used [7]-[9], in the mixed integral equation (1.1), to obtain a system of Urysohn integral equations. Therefore, we divide the interval $[a,b]$ into s subintervals, by means of the points: $a = x_0 < x_1 < \cdots < x_s = b$,where $x = x_p$, $y = x_q$, $p,q = 0,1,2,\dots,s$, then the Fredholm integral term of eq. (1.1) becomes:

$$\int_a^b K(x_p,y)\,\phi(y,t)dy = \sum_{q=0}^{s} u_q\, K_{p,q}\phi_q + O(\hbar^{p+1}),\ \ (\hbar \to 0, p > 0\,)\,, \tag{3.1}$$

where $\hbar = \max_{0\le q\le s} h_q$, $h_q = x_{q+1} - x_q$, h is the step size of integration and u_q are the weights, $u_q = h/2$ if $q = 0$, $q = s, u_q = h$ if $0 < q < s\,, O(\hbar^{p+1})$ in eq. (3.1) is the order of sum of errors of the numerical method after dividing the interval $[a,b]$, and the difference between the integration and summation, where the error is determined by:

$$R_s = \left| \int_a^b K(x,y)\,\phi(y,t)\,dy - \sum_{q=0}^{s} u_q\, K_{p,q}\phi_q(t) \right| . \tag{3.2}$$

Replacing (3.1) in (1.1) and neglecting $O(\hbar^{p+1})$, we get:

$$\mu_p\,\phi_p(t) - \lambda \int_0^t F(|t-\tau|)\gamma_p\left(\tau,\phi_p(\tau)\right)d\tau = g_p(t)\,,\ \ (p = 0,1,\dots,s). \tag{3.3}$$

Here, in (3.3) we used the following notations: $\phi(x_p,t) = \phi_p(t)$, $K(x_p,x_q) = K_{p,q}$, $f(x_p,t) = f_p(t), \gamma(t,x_q,\,\phi(x_q,t)) = \gamma_q(t,\phi_q(t))$, $\mu_p = \mu - \lambda u_p\,K_{p,p}$;

$$g_p(t) = f_p(t) + \lambda \sum_{q=0}^{p-1} u_q\,K_{p,q}\phi_q(t) + \lambda \sum_{q=p+1}^{s} u_q\,K_{p,q}\phi_q(t)\,. \tag{3.4}$$

The formula (3.3) represents a **SUIEs** of the second kind.

Remark 1**:** Let E be the set of all continuous functions $\Phi(t) = \{\phi_0(t)\,,\phi_1(t)\,,\phi_2(t)\,, \dots,\phi_p(t)\,,\dots\}$, where $\phi_p(t)\,\epsilon\,C[0,T]$ for all p, and define on E the norm:

$$\|\Phi(t)\|_E = \sup_p \max_{0\le t\le T}|\phi_p(t)| = \sup_p\|\phi_p(t)\|_{C[0,T]} \quad , \quad \forall p\,.$$

Then E is a Banach space.

Definition 1**:** The following relation determines the estimate local error:

$$R_s = \left| \int_a^b K(x,y)\,\phi(y,t)dy - \sum_{q=0}^{s} u_q\,K_{\rho,q}\phi_q(t) \right| \quad , \quad \lim_{s\to\infty} R_s = 0. \tag{3.5}$$

3.1.The existence of a Unique Solution of SUIEs

In order to guarantee the existence of a unique solution of the **SUIEs** (3.3) in the Banach space E, we write the **SUIEs** (3.3) in the integral operator form:

$$\bar{U}\phi_p(t) = \frac{1}{\mu_p} g_p(t) + U\phi_p(t) \quad ; \ \mu_p \ne 0, \forall p\ , \tag{3.6}$$

where

$$U\phi_p(t) = \frac{\lambda}{\mu_p}\int_0^t F(|t-\tau|)\,\gamma_p\left(\tau,\phi_p(\tau)\right)d\tau\,. \tag{3.7}$$

Then we assume in addition to condition (ii) of theorem (1), the following conditions:

(1) $\sup_p \max_{0\le t\le T}|f_p(t)| = \|f(t)\|_E \le L^*$, ($L^*$is a constant) .

(2) $\sum_{q=0}^{s} \sup_q \left| u_q \, K_{\rho,q} \right| \leq \beta^*$, ($\beta^*$is a constant) .

(3) The known continuous functions $\gamma_p\left(t\,,\ \phi_p(t)\right), \forall p$ satisfy for the constants A_1^*, A_2^* and $A^* \geq max\{\ A_1^*, A_2^*\}$ the following conditions:

(a_1) $\sup_p \max_{0\leq t\leq T} \left|\gamma_p\left(t, \phi_p(t)\right)\right| \leq A_1^* \|\Phi(t)\|_E$.

(b_1) $\left|\ \gamma_p\left(t, \phi_p^{(1)}(t)\right) - \gamma_p\left(t, \phi_p^{(2)}(t)\right)\right| \leq A_2^* \left|\ \phi_p^{(1)}(t) - \phi_p^{(2)}(t)\right|$.

***Theorem 2*:** The **SUIEs** (3.3) has a unique solution in the space E under the following condition:

$$\delta = \left|\frac{\lambda}{\mu^*}\right| (\ \beta^* + MA^*) < 1\,,\quad \left(\ \mu^* = \min_{0\leq p\leq s} \mu_p\ \right).$$

***Proof* :** In view of the two formulas (3.6) and (3.7) ,we have:

$$\left|\bar{U}\phi_p(t)\right| \leq \frac{1}{|\mu_p|}\ \left|g_p(t)\right| + \frac{|\lambda|}{|\mu_p|}\int_a^b |F(|t-\tau|)|\ \left|\gamma_p\left(\tau, \phi_p(\tau)\right)\right|\ d\tau\ .$$

By using the formula (3.4) , the conditions (1) - ((3-a_1) and finally the condition (ii) , the above inequality takes the form:

$$\left\|\bar{U}\phi_p(t)\right\| \leq \frac{L^*}{|\mu^*|} + \delta\left\|\phi_p(t)\right\|\ ,\qquad \left(\delta = \left|\frac{\lambda}{\mu^*}\right| (\ \beta^* + MA^*)\,,\qquad \mu^* = \min_{0\leq p\leq s} \mu_p\ \right). \tag{3.8}$$

The previous inequality (3.8) shows that , the operator $\bar{U}$ maps the ball S_η into itself , where:

$$\eta = \frac{L^*}{[|\mu^*| - |\lambda|\ (\ \beta^* + MA^*)]}\ . \tag{3.9}$$

Since $\eta > 0$ and $L^* > 0$, therefore we have $\delta < 1$. Also, the inequality (3.8) involves the boundedness of the operator U , where:

$$\left\|U\phi_p(t)\ \right\| \leq\ \delta\ \left\|\phi_p(t)\right\|\ . \tag{3.10}$$

Moreover , the inequalities (3.8) and (3.10) define the boundedness of the operator $\bar{U}$.

For the two functions $\phi_p^{(1)}(t)$ and $\phi_p^{(2)}(t)$ in E, the formulas (3.6) and (3.7), after using the conditions (ii) and $(3-b_1)$, yield:

$$\left\|\bar{U}\ \phi_p^{(1)}(t) - \bar{U}\ \phi_p^{(2)}(t)\right\| \leq \delta\ \left\|\ \phi_p^{(1)}(t) - \ \phi_p^{(2)}(t)\right\|\ . \tag{3.11}$$

In view of inequality (3.11), we see that the operator $\bar{U}$ is continuous in the Banach space . Moreover, $\bar{U}$ is a contraction operator under the condition $\delta < 1$.

From Banach space fixed point theorem, $\bar{U}$ has a unique fixed point which is of course, the unique solution of **SUIEs** (3.3) .

4. The Modified Toeplitz Matrix Method (MTMM)

Here, we present the **MTMM** to obtain the numerical solution of a **UIE** of the second kind with singular kernel. Therefore, we assume the **UIE**:

$$\mu\,\phi(t) = g(t) + \lambda\int_0^t F(|t-\tau|)\gamma\big(\tau, \phi(\tau)\big)d\tau\ . \tag{4.1}$$

Following the same way of Abdou et al., [10-13], we can apply the **MTMM** for Urysohn term to obtain the following equation:

$$\mu\,\phi(t) - \lambda\sum_{n=0}^{N} D_n(t)\ \gamma\big(n\,h^*, \phi(n\,h^*)\big) = g(t)\,. \tag{4.2}$$

Putting $t = m\,h^*,\ h^* = T/N$, in (4.2) and using the following notations : $\phi(m\,h^*) = \phi_m$, $D_n(m\,h^*) = D_{mn}$, $g(m\,h^*) = g_m$,$\gamma\left(m\,h^*, \phi(m\,h^*)\right) = \gamma_m(\phi_m)$, we get the following **NAS:**

$$\mu\,\phi_m - \lambda\sum_{n=0}^{N} D_{mn}\gamma_n(\,\phi_n) = g_m\quad,\quad 0\leq n\leq m\leq N\quad, \tag{4.3}$$

where:

$$D_{mn} = \begin{cases} A_0(m\,h^*) & , \quad n = 0 \\ A_n(m\,h^*) + B_{n-1}(m\,h^*) & , \quad 0 < n < N \\ B_{N-1}(m\,h^*) & , \quad n = N\ . \end{cases}$$

$$A_n(t) = \frac{\gamma(n\,h^* + h^*, n\,h^* + h^*)\,I(t) - \gamma(n\,h^* + h^*, 1)\,J(t)}{h_1}\ ,$$

$$B_n(t) = \frac{\gamma(n\,h^*, 1)\,J(t) - \gamma(n\,h^*, n\,h^*)\,I(t)}{h_1}\ , \qquad (4.4)$$

$$h_1 = \gamma(n\,h^*, 1)\,\gamma(n\,h^* + h^*, n\,h^* + h^*) - \gamma(n\,h^*, n\,h^*)\,\gamma(n\,h^* + h^*, 1)\ ,$$

and:

$$I(t) = \int_{n\,h^*}^{n\,h^*+h^*} F(|t-\tau|)\gamma(\tau,1)d\tau, \qquad J(t) = \int_{nh^*}^{nh^*+h^*} F(|t-\tau|)\gamma(\tau,\tau)\,d\tau\ .$$

The matrix D_{mn} can be written in the Toeplitz matrix form : $D_{mn} = G_{mn} - E_{mn}$, here, the matrix $G_{mn} = A_n(m\,h^*) + B_{n-1}(m\,h^*),\ 0 \le n \le m \le N,$ is called the Toeplitz matrix of order (N+1) and $E_{mn} = B_{-1}(m\,h^*)$ if $n = 0$, $E_{mn} = 0$ if $0 < n < N$ and $E_{mn} = A_N(m\,h^*)$ if $n = N$, represents a matrix of order $(N+1)$ whose elements are zeros except the first and the last rows (columns).

***Definition 2*:** The **TMM** is said to be convergent of order r, if and only if for sufficiently large N, there exists a constant $d > 0$independent on N such that:

$$\|\,\phi(t) - \phi_N(t)\| \le dN^{-r}, \quad t \in [0,T], T < 1. \qquad (4.5)$$

***Definition 3*:** The estimate local error R_N takes the form:

$$R_N = \left| \int_0^t F(|t-\tau|)\,\gamma\big(\tau,\phi(\tau)\big)d\tau - \sum_{n=0}^{N} D_{mn}\,\gamma_n(\phi_n) \right| \ . \qquad (4.6)$$

***Lemma 3*:** If the kernel $F(|t-\tau|)$ of eq. (1.1) satisfies condition (ii) of theorem (1) and the following condition:

$$\lim_{t'\to t} \int_0^t |F(|t'-\tau|) - F(|t-\tau|)|d\tau = 0 \quad ; \ t', t \in [0,T]\ , \qquad (4.7)$$

then $\sup_N \sum_{n=0}^{N} |D_{mn}|$ exists , and $\lim_{m'\to m} \sup_N \sum_{n=0}^{N} |D_{m'n} - D_{mn}| = 0$.

***Proof*:** From formula (4.4), we have:

$$|A_n(t)| \le \frac{1}{|h_1|}\left[|\gamma(n\,h^* + h^*, n\,h^* + h^*)| \int_{n\,h^*}^{n\,h^*+h^*} |F(|t-\tau|)||\gamma(\tau,1)|\,d\tau \right.$$
$$\left. + |\gamma(n\,h^* + h^*, 1)| \int_{n\,h^*}^{n\,h^*+h^*} |F(|t-\tau|)||\gamma(\tau,\tau)|\,d\tau \right] .$$

Summing from $n = 0$ to $n = N$, then taking in account the continuity of the function $\gamma\big(t,\phi(t)\big)$in the interval $[0,T]$and finally using the condition $(ii-b)$, there exists a small constant E_1, such that

$$\sum_{n=0}^{N} |A_n(t)| \le E_1\ , \quad \forall\, N, \qquad \left(E_1 = \frac{2\,\mathcal{L}\,M}{|h_1|}\ , \big|\gamma\big(t,\phi(t)\big)\big| \le \mathcal{L}\right).$$

Since, each term of $\sum_{n=0}^{N} |A_n(t)|$ is bounded above, hence for $t = mh^*$, we have:

$$\sup_N \sum_{n=0}^{N} |A_n(m\,h^*)| \leq E_1 \quad . \tag{4.8}$$

Similarly, we get:

$$\sup_N \sum_{n=0}^{N} |B_n(m\,h^*)| \leq E_1 \ . \tag{4.9}$$

In the light of (4.4), and the help of (4.8) and (4.9), there exists a small constant E_2 , such that:

$$\sup_N \sum_{n=0}^{N} |D_{mn}| \leq \sup_N \sum_{n=0}^{N} |A_n(m\,h^*)| + \sup_N \sum_{n=0}^{N} |B_n(m\,h^*)| \leq E_2 \ , (E_2 = 2\,E_1).$$

Hence , $\sup_N \sum_{n=0}^{N} |D_{mn}|$ exists .

Also, for $t, t' \in [0, T]$, we get:

$$|A_n(t') - A_n(t)| \leq \frac{1}{|h_1|}\left[|\gamma(n\,h^* + h^*, n\,h^* + h^*)| \int_{n\,h^*}^{n\,h^*+h^*} |F(|t' - \tau|) - F(|t - \tau|)||\gamma(\tau, 1)|\,d\tau \right.$$

$$\left. + |\gamma(n\,h^* + h^*, 1)| \int_{n\,h^*}^{n\,h^*+h^*} |F(t', \tau) - F(t, \tau)||\gamma(\tau, \tau)|\,d\tau \right],$$

summing from $n = 0$ to $n = N$, taking in account the continuity of the function γ, then putting $t = m\,h^*$, $t' = m'h^*$, and using the condition (4.7), we get:

$$\lim_{m' \to m} \sup_N \sum_{n=0}^{N} |A_n(m'h^*) - A_n(m\,h^*)| = 0 \, .$$

Similarly, we can prove that $\lim_{m' \to m} \sup_N \sum_{n=0}^{N} |B_n(m'h^*) - B_n(m\,h^*)| = 0$.

Finally, we have:

$$\lim_{m' \to m} \sup_N \sum_{n=0}^{N} |D_{m'n} - D_{mn}| = 0 \, .$$

5. The Existence of a Unique Solution of NAS

The **SUIEs** (3.3) after using **MTMM** takes the form

$$\mu_p \phi_{p,m} = g_{p,m} + \lambda \sum_{n=0}^{N} D_{mn}^{[p]}\, \gamma_{p,n}(\phi_{p,n}) \, , \tag{5.1}$$

where:

$$g_{p,m} = f_{p,m} + \lambda \sum_{q=0}^{p-1} u_q\, K_{p,q}\ \phi_{q,m} + \lambda \sum_{q=p+1}^{s} u_q\, K_{p,q}\ \phi_{q,m} \ .$$

To guarantee the existence of a unique solution of the **NAS** (5.1) in the Banach space l^∞, we write the **NAS** (5.1) in the integral operator form:

$$\bar{V}\phi_{p,m} = \frac{1}{\mu_p} g_{p,m} + V\phi_{p,m} \quad ; \ \mu_p \neq 0, \forall p \ , \tag{5.2}$$

where:

$$V\phi_{p,m} = \frac{\lambda}{\mu_p} \sum_{n=0}^{N} D_{mn}^{[p]}\, \gamma_{p,n}(\phi_{p,n}) \ , \tag{5.3}$$

we assume the following conditions:

$(\grave{1})\ \sup_{p,m} |g_{p,m}| \le H$, (H is a constant) .

$(\grave{2})\ \sup_{p,N} \sum_{n=o}^{N} \left|D_{mn}^{[p]}\right| \le E^*$, (E^* *is* a constant) .

$(\grave{3})$ The known continuous functions $\gamma_{p,n}(\phi_{p,n})$ satisfy $(\forall\ p\ , n)$for the constants $Q_1\,, Q_2$ and $Q \ge max\{Q_1, Q_2\}$ the following conditions:

$(\grave{a})\sup_{p,n} |\gamma_{p,n}(\phi_{p,n})| \le Q_1 \|\phi_{p,n}\|_{l^\infty}$

$(\grave{b})\ \left|\gamma_{p,n}\left(\phi_{p,n}^{(1)}\right) - \gamma_{p,n}\left(\phi_{p,n}^{(2)}\right)\right| \le Q_2 \left|\phi_{p,n}^{(1)} - \phi_{p,n}^{(2)}\right|$.

Theorem 3: The formula (5.1) has a unique solution in the space l^∞ under the following condition:

$$\alpha = \left|\frac{\lambda}{\mu^*}\right| Q\, E^* < 1\,, \quad \left(\mu^* = \min_{0\le p\le s} \mu_p\right).$$

Proof : From the formulas (5.2) and (5.3) ,we obtain

$$|\bar{V}\phi_{p,m}| \le \frac{1}{|\mu_p|}\, |g_{p,m}| + \frac{|\lambda|}{|\mu_p|} \sum_{n=0}^{N} \left|D_{mn}^{[p]}\right| |\, \gamma_{p,n}(\phi_{p,n})| \ .$$

In view of the conditions $(\grave{1})$ - ($(\grave{3}$-$\grave{a})$, the above inequality takes the form:

$$\|\bar{V}\phi_{p,m}\| \le \frac{H}{|\mu^*|} + \alpha\|\phi_{p,m}\| \ , \quad \left(\alpha = \left|\frac{\lambda}{\mu^*}\right| (\,Q\,E^*)\,, \quad \mu^* = \min_{0\le p\le s} \mu_p\right). \tag{5.4}$$

The previous inequality (5.4) shows that , the operator $\bar{V}$ maps the ball S_ξ into itself , where:

$$\xi = \frac{H}{[|\mu^*| - |\lambda|\,(Q\,E^*)]} \ . \tag{5.5}$$

Since $\xi > 0$ and $H > 0$, therefore we have $\alpha < 1$. Also, the inequality (5.4) involves the boundedness of the operator V , where:

$$\|V\phi_{p,m}\| \le \alpha\, \|\phi_{p,m}\| \ . \tag{5.6}$$

Besides, the inequalities (5.4) and (5.6) define the boundedness of the operator $\bar{V}$.

For the two functions $\phi_{p,m}^{(1)}$ and $\phi_{p,m}^{(2)}$ in l^∞, the formulas (5.2) and (5.3) yield:

$$\left|\bar{V}\ \phi_{p,m}^{(1)} - \bar{V}\ \phi_{p,m}^{(2)}\right| \le \frac{|\lambda|}{|\mu_p|} \sum_{n=0}^{N} \left|D_{mn}^{[p]}\right| \left|\, \gamma_{p,n}\left(\phi_{p,n}^{(1)}\right) - \gamma_{p,n}\left(\phi_{p,n}^{(2)}\right)\right| \ .$$

Using the conditions $(\grave{2})$ and $(\grave{3} - \grave{b})$, the above inequality takes the form:

$$\left\|\bar{V}\ \phi_{p,m}^{(1)} - \bar{V}\ \phi_{p,m}^{(2)}\right\| \le \alpha \left\|\phi_{p,m}^{(1)} - \phi_{p,m}^{(2)}\right\| \ . \tag{5.7}$$

In view of inequality (5.7) , we see that the operator $\bar{V}$ is continuous in the Banach space l^∞ . Moreover, $\bar{V}$ is a contraction operator under the condition $\delta < 1$.

From Banach space fixed point theorem , $\bar{V}$ has a unique fixed point which is of course, the unique solution of **NAS** (5.1) .

***Definition 4*:** The following relation determines the estimate total error R_j:

$$R_j = \left|\int_a^b K(x,y)\,\phi(y,t)dy + \int_0^t F(|t-\tau|)\,\gamma\big(\tau, x, \phi(x,\tau)\big)d\tau - \sum_{q=0}^{s} u_q K_{p,q}\,\phi_{q,m} - \sum_{n=0}^{N} D_{mn}^{[p]}\,\gamma_{p,n}(\phi_{p,n})\right|,$$

when $j = max\{N, s\} \to \infty$, the sums:

$$\left\{\sum_{q=0}^{s} u_q\, K_{p,q}\, \phi_{q,m} + \sum_{n=0}^{N} D_{mn}^{[p]}\, \gamma_{p,n}(\phi_{p,n})\right\} \rightarrow \left\{\int_a^b K(x,y)\, \phi(y,t)dy + \int_0^t F(|t-\tau|)\, \gamma(\tau, x, \phi(x,\tau))d\tau\right\},$$

and the solution of the **NAS** (5.1) becomes the solution of Eq. (1.1) .

***Theorem 4*:** If the sequence of continuous functions $\{ f_j(x,t)\}$ converges uniformly to the function $f(x,t)$ in the space $C([a,b]\times[0,T])$, then under the conditions of theorem (1), the sequence $\{\phi_j(x,t)\}$ converges uniformly to the exact solution of Eq. (1.1) in $C([a,b]\times[0,T])$.

***Proof*:** The formula (1.1) with its approximate solution gives:

$$\left|\phi(x,t)-\phi_j(x,t)\right| \le \frac{1}{|\mu|}\left|f(x,t)-f_j(x,t)\right| + \left|\frac{\lambda}{\mu}\right|\int_a^b |K(x,y)|\left|\phi(y,t)-\phi_j(y,t)\right|dy$$
$$+\left|\frac{\lambda}{\mu}\right|\int_0^t |F(|t-\tau|)|\left|\gamma(\tau,x,\phi(x,\tau))-\gamma\left(\tau,x,\phi_j(x,\tau)\right)\right|d\tau\,.$$

In view of the conditions of theorem 1, we get:

$$\left\|\phi(x,t)-\phi_j(x,t)\right\| \le \frac{1}{[|\mu|-|\lambda|(Bc^*+MA)]}\left\|f(x,t)-f_j(x,t)\right\|\ .$$

Hence, $\left\|\phi(x,t)-\phi_j(x,t)\right\| \to 0$ since $\left\|f(x,t)-f_j(x,t)\right\| \to 0$ as $j\to\infty$.

***Corollary1*:** The total error R_j satisfies $\lim_{j\to\infty} R_j = 0$.

6. Applications

In this section, we will consider the **F-UIE:**

$$\mu\,\phi(x,t) = f(x,t) + \lambda\int_a^b xy\,\phi(y,t)\,dy + \lambda\int_0^t F(|t-\tau|)\,\phi^2(x,\tau)\,d\tau\ . \qquad (6.1)$$

The results are obtained numerically by Maple 18 software, for $x\in[-1,1]$, with $\lambda = 0.16279$, 0.3158 and 0.0684, and the parameter $\mu = 1$. The time interval $[0,T]$ is divided to 40 intervals, where the exact solution is $\phi(x,t) = x^2\, t^2$.

Application 1: In eq. (6.1), the Urysohn kernel takes the logarithmic form $F(|t-\tau|) = \ln|t-\tau|$.

Table (1) , T = 0.006

X	EXACT	$\lambda = 0.0684$		$\lambda = 0.16279$		$\lambda = 0.3158$	
		APP	ERR	APP	ERR	APP	ERR
-1	3.600E-05	3.587E-05	1.302E-07	3.569E-05	3.106E-07	3.540E-05	6.048E-07
-0.6	1.296E-05	1.289E-05	7.434E-08	1.278E-05	1.771E-07	1.262E-05	3.440E-07
-0.2	1.440E-06	1.421E-06	1.936E-08	1.394E-06	4.608E-08	1.351E-06	8.941E-08
0.2	1.440E-06	1.459E-06	1.895E-08	1.485E-06	4.509E-08	1.527E-06	8.749E-08
0.6	1.296E-05	1.300E-05	4.064E-08	1.306E-05	9.680E-08	1.315E-05	1.880E-07
1	3.600E-05	3.597E-05	3.426E-08	3.592E-05	8.173E-08	3.584E-05	1.592E-07

Table (2) , T = 0.04

X	EXACT	$\lambda = 0.0684$		$\lambda = 0.16279$		$\lambda = 0.3158$	
		APP	ERR	APP	ERR	APP	ERR
-1	1.600E-03	1.602E-03	1.862E-06	1.604E-03	4.441E-06	1.609E-03	8.647E-06
-0.6	5.760E-04	5.760E-04	1.451E-08	5.760E-04	3.456E-08	5.759E-04	6.713E-08

-0.2	6.400E-05	6.392E-05	8.215E-08	6.380E-05	1.955E-07	6.362E-05	3.794E-07
0.2	6.400E-05	6.409E-05	8.810E-08	6.421E-05	2.097E-07	6.441E-05	4.068E-07
0.6	5.760E-04	5.765E-04	4.965E-07	5.772E-04	1.183E-06	5.783E-04	2.297E-06
1	1.600E-03	1.602E-03	2.288E-06	1.605E-03	5.458E-06	1.611E-03	1.063E-05

Table (3) , T=0.2

X	**EXACT**	$\boldsymbol{\lambda = 0.0684}$		$\boldsymbol{\lambda = 0.16279}$		$\boldsymbol{\lambda = 0.3158}$	
		APP	ERR	APP	ERR	APP	ERR
-1	4.000E-02	4.001E-02	1.224E-05	4.003E-02	2.906E-05	4.006E-02	5.615E-05
-0.6	1.440E-02	1.440E-02	9.506E-07	1.440E-02	2.260E-06	1.440E-02	4.378E-06
-0.2	1.600E-03	1.600E-03	1.932E-07	1.600E-03	4.597E-07	1.599E-03	8.916E-07
0.2	1.600E-03	1.600E-03	2.324E-07	1.601E-03	5.531E-07	1.601E-03	1.073E-06
0.6	1.440E-02	1.440E-02	2.227E-06	1.441E-02	5.294E-06	1.441E-02	1.026E-05
1	4.000E-02	4.001E-02	1.331E-05	4.003E-02	3.159E-05	4.006E-02	6.102E-05

Application 2: In eq. (6.1), we assume that the Urysohn kernel takes the Carleman function $F(|t-\tau|) = |t-\tau|^{-\upsilon},\ 0 < \upsilon < 1/2$, where υ is called Poisson ratio.

Table (4) , T=0.006

X	**EXACT**	$\boldsymbol{\lambda = 0.0684}$		$\boldsymbol{\lambda = 0.16279}$		$\boldsymbol{\lambda = 0.3158}$	
		APP	ERR	APP	ERR	APP	ERR
-1	3.600E-05	3.594E-05	6.166E-08	3.585E-05	1.471E-07	3.571E-05	2.865E-07
-0.6	1.296E-05	1.292E-05	4.494E-08	1.285E-05	1.071E-07	1.275E-05	2.080E-07
-0.2	1.440E-06	1.428E-06	1.241E-08	1.410E-56	1.440E-06	1.383E-06	5.732E-08
0.2	1.440E-06	1.452E-06	1.221E-08	1.469E-06	2.907E-08	1.496E-06	5.641E-08
0.6	1.296E-05	1.299E-05	2.898E-08	1.303E-05	6.902E-08	1.309E-05	1.341E-07
1	3.600E-05	3.601E-05	1.370E-08	3.609E-05	9.181E-07	3.607E-05	7.218E-07

Table (5) , T=0.04

X	**EXACT**	$\boldsymbol{\lambda = 0.0684}$		$\boldsymbol{\lambda = 0.16279}$		$\boldsymbol{\lambda = 0.3158}$	
		APP	ERR	APP	ERR	APP	ERR
-1	1.600E-03	1.602E-03	2.192E-06	1.605E-03	5.229E-06	1.610E-03	1.018E-05
-0.6	5.760E-04	5.761E-04	1.195E-07	5.763E-04	2.847E-07	5.766E-04	5.529E-07
-0.2	6.400E-05	6.395E-05	5.122E-08	6.388E-05	1.219E-07	6.376E-05	2.366E-07
0.2	6.400E-05	6.406E-05	5.823E-08	6.414E-05	1.386E-07	6.427E-05	2.689E-07
0.6	5.760E-04	5.764E-04	4.481E-07	5.771E-04	1.067E-06	5.781E-04	2.073E-06
1	1.600E-03	1.602E-03	2.466E-06	1.606E-03	5.883E-06	1.611E-03	1.146E-05

Table (6) , T=0.2

X	**EXACT**	$\boldsymbol{\lambda = 0.0684}$		$\boldsymbol{\lambda = 0.16279}$		$\boldsymbol{\lambda = 0.3158}$	
		APP	ERR	APP	ERR	APP	ERR
-1	4.000E-02	4.007E-02	6.604E-05	4.016E-02	1.569E-04	4.030E-02	3.032E-04
-0.6	1.440E-02	1.441E-02	8.138E-06	1.442E-02	1.929E-05	1.444E-02	3.716E-05
-0.2	1.600E-03	1.600E-03	3.133E-08	1.600E-03	7.576E-08	1.600E-03	1.510E-07
0.2	1.600E-03	1.600E-03	2.423E-07	1.601E-03	5.755E-07	1.601E-03	1.113E-06
0.6	1.440E-02	1.441E-02	8.960E-06	1.442E-02	2.125E-05	1.444E-02	4.097E-05
1	4.000E-02	4.007E-02	6.673E-05	4.016E-02	1.585E-04	4.031E-02	3.064E-04

From the numerical results, we notice that:

1) The error values increasing at the end points of the interval $[-1,1]$.
2) The maximum value of the error for the Carleman case is 3.064E-04 , at $\lambda = 0.3158$, $T = 0.2$ and $x = 1$.
3) The maximum value of the error for the logarithmic case is 6.102E-05, at $\lambda = 0.3158$, $T = 0.2$ and $x = 1$.
4) For fixed values of λ , the error values are increasing with the increase of time , since the increasing of time causes more deformation of the materials.

5) The change in the values of the elasticity modules λ or in the values of the Poison ratio υ cause lightly effective in the numerical results.
6) In general, the error values of the logarithmic kernel are lower than the error values of the Carleman kernel.

7. Conclusions

By the current research, we investigated a mixed integral equation of the second kind with two integral terms; the first is a linear integral term of Fredholm considered in position with continuous kernel, while the second is a nonlinear integral term of Urysohn in time with singular kernel. The Banach fixed point theorem has been used to prove the existence and uniqueness solution of Fredholm-Urysohn integral equation of the second kind in the space $C([a,b]\times[0,T])$. A quadratic numerical method has been used to obtain a system of Urysohn integral equations of the second kind in time . Moreover, the modified Toepliyz matrix method, as a best numerical method, has been used to obtain a nonlinear algebraic system. Many important theorems related the existence of a unique solution of the system of Urysohn integral equations, the nonlinear algebraic system and the estimate error are considered. Alo, the stability of the solution of Fredholm- Urysohn integral equation is proved.

Many different cases can be established when the mixed integral equation takes special forms . For example the mixed integral equation of the second kind in n- dimensional:

$$\mu\phi(x,t)=\lambda\int_{\Omega} K(x,y)\,\phi(y,t)\,dy+\lambda\int_0^t F(|t-\tau|)\gamma\big(\tau,x,\phi(x,\tau)\big)d\tau$$
$$+\lambda\int_{\Omega}\int_0^t K(x,y)\,F(|t-\tau|)\gamma\big(\tau,x,\phi(x,\tau)\big)d\tau\,dy\,,$$

where $x=\bar{x}(x_1,x_2,\dots,x_n)$, $y=\bar{y}(y_1,y_2,\dots,y_n)$, and the domain of integration Ω is a closed bounded set depends on the vector of position .

References

1. *Francel J. I.*, ‘A Galerkin solution to a regularized Cauchy singular integro differential equation’, *Quart . Appl. Math. L.* ***III (2) b*** *(1995), 245 – 258.*
2. *Abdou M. A.* , ‘Fredholm – Volterra integral equation of the first kind and contact problem’, *Appl. Math. Compute. ,* ***125*** *(2002), 177 -193.*
3. *Abdou M. A.*, ‘Fredholm – Volterra integral equation with singular kernel’, *Appl. Math. Compute.,* ***137*** *(2003), 231 – 243.*
4. *Abdou M. A., Mohamed K. I. and Ismail A.S.*, ‘On the numerical solutions of integral equation of mixed type’, *Appl. Math. Compute.,* ***138*** *(2003), 172–186.*
5. *Abdou M. A., Nasr A. A.*, ‘On the numerical treatment of the singular integral equation of the second kind, *J. Appl. Math. Compute.,* ***146*** *(2003), 373–380.*
6. *Abdou M. A.*, ‘On the solution of linear and nonlinear integral equation’, *Appl. Math. Compute.* ***146*** *(2003), 857 – 871.*
7. *Atkinson K.E., 'The Numerical Solution of Integral Equation of the Second Kind'*, Cambridge University, Cambridge, *1997.*
8. *Atkinson K.E., 'A Survey of Numerical Method for the Solution of Fredholm Integral Equation of the Second Kind'*, Philadelphia, *1976.*
9. *Delves L.M., Mohamed J.L., 'Computational Methods for Integral Equations*', Cambridge University, Cambridge, *1985.*
10. *Abdou M. A., Mohamed K. I. and Ismail A.S.*, ‘Toeplitz matrix and product Nyström methods for solving the singular integral equation’, *Le Matematica.* ***LVII*** *(2002), 21 – 37.*
11. *Abdou M. A., Raad S.A.*, ‘Fredholm–Volterra integral equations with logarithmic kernel’, *J. Appl. Math. Compute.* ***176*** *(2006), 215–224.*
12. *Abdou M. A., El-Borae M. M., El-Kojok M. M.*, ‘Toeplitz matrix method and nonlinear integral equation of Hammerstein type’, *j.CAM,* ***223*** *(2009),765-776.*
13. *Abdou M. A., El-Kojok M. M., Raad S.A.*, ‘On the solution of Fredholm-Volterra integral equation with discontinuous kernel in time’, *J. Adv. Math.* Vol. 9/4 *(2014), 2552-2562.*