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Cracks with Interfacial Bonds Bending of a Strip (Beam) by Non-linear Singular Integrodifferential Equations

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Abstract
We consider a failure mechanics problem for a strip (beam) which bends in its plane by a system of external loads. It is assumed that the strip is weakened by arbitrarily arranged through cracks with bonding at the tip zones. A model of a crack with areas where its faces interact with each other is investigated. This interaction is modeled by introducing bonds (cohesive forces) between the faces in the crack tip zones. The boundary-value problem on equilibrium of the crack under the action of external loads is reduced to a system of nonlinear singular integrodifferential equations, from the solution of which the tractions in the bonds are found. The condition of crack growth is formulated taking account of the criterion of the limit traction of the bonds in the material.

Key Word and Phrases
Strip, Cohesive Forces, Cracks with Interfacial Bonds, Bending

1. Introduction

Adequate methods of estimating the load-bearing capacity of plates (strips) with cracks have not been yet developed. It is important to study the process of strip (beam) fracture solve problems of practical importance. Many researchers have given much attention to studying the failure of strips [1, 2]. In these studies, the investigations were limited to the consideration of Griffit's crack (model), i.e., a crack with noninteracting edges. The singularity of stress and strain fields at the tip of such a crack is caused not by physical reasons (stresses and strains cannot be infinite), but by its mathematical description. This description can change so that the singularity disappears [3-5]. In structurally heterogeneous materials, in the presence of zones with a distributed structure near the crack, a significant part of the crack is involved into the process of failure. In this case, the area of failure can be regarded as a tip zone adjacent to the crack, whose material contains partly broken bonds between the particles.

In the present study, it is assumed that the crack faces interact with each other, while the forces of this interaction (cohesive forces) are distributed in such a manner that the crack tip is no longer a singular point of stress-strain state. The physical nature of the cohesive forces depends on the material, as well as on sizes of the crack and its tip zone. The mathematical description of the cohesive forces in the crack tip zone is based on the concept that these forces are continuously distributed in the tip zone, are unknown beforehand, and must be determined in solving the problem. Moreover, the unknown size of the zone of action of the cohesive forces is commensurable with crack length. In this case, the boundary-value problems of material mechanics considered prove to the problems of elasticity theory with an unknown boundary, which must be determined solving the boundary-value problem. For solving of mechanics problems are successfully used methods of singular integral equations [6-8].

2. Formulation of the Problem

Let us consider a homogeneous isotropic strip (beam). The strip width and thickness are denoted by $2c$ and $2h$, respectively (Fig. 1). The mid-surface (x,y) is the plane symmetry. The strip is in a generalized plane stress state. Consider further the following assumptions. The strip is subjected to external loading (bending moments, pressure uniformly distributed along the strip, or concentrated forces) located in the mid-plane of the strip (see Fig. 1). The strip faces parallel to the to the plane *xy* are free from external loading. The crack is aligned in the direction of the maximum tensile stresses. The *x* axis of the coordinate system (*xy*) does not coincide with line along which the crack is located ($a \le x \le b$).

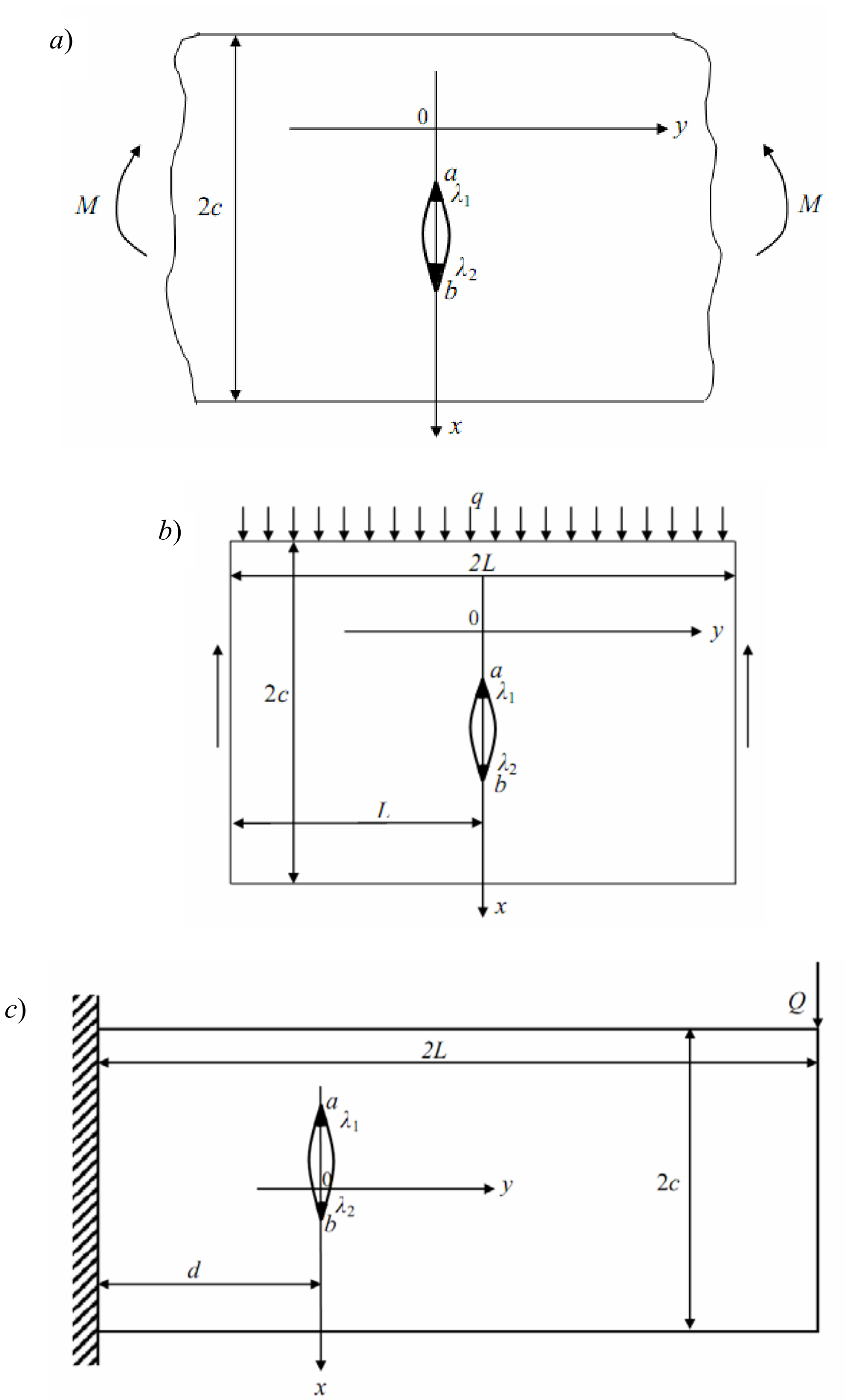

Fig. 1 Computational diagram of problem.

It is also assumed that the strip is weakened by a rectilinear crack. A crack model with areas where its faces interact with each other is investigated. These areas are adjacent to tips of the crack, while their sizes are unknown beforehand are commensurable with the crack size. The interaction between crack faces in the tip zone is modeled by introducing interfacial bonds (cohesive force) with a given deformation diagram. Under the action of external loads on the strip induces normal $q_y(x)$ and shear $q_{xy}(x)$ forces in the bonds connecting the crack faces in tip zones in the general case. Therefore, normal $q_y(x)$ and shear $q_{xy}(x)$ stresses respectively will be applied to the crack faces in the end zones of attenuated interparticle bonds of the material. These stresses are not known in advance and have to be determined in the course of solving the boundary-value problem of fracture mechanics for the strip. The crack faces outside the end zones are free from external loads. We now separate out the parts of the crack λ_1 and λ_2 (the end zones) adjacent to its tips where the crack face interact. We recall that, in the case being considered, each crack consists of an internal domain, that is, the opposite faces of the crack, which are load-free, and end zones (a,λ_1) and (b, λ_2) with bond between the faces.

The boundary conditions in the considered problem have the form

$$\sigma_y - i\tau_{xy} = 0 \quad \text{for } y = 0, \quad \lambda_1 \le x \le \lambda_2 \tag{1}$$

$$\sigma_y - i\tau_{xy} = q_y(x) - iq_{xy}(x) \quad \text{for } y = 0, \quad a \le x < \lambda_1 \text{ and } \lambda_2 < x \le b$$

The basic relations of the problem stated must be complemented with an equation connecting the displacements of crack opening with tractions in the bonds. Without the loss of generality, this equation can be presented in the form

$$v^+(x,0) - v^-(x,0) - i(u^+(x,0) - u^-(x,0)) = C(x,\sigma)\left[q_y(x) - iq_{xy}(x)\right] \tag{2}$$

$$\sigma = \sqrt{q_y^2 + q_{xy}^2}$$

The function $C(x,\sigma)$ can be regarded as the effective compliance of the bonds which depends on their tension, σ is modulus of the vector of the bond tractions, $(v^+ - v^-)$ is the normal and $(u^+ - u^-)$ is the tangential component of expansion of the crack.

For determining the ultimate equilibrium state of the crack, we use the condition of critical opening. Let we assume that rupture of bonds in the crack tip zone $x = \lambda$ occurs when the condition

$$\left|(v^+ - v^-) - i(u^+ - u^-)\right| = \delta_c \tag{3}$$

is satisfied, where δ_c is a characteristic of the fracture toughness of the strip material.

3. The Case of a Single Bridged Crack

The stress-strain state under plane problem condition with a cut along the abscissa is described by two analytic functions $\Phi(z)$ and $\Psi(z)$ [9]

$$\sigma_x + \sigma_y = 2\left[\Phi(z) + \overline{\Phi(z)}\right] \tag{4}$$

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}$$

$$2\mu\frac{\partial(u + iv)}{\partial x} = \kappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}$$

$$\Omega(z) = \overline{\Phi}(z) + z\overline{\Phi'}(z) + \overline{\Psi}(z)$$

where $\kappa = (3-\nu)/(1+\nu)$, ν is Poisson's constant of the strip material, μ is shear modulus of the material.

The stress-strain state in the vicinity of the crack is determined approximately [9]: the boundary conditions of the problem (conditions (1)) are satisfied on the contour of the crack, and the stress state in the strip at a significant distance from the crack coincides with the stress state determined by the functions

$$\Phi_0(z) = \lim_{z\to\infty} \Phi(z) = A_0 z^3 + A_1 z^2 + A_2 z + A_3 \tag{5}$$

$$\Omega_0(z) = \lim_{z\to\infty} \Omega(z) = B_0 z^3 + B_1 z^2 + B_2 z + B_3$$

For different values of the coefficients A_j and B_j (j=0,1,2,3), these functions determine the stress state in the strip (beam) in the absence of crack. For instance, assuming that

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = M/(4I), \quad A_3 = 0 \tag{6}$$
$$B_0 = 0, \quad B_1 = 0, \quad B_2 = 3M/(4I), \quad B_3 = 0$$

in eqs (5) (I is the moment of inertia of the strip cross section), we can show that the functions $\Phi_0(z)$ and $\Omega_0(z)$ determine the solution of the problem of pure bending of an infinite strip (beam) by moments M in the absence of the bridged crack (see Fig. 1a).

Similarly, for

$$A_0 = q/(24I), \quad A_1 = 0, \quad A_2 = q(L^2 + 3c^2/5)/(8I), \quad A_3 = -qc^3/(12I) \tag{7}$$
$$B_0 = 7q/(24I), \quad B_1 = 0, \quad B_2 = q(3L^2 - 11c^2/5)/(8I), \quad B_3 = qc^3/(12I)$$

the functions $\Phi_0(z)$ and $\Omega_0(z)$ determine the solution of the problem of bending of a beam of length 2L loaded by uniform pressure of intensity q in the absence of the bridged crack (see Fig. 1b). In this case, the beam is assumed to freely located on two supports, and the support responses are determined as shear force applied to the end faces of the beam. For

$$A_0 = 0, \quad A_1 = -iQ/(8I), \quad A_2 = -Q(2L-d)/(4I), \quad A_3 = 0 \tag{8}$$
$$B_0 = 0, \quad B_1 = 5iQ/(8I), \quad B_2 = -3Q(2L-d)/(4I), \quad B_3 = -iQc^2/(2I)$$

the functions $\Phi_0(z)$ and $\Omega_0(z)$ determine the solution of the problem of bending of a rigidly fixed cantilever beam in the absence of the bridged crack under the action of a constant transverse force Q applied to the fee and of the beam (see Fig. 1c).

The presence of the bridged crack in the strip disturbs the field of elastic stresses in its vicinity. The stress-strain state in the strip far from the bridged crack under above-mentioned loads is determined by eqs.(5) if the values of the coefficients A_j and B_j are determined by equalities (6)-(8).

The boundary-value problem (1) is reduced to the linear conjugation problem [9]

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 2f(t) \tag{9}$$
$$[\Phi(t) - \Omega(t)]^+ - [\Phi(t) - \Omega(t)]^- = 0$$

where $a \le t \le b$, t is the affix of the points of the bridged crack contour, and

$$f(t) = \begin{cases} 0 & \text{for} \quad \lambda_1 \le t \le \lambda_2 \\ q_y(t) - q_{xy}(t) & \text{for} \quad a \le t < \lambda_1 \quad \text{and} \quad \lambda_2 < t \le b \end{cases}$$

Solving problem (9) in the class of everywhere bounded functions (stresses), we find

$$\Phi(z) = \frac{\sqrt{(z-a)(z-b)}}{2\pi i}\int_a^b \frac{f(t)\,dt}{\sqrt{(t-a)(t-b)}(t-z)} + \sqrt{(z-a)(z-b)}P_n(z) + \frac{1}{2}[\Phi_0(z) - \Omega_0(z)] \tag{10}$$
$$\Omega(z) = \frac{\sqrt{(z-a)(z-b)}}{2\pi i}\int_a^b \frac{f(t)\,dt}{\sqrt{(t-a)(t-b)}(t-z)} + \sqrt{(z-a)(z-b)}P_n(z) - \frac{1}{2}[\Phi_0(z) - \Omega_0(z)]$$

where the functions $\Phi_0(z)$ and $\Omega_0(z)$ are determined by equalities (5); the polynomial $P_n(z)$ has the form

$$P_n(z) = D_n z^n + D_{n-1} z^{n-1} + \ldots + D_0 \tag{11}$$

The power of polynomial (11) and its coefficients D_0, D_1,…, D_n are determined by the behavior of the functions $\Phi(z)$ and $\Omega(z)$ in the neighborhood $|z| = \infty$. The functions $\Phi(z)$ and $\Omega(z)$ are analytical in the domain outside the crack and have the following form at large values of $|z|$:

$$\Phi(z) = \Phi_0(z) + O(1/z^2), \qquad \Omega(z) = \Omega_0(z) + O(1/z^2) \tag{12}$$

To determine the coefficients D_0, D_1,…, D_n and the values of a and b, it is necessary to expand the function $\Phi(z)$ into a series with respect to the powers of z in the neighborhood of the point $|z| = \infty$ and to compare this expansion with Eq. (12). Taking into account the above-given relations and performing necessary calculations, we obtain the system of equations

$$D_2 + \frac{1}{2}(A_0 - B_0) = A_0 \tag{13}$$

$$D_1 - \frac{1}{2}(a+b)D_2 + \frac{1}{2}(A_1 - B_1) = A_1,$$

$$D_0 - \frac{1}{2}(a+b)D_1 - \frac{1}{8}(a-b)^2 D_2 + \frac{1}{2}(A_2 - B_2) = A_2,$$

$$-C_1 - \frac{1}{2}(a+b)D_0 - \frac{1}{2}(a-b)^2 D_1 + \frac{1}{2}(A_3 - B_3) = A_3,$$

$$\frac{1}{2}(a+b)C_1 - C_2 - \frac{1}{8}(a-b)^2 D_0 = 0,$$

$$D_n = 0, \qquad n \ge 3,$$

where $C_1 = \frac{1}{2\pi i}\int\limits_a^b \frac{f(t)dt}{(t-a)(t-b)}$, $C_2 = \frac{1}{2\pi i}\int\limits_a^b \frac{t f(t)dt}{(t-a)(t-b)}$.

The last terms in (13) allow us to determine the parameters a and b. Now, let us construct integral equation for finding the unknown forces $q_y - q_{xy}$. Additional relation (2) is a condition determining unknown tractions in bonds between the faces in end zones of the crack. In the considered problem, it is convenient to write this additional condition for a derivative of opening of displacements of end zones faces.

Using relation (10) and boundary values of the functions $\Phi(z)$ and $\Omega(z)$, we obtain the following equality on the segment $a \le x \le b$

$$\Phi^+(x) - \Phi^-(x) - \frac{1}{2}F_n(x)\sqrt{(x-a)(x-b)} = \tag{14}$$

$$= \frac{2\mu}{1+\kappa}\left[\frac{\partial}{\partial x}(u^+ - u^-) + i\frac{\partial}{\partial x}(v^+ - v^-)\right]$$

where $F_n(x) = d_3 x^3 + d_2 x^2 + d_1 x + d_0$.

Using the Sokhotski-Plemelj formulas [9] and taking into account (10), we find

$$\Phi^+(x) - \Phi^-(x) = \sqrt{(x-a)(x-b)}\left[\frac{1}{\pi i}\int\limits_a^b \frac{f(t)dt}{\sqrt{(t-a)(t-b)}(t-x)} + 2P_n(x)\right] \tag{15}$$

Substituting (15) into the right side of (14), taking into account (2), and applying some transformations, we obtain the complex nonlinear integrodifferential equation

$$-\frac{i(1+\kappa)}{2\mu}\sqrt{(b-x)(x-a)}\left(-\frac{1}{\pi}\int\limits_a^b \frac{f(t)dt}{\sqrt{(b-x)(x-a)}(t-x)} + 2\left[P_n(x) - \frac{1}{4}F_n(x)\right]\right) =$$

$$= \frac{d}{dx}\left[C(x,\sigma)(q_y(x) - iq_{xy}(x))\right] \tag{16}$$

In the case of pure bending (see Fig. 1a) and in the case of strip bending by a uniformly distributed load (see Fig. 1b), we have $q_{xy}(x) = 0$ by virtue of load symmetry. In the case of cantilever beam bending (see Fig. 1c), normal q_y and shear q_{xy} force arise in the bonds between the faces. In this case

$$d_0 = -\frac{iQ}{16I}(b+a)\left[(b-a)^2 - 8c^2\right], \qquad d_1 = -\frac{iQ}{8I}\left[(b-a)^2 + 8c^2\right] \tag{17}$$

$$d_2 = -\frac{iQ}{2I}(b+a), \quad d_3 = \frac{iQ}{I}$$

In the general case, separating the real and imaginary parts of (16), we obtain a system of nonlinear singular integrodifferential equations with respect to $q_y(x)$ and $q_{xy}(x)$ with a kernel of the Cauchy type:

$$-\frac{1}{\pi}\sqrt{(b-x)(x-a)}\left(\int_a^b \frac{q_y(t)dt}{\sqrt{(b-x)(x-a)}(t-x)} + f_y(x)\right) = \frac{2\mu}{1+\kappa}\frac{d}{dx}\left[C(x,\sigma)q_y(x)\right] \quad (18)$$

$$-\frac{1}{\pi}\sqrt{(b-x)(x-a)}\left(\int_a^b \frac{q_{xy}(t)dt}{\sqrt{(b-x)(x-a)}(t-x)} + f_{xy}(x)\right) = \frac{2\mu}{1+\kappa}\frac{d}{dx}\left[C(x,\sigma)q_{xy}(x)\right] \quad (19)$$

,

where $f_y(x) = 2\,\mathrm{Re}\left[P_n(x) - \frac{1}{4}F_n(x)\right]$, $f_{xy}(x) = 2\,\mathrm{Im}\left[P_n(x) - \frac{1}{4}F_n(x)\right]$.

The equations can be solved by using a collocation scheme with approximation of unknown functions.

Now, pass to algebraization of integrodifferential equations (18) and (19) with additional conditions (13). At first, in integral equations (18) and (19) and in additional conditions (13), all integration intervals are reduced to one segment $[-1,1]$ by means of change of variables

$$t = \frac{a+b}{2} + \frac{b-a}{2}\tau, \quad x = \frac{a+b}{2} + \frac{b-a}{2}\eta$$

With such replacement of variables, the left-hand side of the integrodifferential equation (18) acquires the form

$$-\frac{1}{\pi}\sqrt{1-\eta^2}\left(\int_{-1}^{1} \frac{q_y(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} + f_y(\eta)\right) \quad (20)$$

For the left-hand side Eq. (19), we will have

$$-\frac{1}{\pi}\sqrt{1-\eta^2}\left(\int_{-1}^{1} \frac{q_{xy}(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} + f_{xy}(\eta)\right) \quad (21)$$

changing a derivative contained in the right-hand side Eq. (18), arbitrary internal node by finite difference approximation

$$\frac{d}{dx}\left[C(x,\sigma)q_y(x)\right] = \frac{1}{2\Delta x}\left[C(x_{i+1},\sigma(x_{i+1}))q_y(x_{i+1}) - C(x_{i-1},\sigma(x_{i-1}))q_y(x_{i-1})\right] \quad (22)$$

where $\Delta x = (b-a)/n$.

With the right-hand side of (19), we act in the same way. Here, account has been taken of the boundary conditions for $\eta_0 = \pm 1$ $q_y(a) = q_y(b) = 0$; $q_{xy}(a) = q_{xy}(b) = 0$ (which correspond to the conditions $v^+(a,0) - v^-(a,0) = 0$; $v^+(b,0) - v^-(b,0) = 0$; $u^+(a,0) - u^-(a,0) = 0$; $u^+(b,0) - u^-(b,0) = 0$).

Using the quadrature formula

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{g(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} = \frac{1}{n\sin\theta}\sum_{k=1}^{n} g_k \sum_{m=0}^{n-1}\cos m\theta_k \sin m\theta$$

$$\tau = \cos\theta; \quad \eta_m = \cos\theta_m; \quad \theta_m = \frac{2m-1}{2n}\pi \quad (m=1,2,\ldots,M)$$

all the integrals in eqs. (18) and (19) are substituted by finite sums and derivatives in the right-hand sides are replaced by finite difference approximations.

The cited formulae enable to change each singular integrodifferential equation by a system of algebraic equations with respect to approximate values of desired function at the nodal points. As a result, we obtain

$$-\frac{2}{n}\left[\sum_{\nu=1}^{n} q_{y,\nu}\sum_{k=0}^{n-1}\cos k\theta_k \sin k\theta_m\right]-\frac{1}{\pi}\sqrt{1-\eta_m^2}\,f_{y,m}= \tag{23}$$

$$=\frac{n(1+\kappa)}{4(b-a)\mu}\left[C(x_{m+1},\sigma(x_{m+1}))q_{y,m+1}-C(x_{m-1},\sigma(x_{m-1}))q_{y,m-1}\right]\quad (m=1,2,..,n),$$

$$-\frac{2}{n}\left[\sum_{\nu=1}^{n} q_{xy,\nu}\sum_{k=0}^{n-1}\cos k\theta_k \sin k\theta_m\right]-\frac{1}{\pi}\sqrt{1-\eta_m^2}\,f_{xy,m}= \tag{24}$$

$$=\frac{n(1+\kappa)}{4(b-a)\mu}\left[C(x_{m+1},\sigma(x_{m+1}))q_{xy,m+1}-C(x_{m-1},\sigma(x_{m-1}))q_{xy,m-1}\right]\quad (m=1,2,..,n).$$

If we take into account the equality

$$2\sum_{k=0}^{n-1}\cos k\theta_k \sin k\theta_m=\cot\frac{\theta_m\mp\theta_k}{2}$$

the systems will take the forms

$$\sum_{\nu=1}^{n} A_{m\nu}q_{y,\nu}-\frac{1}{\pi}\sqrt{1-\eta_m^2}\,f_{y,m}= \tag{25}$$

$$=\frac{n(1+\kappa)}{4(b-a)\mu}\left[C(x_{m+1},\sigma(x_{m+1}))q_{y,m+1}-C(x_{m-1},\sigma(x_{m-1}))q_{y,m-1}\right]\quad (m=1,2,..,n)$$

$$\sum_{\nu=1}^{n} A_{m\nu}q_{xy,\nu}-\frac{1}{\pi}\sqrt{1-\eta_m^2}\,f_{xy,m}= \tag{26}$$

$$=\frac{n(1+\kappa)}{4(b-a)\mu}\left[C(x_{m+1},\sigma(x_{m+1}))q_{xy,m+1}-C(x_{m-1},\sigma(x_{m-1}))q_{xy,m-1}\right]\quad (m=1,2,..,n)$$

Here $q_{y,\nu}=q_y(\tau_\nu)$; $q_{xy,\nu}=q_{xy}(\tau_\nu)$; $f_{y,m}=f_y(\tau_m)$; $f_{xy,m}=f_{xy}(\tau_m)$; $x_{m+1}=\frac{a+b}{2}+\frac{b-a}{a}\eta_{m+1}$; $A_{m\nu}=-\frac{1}{n}\cot\frac{\theta_m\mp\theta_\nu}{2}$. The upper sign is taken when number $|m-\nu|$ is odd; the lower one when it is even.

Now, pass to algebraization of solvability conditions of boundary-value problem eq. (13). Separating therein real and imaginary parts and using change of variables and Gauss quadrature formula, we get solvability condition on the problem in the following form

$$\frac{Q(a+b)}{4I}(2L-d)=-\frac{1}{2n}\sum_{\nu=1}^{n}q_y(\tau_\nu) \tag{27}$$

$$-\frac{1}{16}(a-b)^2(2L-d)Q=-\frac{1}{n}\sum_{\nu=1}^{n}\tau_\nu q_y(\tau_\nu)$$

$$-\frac{Q(a+b)^2}{16I}-\frac{Q(a-b)^2}{8I}+\frac{Qc^3}{4I}=-\frac{1}{2n}\sum_{\nu=1}^{n}q_{xy}(\tau_\nu) \tag{28}$$

$$\frac{Q(a-b)^2(a+b)}{64I}=-\frac{1}{n}\sum_{\nu=1}^{n}\tau_\nu q_{xy}(\tau_\nu)$$

As a result of algebraization, instead of integrodifferential equation with appropriate additional conditions, we get M+2 algebraic equations for determining stresses at nodal points and sizes of the end zones of bridged crack.

With allowance for (3), the critical condition is written in the following form (for $x=\lambda$):

$$C(\lambda,\sigma(\lambda))\sigma(\lambda)=\delta_c \tag{29}$$

where $\lambda=\lambda_1$ for the upper end zone and $\lambda=\lambda_2$ for the lower end zone.

As the size of the end zones is unknown even in the particular case of linearly elastic body, the resultant algebraic systems are nonlinear. A method of consecutive approximations [10] is used to solve these systems of equations in the case of linear bonds. At each approximation to algebraic system of equation was solved by Gauss method with choice of the principal element for different values of n of order to *n*=30.

In the case of nonlinear law of deformation of bonds, an iterative method is used to determine the tractions in the end domains, which is similar to the method of elastic solutions [11]. It is assumed that the law of deformation of interparticle bonds (cohesive forces) in linear even for

$$V = \left|(u^+ - u^-) - i(v^+ - v^-)\right| \le V_*$$

The first step of the iterative calculation procedure consists of solving the system for linearly elastic interparticle bonds. Subsequent iterations are only carried out when the inequality $V > V_*$ holds on a part of end domains of crack.

The system of solving equations for quasi-elastic bonds (cohesive forces) with an effective compliance which is variable along the end zones and depend on the magnitude of the modulus of the tractions vector in bonds obtained in the preceding step of the calculation. The effective compliance is calculated in a similar manner that used to find the secant modulus elastic in the method of variable parameters [12]. It is accepted that successive approximations process is terminated only when the tractions in the end areas of crack obtained in two successive steps very little differ. The nonlinear part of the curve of deformation of bonds was represented in the form of a bilinear relation [13, 14], whose ascending part corresponded to the elastic deformation of bonds $(0 < V(x) < V_*)$ with maximal tension of bonds. For $V(x) > V_*$, the deformation law was described by a linear dependence defined by two points (V_*, σ_*) and (δ_c, σ_c); moreover, for $\sigma_c \ge \sigma_*$, we have increasing linear dependence (linear strengthening corresponding to the elastic-plastic deformation of bonds).

The calculations show that, in the case of a linear low of deformation of the bonds, the bond tractions of always have a maximum value at the edge of an end zone. A similar pattern is also observed for the magnitudes of the openings of the cracks. In fact, the opening of a crack at the edge of an end zone has a maximum in the case of both linear and nonlinear deformation laws and, moreover, the opening of a crack increases as the relative compliance of the bonds increases.

The simultaneous solution of joint algebraic system of equations and conditions (29) allows to determine the critical external load, tractions in bonds, dimensions of end zones for limit equilibrium state under given characteristics of bonds.

Figure 2 shows the $d_1 = (\lambda_1 - a)/2c$ as a function of the dimensionless load M/M_s ($M_s = \sigma_s h^2/4$; σ_s is the yield stress of the strip material under tension) in the case pure bending of the beam. The following values of the parameters were used in the calculations $\nu = 0.3$ and *n*=30.

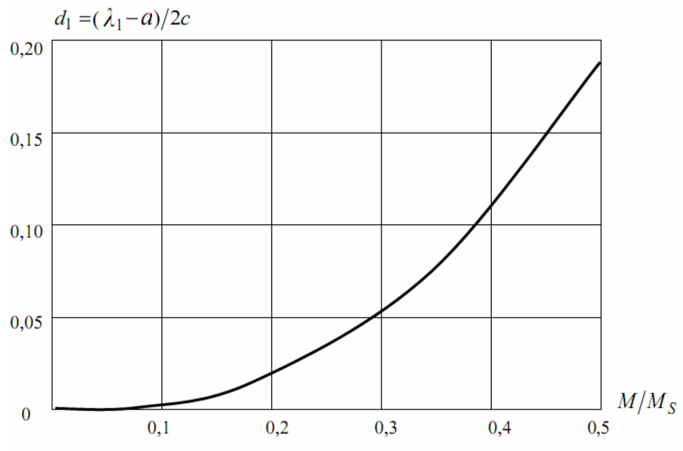

Fig. 2 Dependence of the end zone length $d_1 = (\lambda_1 - a)/2c$ on the dimensionless load M/M_s.

Figure 3 shows the dimensionless critical load

$$M_c = \frac{1}{\sqrt{2}} \frac{M}{hc^{3/2}} \frac{1}{\sqrt{E\delta_c \sigma_s}}$$

as a function of the dimensionless crack length $(\lambda_2 - \lambda_1)/c$.

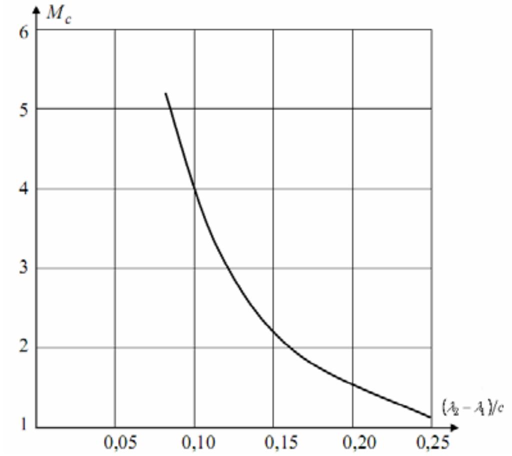

Fig. 3 Dependence of the critical load M_c on the dimensionless crack length $(\lambda_2 - \lambda_1)/c$.

The distributions of the normal forces in the bonds between the crack faces as a function of the dimensionless coordinate $x/(b-a)$ are shown in Figure 4. Curve 1 corresponds to a linear bond and curve 2 to a bilinear bond, $d_1/(b-a) = 0.3$.

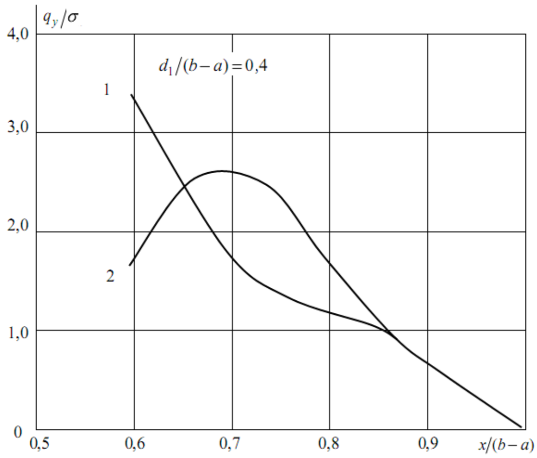

Fig. 4 Distribution of tractions q_y/σ_S in the bonds between the crack faces (curve 1 corresponds to a linear law of deformation of bonds; 2 to a nonlinear deformation).

4. The Case of an Arbitrary Number of Bridged Cracks

Now suppose there are N rectilinear bridged cracks of length $2l_k$ (k=1,2,…,N) in the strip (beam). The faces of cracks off the end zones are free from loads. At the center of end zone cracks we locate the origin of local systems of coordinates $x_kO_ky_k$ of the axis x_k that coincide with the crack lines are make the angles α_k with the axis x (Fig. 5). We will consider practically important case when the cracks are of small length. In this case, we can find the stress-strain state in the vicinity of cracks with sufficient accuracy by means of the solution of appropriate problem for the plane $(c \to \infty)$ with cracks in those surfaces act the efforts determined in the solution process.

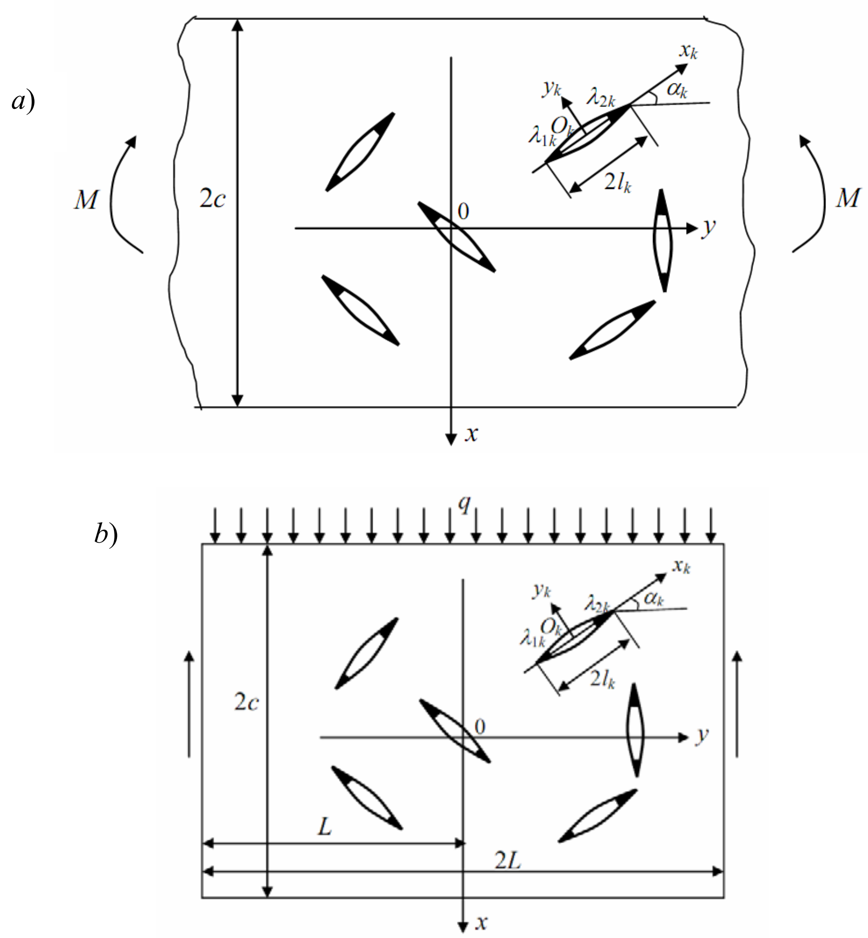

Fig. 5 Computational diagram of problem for the case of an arbitrary number of bridged cracks.

At the strip is loaded in the bonds connecting the crack faces at the areas, there will arise normal $q_{y_k}(x_k)$ and tangential $q_{x_k y_k}(x_k)$ (k=1,2,…,N) tractions. The quantities of these stresses and the dimensions λ_{1k} and λ_{2k} of end zones are not known beforehand and they are to be determined in the course of solution of the fracture mechanics problem. Since the end zones are small in comparison with remaining part of the strip, we can mentally remove them and change by the sections whose surfaces interact between themselves by a certain law corresponding to the action of removed material. Thus, normal and tangential stresses that numerical equal $q_{y_k}(x_k)$ and $q_{x_k y_k}(x_k)$, respectively, will be applied to cracks faces at the end zones. By $L' = \sum_{k=1}^{N} L'_k$ we denote a totality of faces outside of end zones of cracks, and by $L'' = \sum_{k=1}^{N} L''_k$ a totality of prefracture end zones, wherein the faces interact with bonding.

In considered problem the boundary conditions are of the form

$$\sigma_{y_k} - i\tau_{x_k y_k} = 0 \quad \text{on } L'_k, \ (k=1,2,\ldots,N) \tag{30}$$

$$\sigma_{y_k} - i\tau_{x_k y_k} = q_{y_k}(x_k) - iq_{x_k y_k}(x_k) \quad \text{on } L''_k.$$

The equations for opening displacements of the end zones lips and bonding forces, may be represented in the form

$$(v_k^+ - v_k^-) - i(u_k^+ - u_k^-) = C(x_k, \sigma_k)(q_{y_k}(x_k) - iq_{x_k y_k}(x_k)) \tag{31}$$

where the function $C(x_k, \sigma_k)$ may be considered as effective compliance of appropriate bond dependent of the degree its tension; $\sigma_k^2 = q_{y_k}^2 + q_{x_k y_k}^2$ is the modulus of the vector of forces appropriate bonds.

By means of Kolosov-Muskhelishvili formulae [9] we can the boundary conditions of problem (30) for complex potentials $\Phi(z)$ and $\Psi(z)$

$$\Phi(t_k) + \overline{\Phi(t_k)} + t_k \overline{\Phi'(t_k)} + \overline{\Psi(t_k)} = f_k \tag{32}$$

$$f_k = \begin{cases} 0 & \text{on } L_k' \\ q_{y_k}(t_k) - iq_{x_k y_k}(t_k) & \text{on } L_k'' \end{cases}$$

where t_k is an affix of the faces points of the k-th crack with end zones.

We look for the complex potentials $\Phi(z)$ and $\Psi(z)$ giving the solution of boundary-value problem (32) in the form

$$\Phi(z) = \Phi_0(z) + \Phi_1(z), \qquad \Psi(z) = \Psi_0(z) + \Psi_1(z) \tag{33}$$

Here the functions

$$\Phi_0(z) = A_0 z^3 + A_1 z^2 + A_2 z + A_3 \tag{34}$$

$$\Psi_0(z) = B_0 z^3 + B_1 z^2 + B_2 z + B_3$$

describe the stress-strain state of defectless strip (beam) depending on the values of the coefficients A_j and B_j (j=0,1,2,3).

Then we can write boundary conditions on the surfaces of cracks with the end zones in the form

$$\Phi_1(t_k) + \overline{\Phi_1(t_k)} + t_k \overline{\Phi_1'(t_k)} + \overline{\Psi_1(t_k)} = f_k - F_k, \quad (k=1,2,\ldots,N) \tag{35}$$

where

$$F_k = \Phi_0(t_k) + \overline{\Phi_0(t_k)} + t_k \overline{\Phi_0'(t_k)} + \overline{\Psi_0(t_k)}.$$

We will seek the complex potentials $\Phi_1(z)$ and $\Psi_1(z)$ in the form

$$\Phi_1(z) = \frac{1}{2\pi} \sum_{k=1}^{N} \int_{-l_k}^{l_k} \frac{g_k(t)dt}{t - z_k} \tag{36}$$

$$\Psi_1(z) = \frac{1}{2\pi} \sum_{k=1}^{N} e^{-2i\alpha_k} \int_{-l_k}^{l_k} \left[\frac{\overline{g_k(t)}}{t - z_k} - \frac{\overline{T}_k e^{i\alpha_k}}{(t - z_k)^2} g_k(t) \right] dt$$

where $T_k = te^{i\alpha_k} + z_k^0$, $z_k = e^{-i\alpha_k}(z - z_0)$.

Here $g_k(x_k)$ are the desired functions, which characterizes the expansion of displacements when passing through the line of appropriate crack with end zones

$$g_k(x_k) = \frac{2\mu}{i(1+\kappa)} \frac{\partial}{\partial x_k} \left[u_k^+(x_k,0) - u_k^-(x_k,0) + i(v_k^+(x_k,0) - v_k^-(x_k,0)) \right] \tag{37}$$

On the satisfying the boundary conditions (35) by complex potentials, after some transformations we get a system of N singular integral equations in the form

$$\sum_{k=1}^{N} \int_{-l_k}^{l_k} \left[R_{nk}(t, x_n) g_k(t) + S_{nk}(t, x_n) \overline{g_k(t)} \right] dt = \pi \left[f_n(x_n) - F_n(x_n) \right] \tag{38}$$

$$|x_n| \le l_n \quad (n=1,2,\ldots,N).$$

Here R_{nk} and S_{nk} are determined by the relations

$$R_{nk} = \frac{e^{i\alpha_k}}{2} \left[\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\overline{T}_k - \overline{X}_n} \right],$$

$$S_{nk} = \frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k - \overline{X}_n} + \frac{(T_k - X_n)e^{-2i\alpha_n}}{(\overline{T}_k - \overline{X}_n)^2}\right]$$

$$X_n = xe^{i\alpha_n} + z_n^0$$

To the system of singular integral equations (38) for the internal cracks with end zones, we should add equalities

$$\int_{-l_k}^{l_k} g_k(t)dt = 0 \qquad (39)$$

providing uniqueness of displacements in tracing the boundary of cracks with end zones.

The system of complex singular integral equations (38) with additional conditions (39) by means of agebraization procedure [1, 10] is reduced to the system of $N\times M$ algebraic equations for determining $N\times n$ unknowns $g_k(t_m)$ (k=1,2,…,N; m=1,2,…,n)

$$\frac{1}{n}\sum_{m=1}^{n}\sum_{k=1}^{N} l_k\left[R_{nk}(l_k t_m, l_n x_r)g_k(t_m) + S_{nk}(l_k t_m, l_n x_r)\overline{g_k(t_m)}\right] = f_n^*(x_n) \qquad (40)$$

$$\sum_{m=1}^{n} g_n(t_m) = 0 \quad (n=1,2,\ldots,N;\ r=1,2,\ldots,n\text{-}1)$$

Here the values of t_m and x_k are determined by the formulae

$$t_m = \cos\frac{2m-1}{2n}\pi, \qquad x_r = \cos\frac{r}{n}\pi$$

If in (40) we pass to complex adjoint values, we get $N\times n$ algebraic equations more. To close system (40), we need $2N$ equations determining the sizes of the end zones. The conditions of stresses finiteness in the vicinity of vertices of each crack are the conditions for finding the sizes of the end zones. Writing the stress finiteness conditions, we get $2N$ missing equations in the following form

$$\sum_{m=1}^{n}(-1)^m g_k(t_m)\cot\frac{2m-1}{4n}\pi = 0 \qquad (41)$$

$$\sum_{m=1}^{n}(-1)^{n+m} g_k(t_m)\tan\frac{2m-1}{4n}\pi = 0$$

The right-hand side of the system of integral equations contains the unknown values of stresses $q_{y_k}(x_k)$ and $q_{x_k y_k}(x_k)$. For the left-hand side of the relations (31) we have

$$(v_k^+ - v_k^-) - i(u_k^+ - u_k^-) = -\frac{1+\kappa}{2\mu}\int_{-l_k}^{l_k} g_k(x_k)dx_k \qquad (42)$$

Allowing for (42), relations (31) accepts the form

$$-\frac{1+\kappa}{2\mu}\int_{-l_k}^{l_k} g_k(x_k)dx_k = C(x_k,\sigma_k)\left[q_{y_k}(x_k) - iq_{x_k y_k}(x_k)\right] \qquad (43)$$

Separating in (43) the real and imaginary parts, we get

$$-\frac{1+\kappa}{2\mu}\int_{-l_k}^{x_k} v_k^0(x_k)dx_k = C(x_k,\sigma_k)q_{y_k}(x_k) \qquad (44)$$

$$-\frac{1+\kappa}{2\mu}\int_{-l_k}^{x_k} u_k^0(x_k)dx_k = C(x_k,\sigma_k)q_{x_k y_k}(x_k)$$

$$v_k^0 = v_k^+(x_k,0) - v_k^-(x_k,0), \qquad u_k^0 = u_k^+(x_k,0) - u_k^-(x_k,0)$$

For constructing missing equations for the determination of tractions at the end zone bonds, the conditions (44) should be fulfilled at the nodal points $t_{m,k}$ contained at the crack's end zones. As a result, we get $2\times n$ systems more from $n_{1,k}$ equations for determining approximate values $q_{y_k}(t_{m,k})$ and $q_{x_k y_k}(t_{m,k})$ (k=1,2,…,N; m_1=1,2,…,$n_{1,k}$)

$$C_0 v_k^0(t_{1,k}) = C(t_{1,k}, \sigma_k(t_{1,k})) q_{y_k}(t_{1,k}) \quad (45)$$

$$C_0 (v_k^0(t_{1,k}) + v_k^0(t_{2,k})) = C(t_{2,k}, \sigma_k(t_{2,k})) q_{y_k}(t_{2,k})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C_0 \sum_{m_1=1}^{n_{1,k}} v_k^0(t_{m_{1,k}}) = C(t_{n_{1,k}}, \sigma_k(t_{n_{1,k}})) q_{y_k}(t_{n_{1,k}})$$

$$C_0 u_k^0(t_{1,k}) = C(t_{1,k}, \sigma_k(t_{1,k})) q_{x_k y_k}(t_{1,k}) \quad (46)$$

$$C_0 (u_k^0(t_{1,k}) + u_k^0(t_{2,k})) = C(t_{2,k}, \sigma_k(t_{2,k})) q_{x_k y_k}(t_{2,k})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C_0 \sum_{m_1=1}^{n_{1,k}} u_k^0(t_{m_{1,k}}) = C(t_{n_{1,k}}, \sigma_k(t_{n_{1,k}})) q_{x_k y_k}(t_{n_{1,k}})$$

where $C_0 = -\frac{1+\kappa}{2\mu}\frac{\pi d_k}{n}$.

Additional critical condition is necessary for determination of limit-equilibrium state of crack. In place of such a condition we accept the condition of critical opening of the crack at the end zone edge. Accept that the break of bonding at the edge of end zone $x_k^0 = l_k - d_k$ happens while fulfilling the condition

$$V(x_k^0) = \sqrt{\left(u_k^+(x_k^0,0) - u_k^-(x_k^0,0)\right)^2 + \left(v_k^+(x_k^0,0) - v_k^-(x_k^0,0)\right)^2} = \delta_c \quad (47)$$

Joint solution of equations (40), (41), (45), (46), (47) enables (for the given characteristics of bonds) to find critical external load, efforts in the end zone bonds, the sizes of end zones for limit-equilibrium state of the strip. Because of unknown sizes of the end zones of the crack, the systems of algebraic equations became nonlinear even for linear-elastic bonds. For linear-elastic relations the nonlinear algebraic system was solved by the sequential approximations method. In the case of nonlinear law of deformation of bonds, for determining the efforts at the end zones of the crack, an iteration algorithm similar to the method of elastic solution [11] is also used.

It is accepted that the law of deformation of interparticle bonds (interparticle adhesion forces) is linear for $V(x) \le V_*$. The first step of iterative calculations process is solving the system of equations for linear elastic interparticle bonds. The consequent iterations are fulfilled in the cases when $V(x_k) \le V_*$ holds on a part of the end zone. For such iterations the system of equations is solved at each approximation for quasielastic bonds with effective compliance variable along the end zone of the crack and dependent on the quantity of modulus of vector of forces bonds in the obtained in previous calculation step. The effective compliance calculation is carried out similar to determination of a cutting modulus in the method of variables of elasticity parameters [12]. It is accepted that the successive approximations process comes to end as soon as long the end zone, the cracks obtained in two successive iterations differ from each other very little.

The calculation show that for linear law of deformation of bonds the tractions in bonding have always maximal values at the edge of the end zone. The similar picture is obtained for the quantities of opening of the crack with end zones.

Thus, an analysis of the model on development of cracks with bonding at the end zones in the strip under action of external loadings is reduced to parametric joint investigation of algebraic systems (40), (41), (45), (46) and criteria on advance of the crack (47) for different values of physical and geometrical parameters of the strip (beam).

The constructed calculated model allows to investigate different cases of arrangement of small cracks in the section of the strip on its load-carrying capacity by varying the parameters α_k and z_k^0 (k=1,2,…,N).

5. Conclusions

Thus, a closed system of algebraic equations and condition allowing predicting ultimately admissible size of crack by numerical calculation for each specific kind of strip material is obtained. From the solution of this system normal and tangential stresses in connections are found. The model of a crack with interfacial bonds allows one to investigate the basic laws of distribution of tractions in the bonds for different laws of their deformation, to estimate the hardening of a strip caused by the bonds in crack tip zone, and to analyze the ultimate equilibrium state of the crack by using a deformation criterion. Such an analysis made in possible to determine the ultimate size of the crack tip zone, the allowable external load, and the residual strength of a strip (beam). It should be noted that the model of crack with bonds between its faces enables one to consider, from unified positions, the process of failure, including the origination of crack-type defects, as well as the formation and development of micro- and macrocracks.

References

1. *Savruk M.P.*, '*Two-dimensional problem of elasticity for bodies with cracks',* Naukova Dumka, Kiev, *1981.*
2. *Krenk S.,* 'On the elastic strip an internal crack'*, Int. J. Solids and Struct.,* ***11****(6) (1975), 693–708.*
3. *Rubinstein, A.A., Wang, P.* 'The fracture toughness of a particulate-reinforced brittle matrix', *J. Mech. Phyc. Solids,* ***46*** *(7) (1998), 1139-1154.*
4. *Cox B.N., Marshall D.B.* 'Concepts for bridged cracks fracture and fatigue', *Acta Met. Mater.,* ***42*** *(2) (1994), 341–363.*
5. The special issue: Cohesive models. *Eng.Fract.Mech.,* ***70*** *(14) (2003), 1741–1987.*
6. *Mirsalimov V.M. 'Fracture of elasto - and elastoplastic bodies with cracks'.* Science, Baku, *1984.*
7. *Ladopoulos E.G.,* 'Collocation-approximation methods for nonlinear singular integrodifferential equations in Banach spaces'*, Journal of Computational and Applied Mathematics,* **79** *(1997), 289-297.*
8. *Ladopoulos E.G., 'Singular Integral Equations, Linear and Non-Linear Theory and its Applications in Science and Engineering',* Springer Verlag, New York, Berlin, *2000.*
9. *Muskhelishvili N.I. 'Some Basic Problems in the Mathematical Theory of Elasticity',* Kluver, Amsterdam, *1977.*
10. *Mirsalimov V.M.' Non-one-dimensional elastoplastic problems',* Nauka, Moscow, *1987.*
11. *Il'yushin A.A. 'Plasticity',* Gostexhizd Moscow and Leningrad*, 1948.*
12. *Birger I.A.* 'The design of structures allowings for plasticity and creep', *Izv. Akad. Nauk SSSR Mekhanika.,* ***2*** *(1965), 113–119.*
13. *Goldstein R.V., Bakirov V.F. and Perelmuter M.N.* 'Modeling of the cohesive strength and fracture kinetics of the microelectronic package polymer-polymer joints', *Ins. Proc. Inst. Phys. Techol. Russian Ac. of Sc.,* ***13*** *(1997), 115–125.*
14. *Mirsalimov, V.M.* 'The solution of a problem in contact fracture mechanics on the nucleation and development of a bridged crack in the hub of a friction pair', *J. of Applied Mathematics and Mechanics,* ***71*** *(1)(2007), 120–136.*