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Calculation of Cylindrical Functions using Correction of Asymptotic Expansions

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Abstract
A new method – the method of correction of asymptotic expansions, as it is called, – is suggested and applied to cylindrical functions $J_\nu(x)$ and $Y_\nu(x)$ of a real order ν and a real argument x for constructing formulas which improve significantly the precision of the asymptotic expansions whereas the correction itself contains few terms. The method includes a specific interpolation with terms decreasing at infinity for an interval adjacent to the initial point $a > 0$ where the whole interval (a,∞) begins; for the remaining part of the interval, the corrected asymptotic expansion goes more and more closely to the usual asymptotic expansion. The obtained formulas include an explicit dependence on ν.

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Key Word and Phrases
Cylindrical functions**,** Taylor's series, Asymptotic expansions.

1. Introduction

The problem of calculation of cylindrical functions is addressed in large amount of publications. Many effective algorithms are known which are based on applying recurrent relationships and continues fractions [2,3,7,9]; approximations with Chebyshev polynomials [4] or rational approximations [8]. A description of various alternatives and the corresponding bibliography one can find in [1, 5, 6, 8, 10, 11]. Among the known methods, rather attractive remain the classical methods: application of Taylor's series for small values of arguments and asymptotic expansions for large argument's values. Overlapping of the corresponding domains could ensure a possibility to perform calculations for all values of arguments. It is important that considering these methods for cylindrical functions we obtain formulas with explicit dependence on the order ν of the functions (stipulating that values of ν are not large). Note that such a possibility does not exist in the case of the above mentioned algorithms which result in calculations performed for specific values of ν (most frequently for $\nu = 0, 1$). Unfortunately, the higher the required precision, the more difficult to ensure the overlapping, especially if we remain in the frame of the double precision arithmetic inherent in common programming languages. For example the Hankel asymptotic expansions lead to an error less than 10^{-15} only for argument values $x \geq 16$ (for order $\nu = 0$), however the application of the Taylor series leads to the error of 2×10^{-11} for $x = 16$ (using double precision arithmetic) because of the loss of accuracy. The suggested correction of asymptotic expansions significantly improves the corresponding precision which allows us to apply the corrected expansions beginning from sufficiently small values of argument. As a result achieving the overlapping of the domains of application Taylor series and corrected asymptotic expansions becomes simpler. Emphasize that contrary to previous publications, final formulas which are suggested in the present paper (being rather simple and competitive with known methods regarding the precision and computational work) include the explicit dependence on the order of cylindrical functions for some interval of ν variation which can be widened using well known recurrent relationships. Note that auxiliary high-precision computations needed for constructing our approximations and their testing can be performed using other known methods. In our study the package Wolfram Mathematica has been used.

2. Description of the Method

Let for a function $\Phi(x)$ of the real argument x an asymptotic expansion be known which delivers an approximation for $x > x_1 > 0$:

$$\Phi(x) \approx A_n(x) = \sum_{j=0}^{n} \frac{a_j}{x^j} \tag{2.1}$$

Here n denotes the maximal accounted power of x in the expansion. Generally the more the value x_1 is the more n can be taken for achieving minimal errors of approximation (2.1). Consider a deviation, $\varphi_n(x)$, of the functions $\Phi(x)$ from the corresponding asymptotic expansion. An effective approximation for $\varphi_n(x)$, and thus $\Phi(x)$, can be realized as follows. Let us take s points, $x_1 < x_2 < \ldots < x_s$. Introducing additional parameters, w and λ, we construct the following function $G_\varphi(x)$ which interpolates the deviation $\varphi_n(x)$ at the interval $[x_1, x_s]$ and extrapolates it for $x > x_s$:

$$\varphi_n(x) = \Phi(x) - A_n(x) \approx G_\varphi(x) = \sum_{j=1}^{s} \left(\frac{w + x_j / x_1}{w + x / x_1} \right)^{\lambda} L_j(x) \varphi_n(x_j) \tag{2.2}$$

Here $L_j(x)$ are polynomials of degree $s - 1$ known from Lagrange interpolation method, they are equal to 1 for $x = x_j$ ($j = 1,2,\ldots,s$) and to 0 for other interpolation points. Calculations show that the optimum value of λ should be an integer greater than $n + s - 1$; this leads to a suitable rate of decreasing of the function (2.2) for $x > x_s$ (faster than the last term in (2.1) decreases). It is advisable to choose points $x_2, x_3, \ldots, x_s$ using the geometric progression,

$$x_j = x_{j-1} + q^{j-2} h \quad (j = 2,\ldots,s) \tag{2.3}$$

where $h > 0$ is an initial step, and $q > 1$; actually, good results are achieved also in the case of a constant step ($q = 1$). The parameters h, q, λ and w should minimize, wherever possible, the maximum of absolute values of $\delta_\varphi = \varphi_n(x) - G_\varphi(x)$ at the interval $x \ge x_1$, consequently we obtain rather good approximation, $A_n(x) + G_\varphi(x)$, for the function $\Phi(x)$. As is seen from (2.2), the auxiliary values $\varphi_n(x_j)$ ($j = 1,\ldots,s$) are needed for constructing the approximation. These values as well the values of the parameters close to optimal can be found employing high precision calculations. As an illustration consider the gamma function $\Gamma(x)$. According to Stirling's formula for $\ln(\Gamma(x))$:

$$\Phi(x) = \ln(\Gamma(x)) + x - (x - 1/2)\ln(x) - \ln\left(\sqrt{2\pi}\right) \approx A_{2m-1}(x) \tag{2.4}$$

$$A_{2m-1}(x) = \sum_{k=1}^{m} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \qquad (m = 1,2,\ldots.) \tag{2.5}$$

where B_{2k} are the Bernoulli numbers. Rather simple approximation we obtain taking $x_1 = 3$, $n = 5$ ($m = 3$ in (2.4)), $s = 2$, $\lambda = 8$. The value of λ equals the power of x in the next (seventh) term of the asymptotic expansion plus $s - 1$ (degree of Lagrange polynomials). Performing a number of calculations for different values h and w we come to the following nearly optimal values $h = 0.244$, $w = 0.093644$. Using these parameters and two "exact" values of $\varphi_n(x_1) = -2.37486992 \times 10^{-7}$ and $\varphi_n(x_2) = -1.398285024 \times 10^{-7}$ in (2.2) the following representation for the correction $G_\varphi(x)$ can be written:

$$G_\varphi(x) = -\left(\frac{w+1}{w + x/3} \right)^{\lambda} \frac{x - x_2}{h} \varphi_n(x_1) + \left(\frac{w + x_2/3}{w + x/3} \right)^{\lambda} \frac{x - x_1}{h} \varphi_n(x_2) \tag{2.6}$$

Or

$$G_\varphi(x) = -\frac{10^{-7}}{(w + x/3)^8} (2.1613872 + 0.89957994x) \tag{2.7}$$

Applying the well known recurrent relationship for gamma function we obtain the following approximation in the right complex half plane $\mathrm{Re}(z) \ge 0$:

$$\ln(\Gamma(z)) \approx \frac{1}{12u} - \frac{1}{360u^3} + \frac{1}{12u^5} + G_\varphi(u) - u + (u - 1/2)\ln(u) + \ln \frac{\sqrt{2\pi}}{z(z+1)(z+2)} \tag{2.8}$$

where $u = z + 3$. The three terms Stirling's expansion is modified by including the correction $G_\varphi(u)$ and the denominator in the last term in (2.8) which accounts on the recurrent relationship for going till $x = \mathrm{Re}(z) = 0$. At the real axis the errors of the approximation (2.8), (2.7) are less than 1.57×10^{-13}, which is illustrated by Fig. 1a where error δ of approximation (2.8) is shown as a function of x.

Without the correction $G_\varphi(u)$ in (2.8) the error achieves 2.4×10^{-7}; the so successful improvement of the approximation by as low as two term correction is explained by a suitable form of equation (2.2): the chosen parameters in the equation have resulted actually in two additional interpolation points (see Fig 1a). In the whole right complex half-plane the errors are less than 4.75×10^{-11} (the maximum error is achieved at the imaginary axis); this is illustrated in Fig. 1b where $|\delta|$ as functions of y is shown for $z = \xi + iy$ ($\xi = 0, 0.1, 0.2,\ldots 2$). Note that smallness of the function of $G_\varphi(u)$ allows us to consider only 8 significant digits in (2.7) while ensuring the much higher final precision in (2.8). Remember that the error for $\ln(\Gamma(z))$ is very close to the relative error for the gamma function itself.

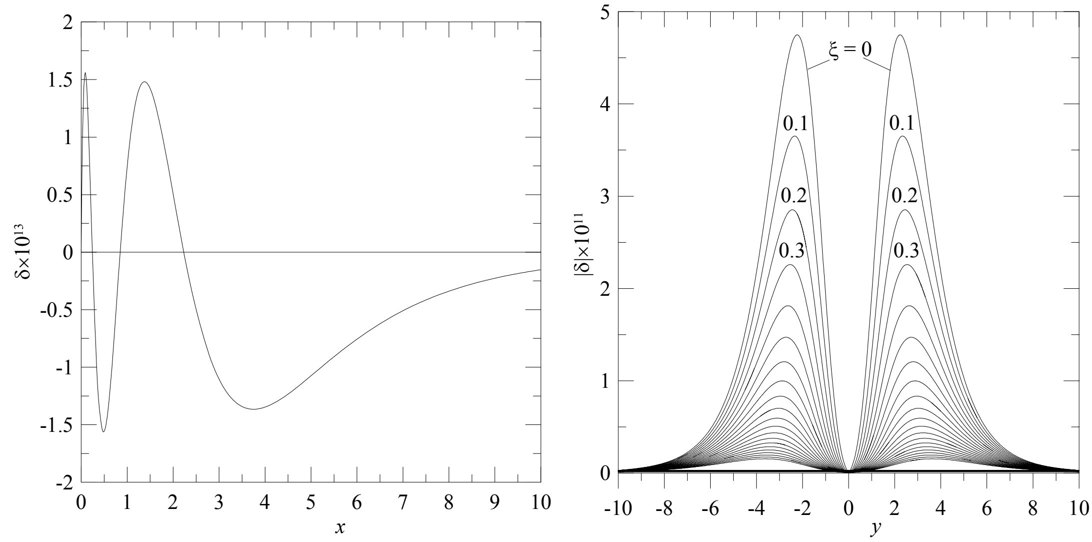

Fig. 1 (a,b) Errors of approximation for $\ln(\Gamma(x))$ for $s = 2$, $x_1 = 3$ at real axis (a) and in right complex half-plane (b): $z = \xi + iy$ ($\xi = 0, 0.1, 0.2,\ldots 2$).

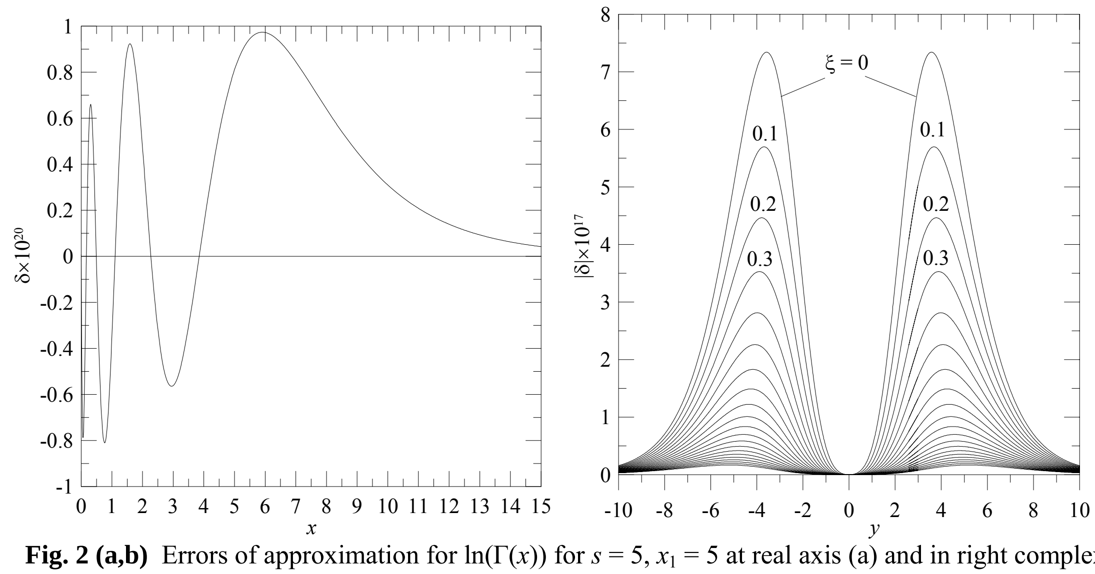

Fig. 2 (a,b) Errors of approximation for $\ln(\Gamma(x))$ for $s = 5$, $x_1 = 5$ at real axis (a) and in right complex half-plane (b): $z = \xi + iy$ ($\xi = 0, 0.1, 0.2,\ldots 2$).

We present also a high precision approximation for the gamma function taking $x_1 = 5$, $s = 5$, $n = 9$, $\lambda = 15$. Nearly optimal values of remaining parameters are: $w = 0.326$, $h = 0.17$, $q = 1.9$. Using equation (2.2) with these parameters and values $\varphi_n(x_j)$ ($j = 1,\ldots,s$) we obtain after transformations

$$G_\varphi(x) = \frac{10^{-13}}{(w+x/5)^{15}}(-6.3589538855 + 2.5880577262x - 5.1677775893x^2 - 3.0568754731x^3 - 0.62873305545x^4) \tag{2.9}$$

The approximation for ln(Γ(z)) in the right complex half plane Re(z) ≥ 0 will be ($u = z + 5$):

$$\ln(\Gamma(z)) \approx \frac{1}{12u} - \frac{1}{360u^3} + \frac{1}{12u^5} - \frac{1}{1680u^7} + \frac{1}{1188u^9} + G_\varphi(u) - u + (u-1/2)\ln(u) + \ln\frac{\sqrt{2\pi}}{z(z+1)(z+2)(z+3)(z+4)} \tag{2.10}$$

Now errors at the real axis are less than 10^{-20} (see Fig. 2a) whereas without the correction $G_\varphi(u)$ in (2.10) the error achieves 3.5×10^{-11}. In the whole right complex half-plane errors are less than 8×10^{-17} (see Fig . 2b where $z = \xi + iy$, ξ = 0, 0.1, 0.2,…2). Approximations (2.8), (2.10) have an advantage in simplicity and precision over known corresponding approximations; in addition, one can merely drop the correction for large enough values of $|z|$ going to the pure Stirling approximation.

3 Correction of Asymptotic Expansions for Bessel, Neumann and Hankel Functions

As a basis for our considerations we use the following representation [1] of cylindrical functions $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, $J_\nu(z)$, $Y_\nu(z)$ of order ν

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\left(P(\nu,z) + \mathrm{i}\,Q(\nu,z)\right)\mathrm{e}^{\mathrm{i}\chi} \tag{3.1a}$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\left(P(\nu,z) - \mathrm{i}\,Q(\nu,z)\right)\mathrm{e}^{-\mathrm{i}\chi} \tag{3.1b}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left(P(\nu,z)\cos(\chi) - Q(\nu,z)\sin(\chi)\right) \tag{3.1c}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\left(P(\nu,z)\sin(\chi) + Q(\nu,z)\cos(\chi)\right) \tag{3.1d}$$

where

$$\chi = z - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi \tag{3.2}$$

For functions $P(\nu, z)$, $Q(\nu, z)$, the Hankel's asymptotic expansions are known which can be represented in the form [1]:

$$P(\nu,z) \approx a_0 + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \ldots$$
$$a_0 = 1, a_n = -a_{n-2}\frac{(\nu^2-(n-3/2)^2)(\nu^2-(n-1/2)^2)}{4n(n-1)} \quad (n = 2,6,8,\ldots) \tag{3.3}$$

$$Q(\nu,z) \approx \frac{b_1}{z} + \frac{b_3}{z^3} + \frac{b_5}{z^5} + \ldots$$
$$b_1 = \frac{\nu^2 - 1/4}{2}, b_n = -b_{n-2}\frac{(\nu^2-(n-1/2)^2)(\nu^2-(n-3/2)^2)}{4n(n-1)}, \quad (n = 3,5,7,\ldots) \tag{3.4}$$

The relationships

$$H_{-\nu}^{(1)}(z) = e^{\mathrm{i}\pi\nu}H_\nu^{(1)}(z), H_{-\nu}^{(2)}(z) = e^{-\mathrm{i}\pi\nu}H_\nu^{(2)}(z) \tag{3.5}$$

show that $P(\nu, z)$ and $Q(\nu, z)$ should be even functions of ν; knowing them for $0 \le \nu \le 1$ one is able, using well known recurrent relationships, to perform easily calculations for other real value of ν. Note that for $\nu = (2k + 1)/2$ where $k \le n$ is an integer the above asymptotic expansions $A_n(z)$ give exact representations for $P(\nu, z)$, $Q(\nu, z)$.

In the present paper, formulas are constructed using the suggested correction for function $\Phi(x) = P(\nu, x)$ and $\Phi(x) = Q(\nu, x)$ at an interval $x \ge x_1$ where x_1 is a positive value significantly smaller than in the case of direct application of asymptotic expansions (3.3), (3.4). We denote the corresponding deviations of the functions $P(\nu, x)$, $Q(\nu, x)$ from their asymptotic expansions as $p_n(\nu, x)$ and $q_m(\nu, x)$:

$$p_n(\nu,x) = P(\nu,x) - \left(1 + \frac{a_2}{x^2} ... + \frac{a_n}{x^n}\right) \quad (n = 2,4,...)$$
$$q_m(\nu,x) = Q(\nu,x) - \left(\frac{b_1}{x} + \frac{b_3}{x^3} ... + \frac{b_m}{x^m}\right), \quad (m = 3,5,...) \tag{3.6}$$

The indexes n, m define the last accounted term in the brackets. Analogously to (2.2) we write

$$p_n(\nu,x) \approx G_p(\nu,x) = \sum_{j=1}^{s}\left(\frac{w_p + x_j/x_1}{w_p + x/x_1}\right)^{\lambda_p} L_j(x) p_n(\nu,x_j) \tag{3.7}$$

$$q_m(\nu,x) \approx G_q(\nu,x) = \sum_{j=1}^{s}\left(\frac{w_q + x_j/x_1}{w_q + x/x_1}\right)^{\lambda_q} L_j(x) q_m(\nu,x_j) \tag{3.8}$$

Calculations show that the optimum values are: $\lambda_p = s + n + 1$ and $\lambda_q = s + m + 1$ (as for gamma function in the previous section). The parameters h, q (which can be different for G_p, G_q) and w_p, w_q (which depend on ν) should minimize, wherever possible, the absolute values of $\delta_p = p_n(\nu, x) - G_p(\nu, x)$ and $\delta_q = q_m(\nu, x) - G_q(\nu, x)$ at the interval $x > x_1$. In distinction to the case with gamma function, we have now the second argument, ν, entering the considered functions, and our objective is to build simple approximations containing this additional argument.

3.1 Function $G_p(\nu, x)$ for $x_1 = 7$, $s = 1$, $n = 14$, $\lambda_p = 16$

Using only one point, x_1, and $L_1(x) = 1$, we determine previously exact values of $p_{14}(\nu, x_1) = 8.487514\times10^{-8}$, -9.071832×10^{-8} for $\nu = 0, 1$, respectively. The corresponding optimum values of w_p we find making a number of calculations; they are 0.069314 and 0.069028 for these values of ν. Thus the correction $G_p(\nu, x)$ in (3.7) becomes known for $\nu = 0, 1$. An interpolation leads to the following equation for w_p:

$$w_p = 0.069314 - 0.000286\nu^2 \tag{3.9}$$

which is suitable also beyond the interval $0 \le \nu \le 1$ (see below). Now for determination of the function $G_p(\nu, x)$ for a value ν only $p_{14}(\nu, x_1)$ is needed. Taking the values $p_{14}(0, x_1)$, $p_{14}(1, x_1)$, $p_{14}(2, x_1)$ we, using notations $u = \nu^2$, $u_j = \nu_j^2$, construct the following interpolation with the weight function $\cos(\pi\nu)\exp(0.06805u)$ at the interval $0 \le \nu \le 2$:

$$p_{14}(\nu,7) \approx \sum_{j=1}^{3} \mathrm{e}^{0.06805(u-u_j)} \cos(\pi\nu)(-1)^{j-1} p_{14}(\nu_j,7) L_j(u) \tag{3.10}$$

Here $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$; $L_j(u)$ are the already mentioned Lagrange polynomials relating to points u_j. Remember that for $\nu = (2k + 1)/2$ ($k = 0,1,\ldots$) values $p_{14}(\nu, x)$ should be zeros; this explains the application of the cosine function. The parameter in the exponent function, 0.06805, has been selected on basis of a number of calculations for minimization of deviation of function (3.10) from the corresponding exact values. After transformations, equations (3.10) can be written in the form

$$p_{14}(\nu,7) \approx \cos(\pi\nu)\,\mathrm{e}^{0.06805\nu^2}(84.87514 - 0.128329\nu^2 + 0.0034935\nu^4)10^{-9} \tag{3.11}$$

The absolute value of error δ_p occurred at the interval $0 \le \nu \le 1$ when using $G_p(\nu, x)$ with equations (3.9), (3.11) remains less than 1.75×10^{-12} which is illustrated by Fig. 3 where the corresponding results are represented for values of ν from the interval [0, 1] with step 0.1 (for $\nu = 0.5$ the error equals zero). The suitable choice of the parameter w_p results practically in appearing of two additional interpolation points (see Fig. 3) and the considered one term correction leads to about 5×10^4 times smaller error compared with the direct application of the corresponding asymptotic expansion (see the given above values of the deviation $p_{14}(v, 7)$ for $v = 0, 1$). Note that the equations (3.9), (3.11) lead to a rather good approximation at the interval $0 \le v \le 3$ with $|\delta_p| < 2\times10^{-12}$. Considering the function $P(v, z)$ we obtain:

$$P(\nu,x) \approx 1+\sum_{k=1}^{7}\frac{a_{2k}}{x^{2k}}+\left(\frac{w_p+1}{w_p+x/7}\right)^{16} p_{14}(\nu,7) \tag{3.12}$$

Here equations (3.3) for a_j, (3.9) for w_p, (3.11) for $p_{14}(v, 7)$ should be applied. In accordance with (3.1), (3.2), (3.5) the function (3.12) is an even function of v. The constructed approximation (3.12) (when applying it for a specific value of v) is competitive (regarding precision and computational work) with other methods (see, e.g., results in [8] for $v = 0,1$ obtained with the rational approximation), an advantage of equation (3.12) consists in including the explicit dependence on v.

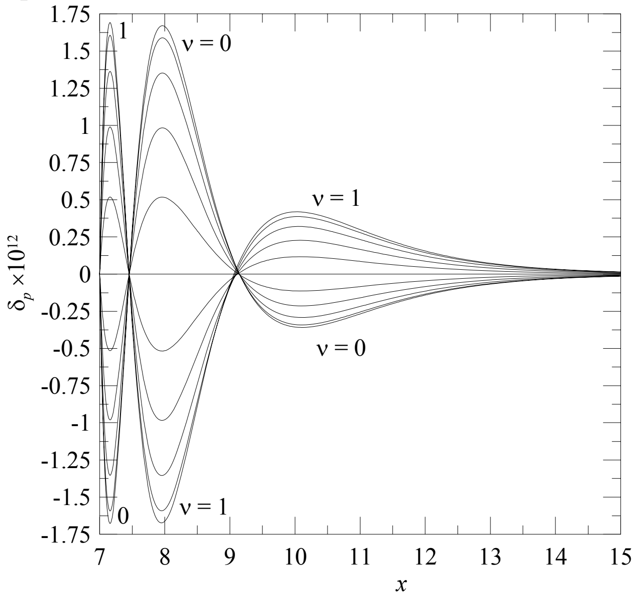

Fig. 3 Errors δ_p of approximation for deviation $p_n(v, x)$ with $n = 14$, $s = 1$, $x_1 = 7$.

3.2 Function $G_q(v, x)$ for $x_1 = 7$, $s = 1$, $m = 13$, $\lambda_q = 15$

Similarly to the previous case we find exact values of $q_{13}(v, x_1) = 8.460892\times10^{-8}$, -9.083417×10^{-8} for $v = 0, 1$, respectively. The corresponding optimum values of w_q are 0.069306, 0.068964. Thus the function $G_q(v, x)$ in (3.8) has been determined for these values of v. An interpolation of the indicated values of w_q leads to the following equation:

$$w_q = 0.069306 - 0.000342\nu^2 \tag{3.13}$$

As above (see (3.10), (3.11)), the three values of $q_{13}(v, x_1)$ for $v = 0, 1, 2$ allow us to achieve the following sufficiently accurate interpolation with the weight function $\cos(\pi v)\exp(0.07276u)$:

$$q_{13}(\nu,7) \approx \cos(\pi\nu)\, e^{0.07276\nu^2}(84.60892 - 0.1533331\nu^2 + 0.0041985\nu^4)10^{-9} \tag{3.14}$$

The absolute value of error δ_q at the interval $0 \le v \le 1$ generated when using $G_q(v, x)$ with equations (3.13) and (3.14) remains less than 2.14×10^{-12} which is illustrated by Fig. 4 where the corresponding results are represented for values of v from 0 to 1 with step 0.1. The equations (3.13), (3.14) lead also to the approximation at the interval $0 \le v \le 3$ with $|\delta_q| < 5.3\times10^{-12}$. Considering the function $Q(v, x)$ itself we obtain:

$$Q(\nu,x) \approx \sum_{k=1}^{7}\frac{b_{2k-1}}{x^{2k-1}}+\left(\frac{w_q+1}{w_q+x/7}\right)^{15} q_{13}(\nu,7) \tag{3.15}$$

Here equations (3.4), (3.13), (3.14) should be applied. The function (3.15) gives correct results also for negative values of ν. Using equations (3.1), (3.12), (3.15) one can calculate cylindrical functions $J_\nu(x)$ and $Y_\nu(x)$ for $x \geq 7$ and $-1 \leq \nu \leq 1$ (or even at the wider interval $-3 \leq \nu \leq 3$ with a small loss of precision) and further go to other values of ν applying well known recurrent relationships. As the calculations show, an error for the functions $J_\nu(x)$ and $Y_\nu(x)$ is less than 6×10^{-13} at the interval $-1 \leq \nu \leq 1$ and less than 10^{-12} at the interval $-3 \leq \nu \leq 3$.

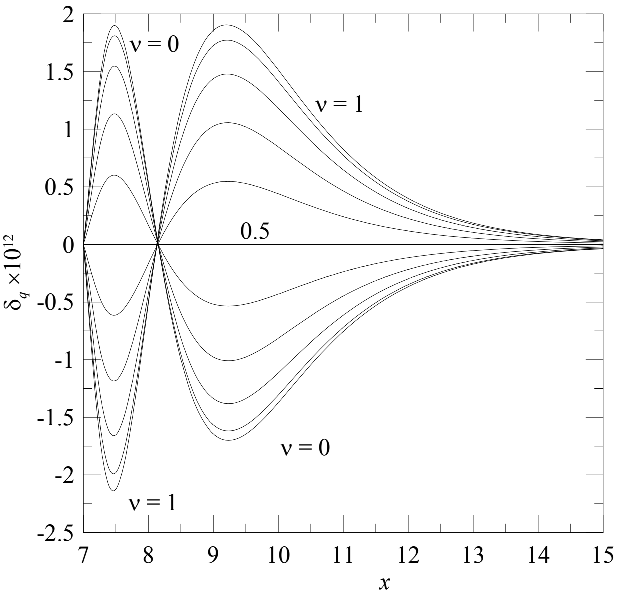

Fig. 4 Errors δ_q of approximation for deviation $q_m(\nu, x)$ with $m = 13$, $s = 1$, $x_1 = 7$.

3.3 Function $G_p(\nu, x)$ for $x_1 = 7$, $s = 4$, $n = 14$, $\lambda_p = 19$

The increase in the number of interpolation points leads to a significantly higher precision. Instead of 1.75×10^{-12} (upper estimation for $|\delta_p|$ for $|\nu| \leq 1$ in the case $x_1 = 7$, $s = 1$) we achieve for $s = 4$, an error less than 5.4×10^{-16}. On basis of calculations we chose $h = 0.204$, $q = 1.92$ for the all considered values of ν, the points x_2, x_3, x_4 are determined according to (3). For two values of $\nu = 0$, 1 we use required four exact values $p_{14}(\nu, x_1)$, $p_{14}(\nu, x_2)$, $p_{14}(\nu, x_3)$, $p_{14}(\nu, x_4)$ and further determine optimum values of $w_p = 0.11286$, 0.11249. Interpolating we obtain

$$w_p = 0.11286 - 0.00037\nu^2 \tag{3.16}$$

which ensures required precision at interval $0 \leq \nu \leq 1$. Now only four values $p_{14}(\nu, x_1)$, $p_{14}(\nu, x_2)$, $p_{14}(\nu, x_3)$, $p_{14}(\nu, x_4)$ are required to construct the approximation $G_p(\nu, x)$ for the corresponding value of ν. Instead of exact values of $p_{14}(\nu, x_j)$ ($j = 1,\ldots4$), the interpolating functions of ν analogous to (3.10), (3.11) can be easily found (as above we use 3 points $\nu = 0, 1, 2$)

$$p_{14}(\nu, x_1) \approx \cos(\pi\nu)\,\mathrm{e}^{0.068003\nu^2}(84.875135496 - 0.124340995\nu^2 + 0.003488375\nu^4)10^{-9} \tag{3.17}$$

$$p_{14}(\nu, x_2) \approx \cos(\pi\nu)\,\mathrm{e}^{0.067901\nu^2}(55.198071686 - 0.081393671\nu^2 + 0.002261041\nu^4)10^{-9} \tag{3.18}$$

$$p_{14}(\nu, x_3) \approx \cos(\pi\nu)\,\mathrm{e}^{0.067717\nu^2}(24.929366564 - 0.037261013\nu^2 + 0.0010156242\nu^4)10^{-9} \tag{3.19}$$

$$p_{14}(\nu, x_4) \approx \cos(\pi\nu)\,\mathrm{e}^{0.067394\nu^2}(6.004356231 - 0.009177469\nu^2 + 0.000242757\nu^4)10^{-9} \tag{3.20}$$

Although the interpolation at the interval $0 \leq \nu \leq 2$ is applied, a required precision is achieved only for the interval $0 \leq \nu \leq 1$. These functions along with function (3.16) allow us to calculate $G_p(\nu, x)$ directly for an arbitrary ν from the interval $0 \leq \nu \leq 1$ without using exact values $p_{14}(\nu, x_j)$. The absolute value of error δ_p occurred when using $G_p(\nu, x)$ with equations (3.16) – (3.20) remains less than 5.4×10^{-16} which is illustrated by Fig. 5 where the corresponding results are represented for values of v from 0 to 1 with step 0.1.

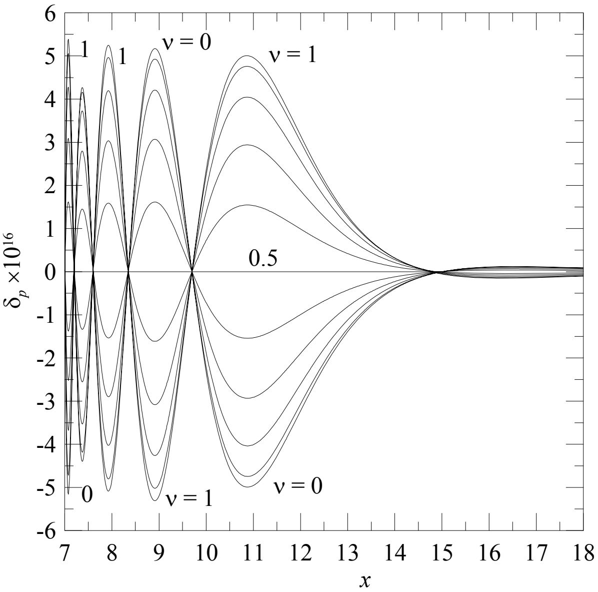

Fig. 5 Errors δ_p of approximation for deviation $p_n(v, x)$ with n = 14, s = 4, x_1 = 7.

The function $P(v, x)$ can be written in the form:

$$P(\nu, x) \approx 1 + \sum_{k=1}^{7} \frac{a_{2k}}{x^{2k}} + G_p(\nu, x) \tag{3.21a}$$

$$G_p(\nu, x) = \sum_{j=1}^{4} \left(\frac{w_p + x_j / x_1}{w_p + x / x_1} \right)^{19} L_j(x) p_{14}(\nu, x_j) \tag{3.21b}$$

Here equations (3.3), (3.16), (3.17) – (3.20) should be applied. The cubic polynomials $L_j(x)$ have the form:

$$\begin{aligned} L_1(x) &= \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}, L_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}, \\ L_3(x) &= \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}, L_4(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \end{aligned} \tag{3.22}$$

In accordance with (3.5) the function (3.21b) is an even function of v. For a chosen value of v equation (3.21b) can be reduced to a cubic polynomial divided by $(w_p + x/7)^{19}$, so for v = 0 we have:

$$G_p(0, x) = 10^{-8} \frac{13.67793804 - 2.390833962x + 1.023945041x^2 + 0.05137405032x^3}{(0.11286 + x/7)^{19}} \tag{3.23}$$

For v = 1:

$$G_p(1, x) = -10^{-8} \frac{14.4223419 - 2.520637967x + 1.086144276x^2 + 0.05469017287x^3}{(0.11249 + x/7)^{19}} \tag{3.24}$$

For v = 1/3:

$$G_p(1/3, x) = 10^{-8} \frac{6.87934176 - 1.202459195x + 0.5153365754x^2 + 0.02586606205x^3}{(0.1128188889 + x/7)^{19}} \tag{3.25}$$

3.4 Function $G_q(v, x)$ for $x_1 = 7$, $s = 4$, $m = 13$, $\lambda_q = 18$

The transition from $s = 1$ to $s = 4$ in the considered case increases the precision significantly: instead of 2.13×10^{-12} (upper estimation for $|\delta_q|$ for $|v| \le 1$ in the case $x_1 = 7$, $s = 1$) errors less than 3.16×10^{-16} occur. On basis of calculations we use $h = 0.467$, $q = 1.5$. For two values of $v = 0, 1$ we use required four values $q_{13}(v, x_1)$, $q_{13}(v, x_2)$, $q_{13}(v, x_3)$, $q_{13}(v, x_4)$ and further determine optimum values of $w_q = 0.11318, 0.11270$. Interpolating we obtain

$$w_q = 0.11318 - 0.00048v^2 \tag{3.26}$$

which ensures required precision at the interval $0 \le v \le 1$. Now only four values $q_{13}(v, x_1)$, $q_{13}(v, x_2)$, $q_{13}(v, x_3)$, $q_{13}(v, x_4)$ are required to construct the approximation $G_q(v, x)$ for the corresponding value of v. Instead of exact values of $q_{13}(v, x_j)$ ($j = 1,\ldots 4$), the interpolating functions analogous to (3.17) – (3.20) can be easily found:

$$q_{13}(v, x_1) \approx \cos(\pi v)\,\mathrm{e}^{0.072642v^2}(84.608924305 - 0.143349092v^2 + 0.004183468v^4)10^{-9} \tag{3.27}$$

$$q_{13}(v, x_2) \approx \cos(\pi v)\,\mathrm{e}^{0.072392v^2}(34.133194725 - 0.058835126v^2 + 0.001677992v^4)10^{-9} \tag{3.28}$$

$$q_{13}(v, x_3) \approx \cos(\pi v)\,\mathrm{e}^{0.072054v^2}(9.620077552 - 0.016966085v^2 + 0.00047015v^4)10^{-9} \tag{3.29}$$

$$q_{13}(v, x_4) \approx \cos(\pi v)\,\mathrm{e}^{0.071612v^2}(1.722369909 - 0.003113071v^2 + 0.000083789v^4)10^{-9} \tag{3.30}$$

These functions along with function (3.26) allow us to calculate $G_q(v, x)$ directly for an arbitrary v from the considered interval. The absolute value of error δ_q occurred when using $G_q(v, x)$ with equations (3.26) and (3.27) – (3.30) remains less than 3.16×10^{-16} which is illustrated by Fig. 6 where the corresponding results are represented for values of v from 0 to 1 with step 0.1. The function $Q(v, x)$ can be written in the form:

$$Q(v, x) \approx \sum_{k=1}^{7} \frac{b_{2k-1}}{x^{2k-1}} + G_q(v, x) \tag{3.31a}$$

$$G_q(v, x) = \sum_{j=1}^{4} \left(\frac{w_q + x_j / x_1}{w_q + x / x_1} \right)^{18} L_j(x) q_{13}(v, x_j) \tag{3.31b}$$

Here equations (3.4), (3.22), (3.26), (3.27) – (3.30) should be applied. In accordance with (3.5) the function (3.31b) is an even function of v. For a chosen value of v equation (3.31b) can be reduced to a cubic polynomial divided by $(w_q + x/7)^{18}$, so for $v = 0$ we have:

$$G_q(0, x) = 10^{-8} \frac{10.65458363 - 1.88832261x + 0.9073732455x^2 + 0.04779090942x^3}{(0.11318 + x/7)^{18}} \tag{3.32}$$

For $v = 1$:

$$G_q(1, x) = -10^{-8} \frac{11.24656984 - 1.995287024x + 0.9649812411x^2 + 0.05111109319x^3}{(0.1127 + x/7)^{18}} \tag{3.33}$$

For $v = 1/3$:

$$G_q(1/3, x) = 10^{-8} \frac{5.359380714 - 0.9499569637x + 0.4567984494x^2 + 0.024074334x^3}{(0.1131266667 + x/7)^{18}} \tag{3.34}$$

Using equations (3.1), (3.21), (3.31) one can calculate Bessel and Neumann functions for $-1 \le v \le 1$ and further go to arbitrary values of v applying well known recurrent relationships. As calculations show, an error for the functions $J_v(x)$ and $Y_v(x)$ is less than 1.6×10^{-16} at the interval $-1 \le v \le 1$; this precision holds when applying the recurrent relationships for cylindrical functions for $v \le 6$.

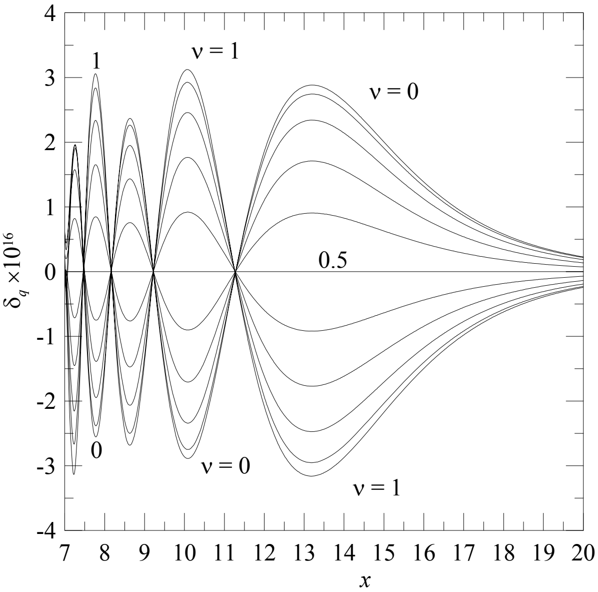

Fig. 6 Errors δ_q of approximation for deviation $q_m(v, x)$ with $m = 13$, $s = 4$, $x_1 = 7$.

3.5 Function $G_p(v, x)$ for $x_1 = 3$, $s = 1$, $n = 6$, $\lambda_p = 8$

The approximation for $x_1 = 3$ is less exact than in the case $x_1 = 7$ considered above, however the accuracy is still acceptable for many applications. We present equations analogous to equations (3.9), (3.11)

$$w_p = 0.15704 - 0.0027v^2 \tag{3.35}$$

$$p_6(v,3) \approx \cos(\pi v)\,\mathrm{e}^{\alpha v^2}((1-v^4)p_6(0,3) - v^4 e^{-\alpha} p_6(1,3)) = \cos(\pi v)\,\mathrm{e}^{0.14198v^2}(36.82385 + 0.01128v^4)10^{-5} \tag{3.36}$$

where α = 0.14198. Here two interpolation points $v_1 = 0$, $v_2 = 1$ are sufficient for approximation $p_6(v, 3)$. The absolute value of error δ_p occurred when using $G_p(v,x)$ with equations (3.35) and (3.36) remains less than 6.5×10^{-8} which is illustrated by Fig. 7 where the corresponding results are represented for values of v from 0 to 1 with step 0.1.

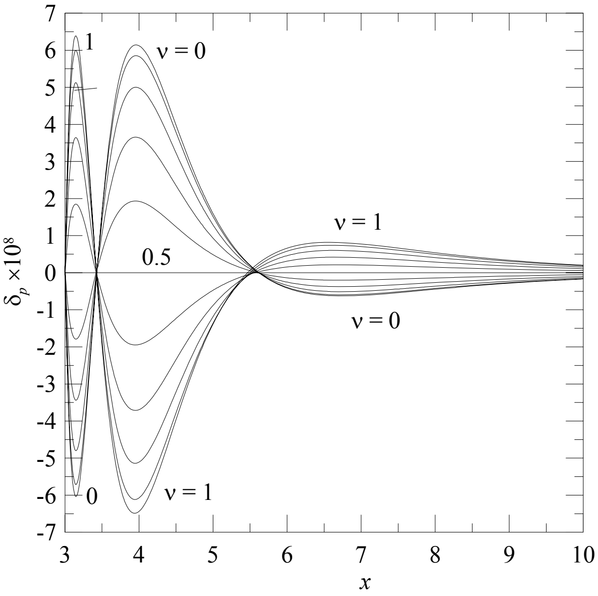

Fig. 7 Errors δ_p of approximation for deviation $p_n(v, x)$ for $n = 6$, $s = 1$, $x_1 = 3$.

Considering the function $P(v, x)$ itself we obtain the following approximation:

$$P(\nu,x) \approx 1 + \frac{a_2}{x^2} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \left(\frac{w_p+1}{w_p+x/3}\right)^8 p_6(\nu,3) \tag{3.37}$$

Here equations (3.3), (3.35), (3.36) should be applied. In accordance with (3.5) the function (3.37) is an even function of v.

3.6 Function $G_q(v, x)$ for $x_1 = 3$, $s = 1$, $m = 5$, $\lambda_q = 7$

We present equations analogous to equations (3.35), (3.36)

$$w_q = 0.15747 - 0.0038\nu^2 \tag{3.38}$$

$$q_5(\nu,3) \approx \cos(\pi\nu)\, e^{0.16182\nu^2}(36.32326 + 0.01621\nu^4)10^{-5} \tag{3.39}$$

The absolute value of error δ_q occurred when using $G_q(v, x)$ with equations (3.38) and (3.39) remains less than 6×10^{-8} which is illustrated by Fig. 8. For the function $Q(v, x)$ we obtain the following approximation:

$$Q(\nu,x) \approx \frac{b_1}{x} + \frac{b_3}{x^3} + \frac{b_5}{x^5} + \left(\frac{w_q+1}{w_q+x/3}\right)^7 q_5(\nu,3) \tag{3.40}$$

Here equations (3.4), (3.38), (3.39) should be applied. When determining the functions $J_v(x)$ and $Y_v(x)$ using (3.1), (3.37), (3.40) error for the interval $-1 \le v \le 1$ is less than 3×10^{-8}.

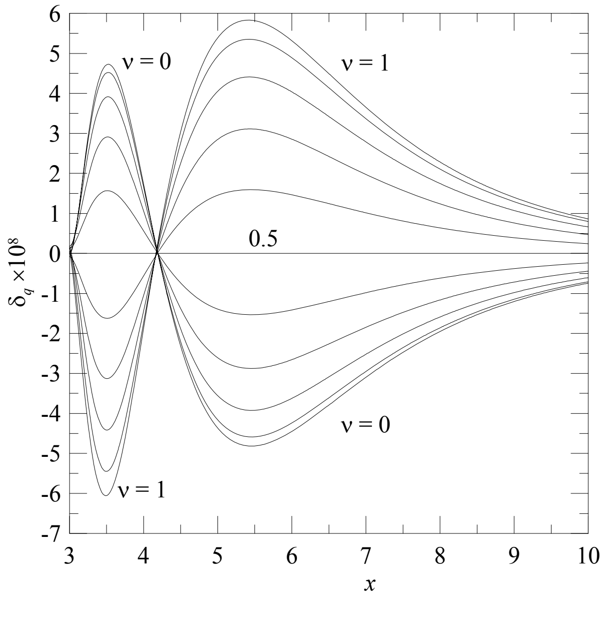

Fig. 8 Errors δ_q of approximation for deviation $q_m(v, x)$ for $m = 5$, $s = 1$, $x_1 = 3$.

4. Conclusions

The correction of asymptotic expansions suggested in the paper significantly improves the precision and as result one can apply the corrected expansions beginning from significantly smaller values of argument than in the case of usual expansions. For cylindrical functions, the method allows obtaining simple formulas which include explicit dependence on order's values. Actually the possibility occurs to apply the corrected asymptotic expansions beginning from a small enough value x_1 of the argument so that for $x < x_1$ the Taylor series can be used.

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