Universal Journal of Integral Equations **4 (2016), 21-29**
www.papersciences.com

Modified Taylor's Method and Nonlinear Mixed Integral Equation

R. T. Matoog
Faculty of Applied Science,
Umm Al – Qurah University
Makah, Kingdom of Saudi Arabia
rmatoog_777@yahoo.com

Abstract
In this paper**,** nonlinear mixed integral equation (**NMIE**) of type Hammerstein - Volterra integral equation (**H- VIES**) of the second kind, under certain conditions, are considered. The Hammerstein integral term is considered in a variable space with continuous kernel; while the Volterra term in time. A quadratic numerical method is used, to obtain a system of Hammerstein integral equations (**SHIEs**) of the second kind. In addition, the modified Taylor's method is applied to obtain a nonlinear algebraic system (**NAS**). Moreover, the **NAS** is solved numerically and the error estimate, in each case is computed.

2010 Mathematics Subject Classification : 45B05, 45G10.
Key Word and Phrases
Nonlinear Integral Equation, Hammerstein (Fredholm (**F**))-Volterra Integral Equation, Modified Taylor's Method (**MTM**), Nonlinear Algebraic System.

1. Introduction

Consider the NIE of the second kind:

$$\mu\varphi(x,t)=f(x,t)+\lambda\int_a^b k(x,y)\gamma(t,y,\varphi(y,t))dy+\lambda\int_0^t F(t,\tau)\varphi(x,\tau)d\tau. \qquad (1.1)$$

The above formula (1.1) is called **H- VIE** of the second kind. The Hammerstein integral term is considered in variable space with continuous kernel $k(x,y)$, while the Volterra integral term is considered in position with continuous kernel $F(t,\tau), t\in[0,T]; T<1$. The constant μ defines the kind of integral equation, while λ is a constant, may be complex, which has a physical meaning. The two functions $f(x,t)$ and $\gamma(t,x,\phi(x,t))$ are known and continuous with its derivatives, while the function $\phi(x,t)$ is unknown. If in (1.1) $\gamma(t,x,\varphi(x,t))=\varphi(x,t),$ we have the Fredholm-Volterra integral equation (**F-VIE**).

The discussion of **F-VIE** of the first kind in one, two and three dimensions, with its applications in the contact problems, was stated in [1]. In [2], [3], the author used asymptotic numerical methods to obtain the solution of **F-VIE** of the second kind. In [4], the relation between the three dimensional contact problem, in the theory of elasticity, and **F-VIE** was considered and the solution of the **IE** was obtained. In [5], the spectral relationships of the **F-VIE** of the first kind and **V-FIE**, when the kernel of position takes a generalized potential form, are discussed and obtained. More information for the physical meaning of the kernel $F(t,\tau)$ in (1.1) and the following **V-FIE**

$$\mu\varphi(x,t)=f(x,t)+\lambda\int_0^t\int_a^b k(x,y)F(t,\tau)\varphi(x,\tau)dyd\tau, \qquad (1.2)$$

can be found in [4], [5].

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for the linear versions of (1.1) and (1.2). The spline collocation method [6] and the iterative method [7] were introduced for obtaining the approximation solution. In [8], a technique based on Homotopy analysis method, is used, for solving **V-HIEs** of the first kind, while Abdou et al. [9] reduced **V-FIE** of the second kind, with discontinuous kernel to a system of **FIEs** using Toeplitz matrix method and Product Nyström method. In [10], the two-dimensional Bernstein operational matrices method, is applied to solve (1.2), while in [11], Dastjerdi et al. used the radial basis function approximation for numerical solution of mixed **V-FIEs**.

Let, in Eq. (1.1), $F(t,\tau)=0,\ t=0,$ to get

$$\mu\,\psi(x)=g(x)+\lambda\int_a^b k(x,y)\,\beta(y,\psi(y))\,dy\,,$$

$$\big(\phi(x,0)=\psi(x)\ ,\ f(x,0)=g(x)\ ,\ \gamma(0,y,\phi(y,0))=\beta(y,\psi(y))\big)\,. \tag{1.3}$$

The integral equation (1.3) is called **HIE** of the second kind. For the continuous kernel of **HIE** and $x\in[0,1]$, we follow the work of some authors who solved (1.2) numerically. Lardy, in [12], used product Nyström method; Kumar, in [13], used discrete collocation–type method. Hacia, in [14], [15] discussed the existence and uniqueness of solution of **SHIEs** in a Banach space and used projection iteration method to solve (1.3), respectively. In addition, Kaneko and Xu, in [16], used the degenerate kernel method to discuss the solution of (1.3) Bannas and Emmanuelle in [17] and [18], [19], respectively, studied the **HIE** of the second kind in $L_1[-1,1]$, where their analysis depended on the technique of noncompactness. In [20], Bugajewski proved the uniqueness theorems for bounded variation solution and continuous bounded variation of Hammerstein and **V-HIEs** in a Banach space. Abdou *et al.* in [21], used Toeplitz matrix method to obtain numerically the solution of **HIE** with discontinuous kernel.

In addition, In [22], Taylor polynomial method has been applied to obtain the approximate solution of **V-FIEs**. On the other hand, in [23] applied new basis functions for approximating the solution of nonlinear VFIEs via direct method is applied. In addition, the matrix based method, the homotopy perturbation method and the modified homotopy perturbation method have been applied for approximating the solution of nonlinear VFIEs in [24], [25] and [26], respectively. The relation between the **H-VIE** and the contact problem, in the theory of elasticity was discussed in [27], [28] and many applications were stated. More information for some different methods to solve integral equations, numerically and its applications in engineering can be found in Ladopoulos [29]-[32].

In order to guarantee the existence of a unique solution of (1.1), we assume the following conditions:

(i) The kernel $k(x,y)\in C([a,b]\times[a,b])$ has m, n derivatives with respect to x, y, respectively.

(ii) The kernel of time $F(t,\tau)\in C[0,T]$, $0\le\tau\le t\le T<1$, is a positive continuous function, and satisfies $F(t,\tau)\le M$, M is a constant, $\forall\, t,\tau\in[0,T]$.

(iii) The given function $f(x,t)$ with its partial derivatives with respect to position and time are continuous in the space $L_2[a,b]\times C[0,T]$, and its norm is defined as:

$$\|f(x,t)\|_{L_2[a,b]\times C[0,T]}=\max_{0\le t\le T}\left|\int_0^t\left\{\int_a^b|f(x,\tau)|^2dx\right\}^{\frac12}d\tau\right|=H\ ,\quad (H\ is\ a\ constant)\ .$$

(iv) The known continuous function $\gamma(t,x,\phi(x,t))$ satisfies, for the constants $A>A_1$ and $A>A_2$, the following conditions:

(a) $$\max_{0\le t\le T}\left|\int_0^t\left\{\int_a^b|\gamma(x,\tau,\varphi(x,\tau))|^2dx\right\}^{\frac12}d\tau\right|\le A_1\|\varphi(x,t)\|$$

(b) $|\gamma(t,x,\phi_1(x,t))-\gamma(t,x,\phi_2(x,t))|\le N(t,x)\,|\phi_1(x)-\phi_2(x)|$,

where $$\|N(t,x)\|_{L_2[a,b]\times C[0,T]}=\max_{0\le t\le T}\left|\int_0^t\left\{\int_a^b|N(x,\tau)|^2dx\right\}^{\frac12}d\tau\right|=A_2\ .$$

In the remainder part of this paper, a numerical method is used, in (1.1) to obtain **SHIEs** of the second kind in position. Moreover, the modified Taylor's method is used to obtain the numerical solution of **SHIEs**, where **NAS** is obtained. The existence and uniqueness of solution of the **NAS** are discussed and proved. Finally, two examples are stated to explain the method.

2. System of Nonlinear Integral Equations

To obtain **SHIEs**, we divide the interval $0 \le t \le T < 1$ *as* $0 = t_0 < t_1 < ... < t_N = T$, when $t = t_i, i = 0,1,2,...,N$. The Volterra integral term of (1.1), in this case, after using the quadrature rule formula (more information is found in [33, 34]), takes the form:

$$\int_0^{t_i} F(t_i,\tau)\phi(x,\tau)\, d\tau = \sum_{j=0}^{i} u_j F(t_i,t_j)\, \phi(x,t_j) + O(\hbar_i^{P+1}) \quad , \qquad (\hbar_i \to 0, p > 0) \tag{2.1}$$

where, $\hbar_i = \max_{0 \le j \le i} h_j$, $\quad h_j = t_{j+1} - t_j$.

The values of the two constants u_j and p depend on the number of derivatives of the function $F(t,\tau)$ with respect to t. For example, if $F(t,\tau) \in C^3[0,T]$, then we have $p = 3$, $p \approx i$ and $u_0 = \frac{1}{2} h_0$, $u_i = \frac{1}{2} h_i$, $u_j = h_j \quad (j \ne 0 , i)$. Here, $O(\hbar_i^{P+1})$ is the order of sum errors of the numerical method of dividing the interval [0, T], and the difference between the integration and summation, where the error is determined by:

$$E_i = \left| \int_0^{t_i} F(t_i,\tau)\ \varphi(x,\tau)\, d\tau - \sum_{j=0}^{i} u_j\, F(t_i,t_j)\ \varphi(x,t_j) \right|. \tag{2.2}$$

Using Eq. (2.1) in (1.1) and neglecting $O(\hbar_i^{P+1})$, we get:

$$\mu_i\, \phi_i(x) - \lambda \int_a^b k(x,y)\gamma_i(y,\phi_i(y))dy = g_i(x) \quad , \qquad (i = 0,1,2,...,N). \tag{2.3}$$

Here, we used the following notations:

$$\mu_i = \mu - \lambda u_i F_{i,i}\,,\ \varphi(x,t_i) = \varphi_i(x)\,,\ g_i(x) = f_i(x) + \lambda \sum_{j=0}^{i-1} u_j F_{i,j}\, \varphi_j(x)\,,\ f(x,t_i) = f_i(x);$$

$$\gamma_i(x,\varphi_i(x)) = \gamma(t_i, x, \varphi(x,t_i)) \tag{2.4}$$

Hence, the formula (2.3) represents **SHIEs** of finite unknown functions $\varphi_i(x)$ corresponding to the time interval $0 = t_0 < t_1 < ... < t_N = T$, and depending on the number of derivatives of $F(t,\tau)$ in $[0,T]$ with respect to time t for all values of τ.

The recurrence relations can be used to obtain the solution of the system (2.3). For this, at i = 0 we have:

$$\mu_0\, \phi_0(x) - \lambda \int_a^b k(x,y)\gamma_0(y,\phi_0(y))dy = g_0(x) \quad , \quad (g_0(x) = f_0(x)) \ . \tag{2.5}$$

3. Modified Taylor's Method

In this section, we develop the Taylor expansion method to obtain numerically the solution of (2.5). The Taylor expansion for solving **IE** had been presented by Kanwal and Liu [35] and then this had been extended, by Sezer, to **VIE** and Volterra differential equations; see [36] and [37], respectively.

Here, the technique is based on differentiating both sides of the **IE** n times. Then, substituting the Taylor polynomial for the unknown function in the resulting equation and later, transforming to

NAS. The existence and uniqueness of solution of the **NAS** are considered, and then the solution of the system will be obtained. .

Consider the solution of (2.5) takes the form:

$$\phi_0(x)=\sum_{n=0}^{r}\frac{1}{n!}\,\phi_0^{(n)}(c_0).(x-c_0)^n \qquad ; \qquad (x,c_0\in(a,b)) \ . \tag{3.1}$$

This is a Taylor polynomial of degree r at $x=c_0$, where $\phi_0^{(n)}(c_0)$, $n=0,1,2,...,r$, are coefficients to be determined.

To obtain the solution of (2.5) in the expression form (3.1), we first differentiate both sides of (2.5), n times with respect to x, to get:

$$\mu_0\,\varphi_0^{(n)}(x)-\lambda\int_a^b\frac{\partial^n k(x,y)}{\partial x^n}\,\psi_0(y)\,dy \;=\; g_0^{(n)}(x); \qquad \psi_0(y)=\gamma_0(y,\varphi_0(y)). \tag{3.2}$$

Then, we put $x=c_0$ in (3.2), and substitute Taylor expansion for $\psi_0(y)$, to have:

$$\mu_0\,\phi_0^{(n)}(c_0)-\lambda\sum_{m=0}^{r}k_{nm}\,\psi_0^{(m)}(c_0)=g_0^{(n)}(c_0) \quad , \tag{3.3}$$

where:

$$k_{nm}=\frac{1}{m!}\int_a^b\frac{\partial^n k(x,y)}{\partial x^n}\Big|_{x=c_0}(y-c_0)^m\,dy\,; \quad (m=0,1,2,...N)\ . \tag{3.4}$$

The quantities $\psi_0^{(m)}(c_0)$, in Eq. (3.3), with the aid of [32] can be found in the form:

$$\psi_0^{(m)}(c_0)=\sum_{\substack{t_1+2t_2+...+\ell t_p=m\\ t_1+t_2+...+t_p=h}}\frac{m!}{t_1!\,t_2!\,...\,t_p!}\,[\gamma_0(c_0,\phi_0(c_0))]^{(h)}\left(\frac{\phi_0'(c_0)}{1!}\right)^{t_1}\left(\frac{\phi_0''(c_0)}{2!}\right)^{t_2}...\left(\frac{\phi_0^{(\ell)}(c_0)}{\ell!}\right)^{t_p}, \tag{3.5}$$

where $t_1, t_2, ..., t_p$ are positive integers and zeros .

If we take $n,m=0,1,2,...,N$, then (3.3) represents **NAS** of $(N+1)$ equations for the (N+1) unknowns $\phi_0^{(0)}(c_0),\phi_0^{(1)}(c_0),...,\phi_0^{(N)}(c_0)$, as following:

$$\mu_0\,\phi_0^{(n)}(c_0)-\lambda\sum_{m=0}^{N}k_{nm}\,\psi_0^{(m)}(c_0)=g_0^{(n)}(c_0) \qquad , \ (n=0,1,2,...,N)\ . \tag{3.6}$$

In order to guarantee the existence of a unique solution of the **NAS** (3.6) firstly, we the lemma:

Lemma 1

If the kernel $k(x,y)$ of (2.5) possess continuous partial derivatives of all order with respect to the variable x in an open neighborhood $(c_0-\varepsilon,c_0+\varepsilon)\subset(a,b)$, then there exists a small constant E such that:

(1) $$\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\left|k_{nm}\right|^2\right)^{\frac{1}{2}}\le E \quad ,$$

where,

$$E=\left\{(b-a)\left(\sum_{n=0}^{N}\int_a^b\left|\frac{\partial^n k(x,y)}{\partial x^n}\Big|_{x=c_0}\right|^2 dy\right)\left(\sum_{m=0}^{N}\left(\frac{\varepsilon^m}{m!}\right)^2\right)\right\}^{\frac{1}{2}} \quad ; \ \varepsilon<<1 \ .$$

Proof

In view of the formula (3.5) we have:

$$\left|k_{nm}\right|^2\le\left\{\frac{1}{m!}\int_a^b\left|\frac{\partial^n k(x,y)}{\partial x^n}\Big|_{x=c_0}\right|\,\left|(y-c_0)^m\right|dy\right\}^2 \quad .$$

Applying Cauchy-Schwarz inequality and summing from $m,n=0$ to $m,n=N$, the above inequality for each $y\in(c_0-\varepsilon,c_0+\varepsilon)$ can be adapted in the form:

$$\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\left|k_{nm}\right|^2\right)^{\frac{1}{2}}\le\left\{(b-a)\left(\sum_{n=0}^{N}\int_a^b\left|\frac{\partial^n k(x,y)}{\partial x^n}\Big|_{x=c_0}\right|^2 dy\right)\left(\sum_{m=0}^{N}\left(\frac{\varepsilon^m}{m!}\right)^2\right)\right\}^{\frac{1}{2}}=E.\bullet \qquad (3.7)$$

Theorem 1

Under the condition (1) of Lemma 1, and the following conditions:

(2) $\left(\sum_{n=0}^{N}\left|g_0^{(n)}(c_0)\right|^2\right)^{\frac{1}{2}}\le G$, (G *is a constant*) .

(3) The known functions $\gamma_0^{(n)}(c_0,\phi_0(c_0))$ for the constants $Q>Q_1$, $Q>Q_2$ satisfy

(3.a) $\left(\sum_{n=0}^{N}\left|\gamma_0^{(n)}(c_0,\phi_0(c_0))\right|^2\right)^{\frac{1}{2}}\le Q_1\left(\sum_{n=0}^{N}\left|\phi_0^{(n)}(c_0)\right|^2\right)^{\frac{1}{2}}$,

(3.b) $\left(\sum_{n=0}^{N}\left|\gamma_0^{(n)}(c_0,\varphi_0(c_0))-\gamma_0^{(n)}(c_0,\xi_0(c_0))\right|^2\right)^{\frac{1}{2}}\le Q_2\left(\sum_{n=0}^{N}\left|\varphi_0^{(n)}(c_0)-\xi_0^{(n)}(c_0)\right|^2\right)^{\frac{1}{2}}$.

the **NAS (**3.6**)** has a unique solution.

Proof

To prove the theorem, we write the **NAS** (3.6) in the operator form:

$$\bar{L}\varphi_0^{(n)}(c_0)=\frac{1}{\mu_0}g_0^{(n)}(c_0)+L\,\varphi_0^{(n)}(c_0), \qquad (3.8)$$

Where:

$$L\,\varphi_0^{(n)}(c_0)=\frac{\lambda}{\mu_0}\sum_{m=0}^{N}k_{nm}\,\gamma_0^{(m)}(c_0,\varphi_0(c_0)), \qquad (n=0,1,2,...,N), \qquad (3.9)$$

Lemma 2

Under the conditions (1) – (3), the operator $\bar{L}$ defined by (3.6) maps the space ℓ_2 into itself.

Proof

Let Y be the set of all functions $\Phi_0=\left\{\phi_0^{(n)}(c_0)\right\}$ in ℓ_2, such that $\left\|\Phi_0\right\|_{\ell_2}\le\beta$. Define the norm of the operator $\bar{L}\Phi_0$ in ℓ_2 by:

$$\left\|\bar{L}\,\Phi_0\right\|_{\ell_2}=\left(\sum_{n=0}^{\infty}\left|\bar{L}\,\varphi_0^{(n)}(c_0)\right|^2\right)^{\frac{1}{2}} . \qquad (3.10)$$

From (3.8) and (3.9) , we get :

$$\left|\bar{L}\varphi_0^{(n)}(c_0)\right|^2\le\left[\frac{\left|g_0^{(n)}(c_0)\right|}{\left|\mu_0\right|}+\left|\frac{\lambda}{\mu_0}\right|\sum_{m=0}^{N}\left|k_{nm}\right|\left|\gamma_0^{(m)}(c_0,\varphi_0(c_0))\right|\right]^2 .$$

Applying Cauchy-Schwarz inequality, then using the conditions (3), and summing from $n=0$ to $n=N$, we obtain:

$$\left(\sum_{n=0}^{N}\left|\bar{L}\,\varphi_0^{(n)}(c_0)\right|^2\right)^{\frac{1}{2}}\le\left\{\sum_{n=0}^{N}\left[\frac{\left|g_0^{(n)}(c_0)\right|}{\left|\mu_0\right|}+\left|\frac{\lambda}{\mu_0}\right|Q\left(\sum_{m=0}^{N}\left|k_{nm}\right|^2\right)^{\frac{1}{2}}\left(\sum_{m=0}^{N}\left|\varphi_0^{(m)}(c_0)\right|^2\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}} .$$

The above inequality, after applying Cauchy-Schwarz inequality, using the conditions (1), (2), and letting $N \to \infty$, takes the form:

$$\left\| \bar{L}\,\Phi_0 \right\|_{\ell_2} \le \frac{G}{|\mu_0|} + \sigma_1 \left\| \Phi_0 \right\|_{\ell_2}, \left(\sigma_1 = \left|\frac{\lambda}{\mu_0}\right| Q\, E\right). \tag{3.11}$$

In view of the inequality (3.11), the operator $\bar{L}$ maps the set Y into itself , where:

$$\beta = \frac{G}{[\,|\mu_0| - |\lambda|\,E\,Q\,]} \quad . \tag{3.12}$$

Since $\beta > 0, G > 0,$ then $\sigma_1 < 1$. In addition, from the inequality (3.11) , we can deduce the boundedness of the operator L , where:

$$\left\| L\,\Phi_0 \right\|_{\ell_2} \le \sigma_1 \left\| \Phi_0 \right\|_{\ell_2} \quad . \tag{3.13}$$

Moreover, the inequalities (3.12) and (3.13) involve the boundedness of the operator $\bar{L}$.

Lemma 3

Under the two conditions (1) and (3-b) , $\bar{L}$ is a contraction operator in the space ℓ_2 .

Proof

Let $\Phi_0 = \left\{ \phi_0^{(n)}(c_0) \right\}$ and $\Xi_0 = \left\{ \xi_0^{(n)}(c_0) \right\}$ be any functions in the space ℓ_2, then in the light of formulas (3.8) and (3.9), we obtain:

$$\left| \bar{V}\,\phi_0^{(n)}(c_0) - \bar{V}\,\xi_0^{(n)}(c_0) \right|^2 \le \left[\left|\frac{\lambda}{\mu_0}\right| \sum_{m=0}^{N} \left| k_{nm} \right| \left| \gamma_0^{(m)}(c_0, \phi_0(c_0)) - \gamma_0^{(m)}(c_0, \xi_0(c_0)) \right| \right]^2 .$$

Applying Cauchy-Schwarz inequality, then summing from $n = 0$ to $n = N$, and using the conditions (1) , (3-b) , the above inequality takes the form:

$$\left(\sum_{n=0}^{N} \left| \bar{L}\,\varphi_0^{(n)}(c_0) - \bar{L}\,\xi_0^{(n)}(c_0) \right|^2 \right)^{\frac{1}{2}} \le \sigma_1 \left(\sum_{m=0}^{N} \left| \varphi_0^{(m)}(c_0) - \xi_0^{(m)}(c_0) \right|^2 \right)^{\frac{1}{2}} .$$

Finally, as $N \to \infty$, the last inequality reduces to

$$\left\| \bar{L}\,\Phi_0 - \bar{L}\,\Xi_0 \right\|_{\ell_2} \le \sigma_1 \left\| \Phi_0 - \Xi_0 \right\|_{\ell_2} \quad . \tag{3.14}$$

Inequality (3.14) shows the continuity of the operator $\bar{L}$ in the space ℓ_2, then $\bar{L}$ is a contraction operator under the condition $\sigma_1 < 1$. Hence, by Banach Fixed Point Theorem $\bar{L}$ has a unique fixed point which is the unique solution of the **NAS** (3.6).

It is obvious that as $N \to \infty$, the system of **NIEs** (3.3) is equivalent to the integral equation (1.1), and consequently the solution is the same.

4. Applications

In this section, we apply the **MTM** to solve the **NIE** of **H-VIE** of the second kind.

Example1. Consider the **H-VIE** of the second kind:

$$\varphi(x,t) - \int_0^1 (x^2 y + x\,y^2)\left[\varphi(y,t)\right]^2 dy - \int_0^t \tau\,\varphi(x,\tau)\,d\tau = f\,(x,t). \quad \{\varphi(x,t) = x^2 - 1 + t\}. \tag{4.1}$$

The free term $f\,(x,t)$ after using the exact solution, yields:

$$f\,(x,t) = [(\frac{5}{6} + \frac{1}{2}t - t^2)x^2 - (\frac{8}{105} - \frac{4}{15}t + \frac{1}{3}t^2)x - (1 - t - \frac{1}{2}t^2 + \frac{1}{3}t^3)]$$

In Table1, for $x \in [0, 0.9]$,$t \in [0, 0.4]$, the numerical computational results of the exact and approximate solution of (4.1) are calculated. In addition, in Figure 1, the relation between the exact solution and the numeric solution for all $x \in [0,1]$, $t \in [0,1)$ is computed.

Example 2: Consider the **H -VIE** of the second kind

$$\varphi(x,t)-\int_0^1[\varphi(y,t)]^3\,dy-\int_0^t(t-\tau)\,\varphi(x,\tau)\,d\tau=(1+t)e^x-\left(\frac{e^3-1}{3}\right)e^{3t}\,\{\varphi(x,t)=e^{x+t}\}. \quad (4.2)$$

The exact and approximate solutions of (4.2) are obtained numerically in Table 2, and Figure 2, with different values of $x\in[0,0.9],\ t\in[0,0.4].$ It can observe from the table that: The error is 0 for $t=0$ at $x=0$. The approximate solution is nearly coincident with the exact solution for $t>0$ at each value of x in the table.

Table 1

x_k	$\varphi_0^E(x_r)_{t=0.0}$	$\varphi_0^N(x_r)$	$\varphi_2^E(x_r)_{t=0.2}$	$\varphi_2^N(x_r)$	$\varphi_3^E(x_r)_{t=0.3}$	$\varphi_3^N(x_r)$	$\varphi_4^E(x_r)_{t=0.4}$	$\varphi_4^N(x_r)$
0.00	-1.000	-1.000	-0.9999999	-0.9999999	-099999998	-0.9999997	-0.9999996	0.9999995
0.08	0.99999	0.99999	-0.9999995	-0.9999994	-0.9999994	-0.9999993	-0.9999993	-0.9999993
0.1	0.99999	-0.99999	-0.9999991	-0.9999990	-0.9999990	-0.9999989	-0.9999988	-0.9999987
0.2	-0.999997	0.999997	-0.9999971	-0.9999970	-0.9999969	-0.9999998	-0.9999967	-0.9999966
0.4	-0.999995	0.999995	-0.9999957	-0.9999956	-0.9999956	-0.9999955	-0.9999955	-0.9999953
0.6	-0.999993	0.999993	-0.9999930	-0.9999929	-0.9999928	-0.9999927	-0.9999927	-0.9999926
0.8	-0.999991	0.999991	-0.9999918	-0.9999917	-0.9999917	-0.9999916	-0.9999915	-0.9999915
0.9	-0.999991	0.999991	-0.9999913	-0.9999913	-0.9999912	-0.9999912	-0.9999911	-0.9999911

In Table 1 we calculate the analytical and approximate solution at different times.
The relation between the exact solution and the numeric solution is obtained.
In Table 2, and Fig. 2, the analytic and approximate solution are computed at time t=0, t=0.2, t=0.3 and t=0.4 for $x\in[0,0.9]$.

Table 2

x_k	$\varphi_0^E(x_r)_{t=0}$	$\varphi_0^N(x_r)$	$\varphi_2^E(x_r)_{t=0.2}$	$\varphi_2^N(x_r)$	$\varphi_3^E(x_r)_{t=0.3}$	$\varphi_3^N(x_r)$	$\varphi_4^E(x_r)_{t=0.4}$	$\varphi_4^N(x_r)$
0.00	1.000	1.0000	1.0000009	1.0000009	1.0000009	1.0000009	1.0000009	1.0000009
0.01	1.0000008	1.0000008	1.0000007	1.0000006	1.0000006	1.0000006	1.0000005	1.0000004
0.4	1.0000001	1.0000001	1.0000099	1.0000098	1.0000098	1.0000097	1.0000096	1.0000096
0..5	1.0000090	1.0000089	1.0000089	1.0000088	1.0000087	1.0000086	1.0000085	1.0000084
0.6	1.0000079	1.0000079	1.0000078	1.0000077	1.0000076	1.0000075	1.0000074	1.0000074
0.8	1.0000070	1.0000069	1.0000068	1.0000067	1.0000066	1.0000065	1.0000064	1.0000063
0.9	1.0000058	1.0000057	1.0000056	1.0000055	1.0000054	1.0000053	1.0000052	1.0000052

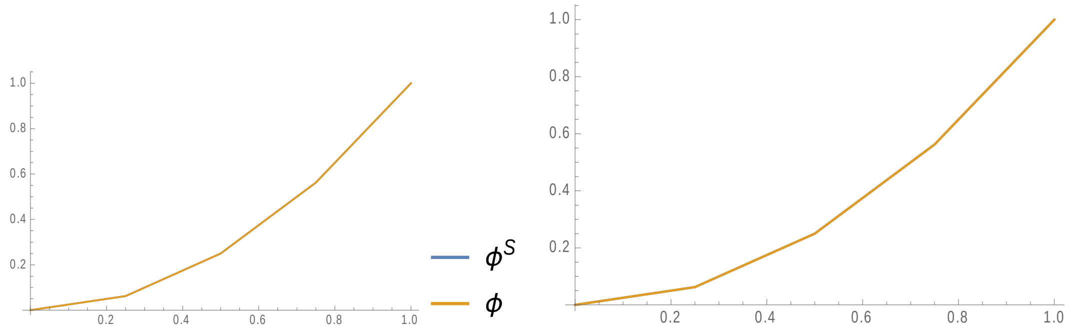

Fig.1 Fig.2

5. Conclusions

From the above discussion and results, we can deduce the following:
1- We consider a general mixed integral equation in the nonlinear form, in time and position.
2- Most the integral equations in the references are considered special cases of this paper.
3- The modified Taylor's method is considered as the best methods to obtain the solution of the **NMIE** with continuous kernel, numerically.

Future work

In future work, the solution of the general form of **MIE** in the nonlinear form will be considered, especially, when the kernel of position takes the discontinuous form.

References

1. *Abdou M. A.*, 'Fredholm – Volterra integral equation of the first kind in contact problem', *Appl. Math. Comput.*, ***125*** *(2002), 177-193.*
2. *Abdou M. A.*,'On asymptotic method for Fredholm – Volterra integral equation of the second kind in contact problem', *J. Com. Appl. Math.*, ***154*** *(2003), 431 – 446.*
3. *Abdou M. A.*, 'Fredholm – Volterra integral equation with singular kernel', *Appl. Math. Comput.*, ***127*** *(2003), 231 – 243.*
4. *Abdou M. A., Mustafa O. L.*, 'Fredholm – Volterra integral equation in contact problem', *Appl. Math. Comput.*, ***138*** *(2003), 199 – 215.*
5. *Abdou M. A.*, 'Fredholm–Volterra integral equation and generalized potential kernel', *Appl. Math. Comput.*, ***131*** *(2002), 81-94.*
6. *Brunner H.*, 'On the numerical solution of the nonlinear Volterra-Fredholm integral equations by collocation methods', *Siam J. Numer. Anal.*, ***27*** *(1990), 987-1000.*
7. *Wang K., Wang O., Guan K.*, 'Iterative method and convergence analysis for a kind of mixed nonlinear Volterra- Fredholm integral equation', *Appl. Math. Compute.*, ***225*** *(2013), 631-637.*
8. *Behzadi Sh. S.*, 'Homotopy approximation technique for solving nonlinear Volterra-Fredholm integral equations of the first kind', *Int. J. Ind. Math.*, ***6*** *(2014), 6-12.*
9. *Abdou M.A., El-Kalla I.L., Al-Bugami A.M.*, 'Numerical solution for Volterra-Ferdholm integral equation with a generalized singular kernel', *J. Mod. Meth. Numer. Math.*, ***2*** *(2011), 1-15.*
10. *Shekarabi F.H., Maleknejad K., Ezzati R.*, 'Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations', *Afr. Mat.*, ***6*** *(2014), 20-27.*
11. *Dastjerdia H.L., Ghainia F.M.M., Hadizadeh M.*, 'A meshless approximate solution of mixed Volterra-Fredholm integral equations', *Inter. J. Compute. Math.*, ***90*** *(2013), 527-538.*
12. *Lardy L. J.*, 'A variation of Nyström's method for Hammerstein integral equations', *Math. Comput.*, ***48*** *(1981), 43 – 60.*
13. *Kumar S.*, 'A discrete collocation – type method for Hammerstein equations', *SIAM, J Numer. Anal.*, ***25*** *(1988), 328 – 341.*
14. *Hacia L.*, 'Approximate solutions of the Hammerstein integral equations', *Fasciculi Mathematici*,***18*** *(1988), 42 – 48.*
15. *Hacia L.*, 'Solving nonlinear integral equations by projection iteration methods', *Nonlin. Vib. Probl.*, ***25*** *(1993), 135 – 141.*
16. *Kaneko H., Xu* Yu., 'Degenerate kernel method for Hammerstein equations', *Math. Comput.*, ***56*** *(1991), 141 – 148.*

17. *Bannas J.*, 'Integrable solution of Hammerstein and Urysohn integral equations', *J. Aust. Math. Soc. A, **46** (1989), 61 – 68.*
18. *Emmanuele G.*, 'Integrable solutions of functional integral equations', *J. Int. Eq. Appl., **4** (1992), 89 – 94.*
19. *Emmanuelle G.*, 'Integrable solutions of Hammerstein equations', *Appl. Anal., **50**, (1993), 277 – 284.*
20. *Bugajewski D.*, 'On BV – solutions of some nonlinear integral equation', *Integral. Eq. Oper. Theory, **46** (2003), 387 – 398.*
21. *Abdou M. A., El- Borai M. M., El-Kojak M. M.*, 'Toeplitz matrix method and nonlinear integral equation of Hammerstein type', *J. Cam. Appl. Math., **223** (2009), 765-776.*
22. *Wang K., Wang Q.* 'Taylor polynomial method and error estimation for a kind of mixed Volterra-Fredholm integral equations', *Appl. Math. Comput., **229** (2014), 53-59.*
23. *Paripour M., Kamyar M.*, 'Numerical solution of nonlinear Volterra-Fredholm integral equations by using new basis functions', *Commun. Numer. Anal., **2013** (2013), 1-11.*
24. *Hussein S.A., Shahmorad S.*, Talati F., 'A matrix based method for two dimensional nonlinear Volterra-Fredholm integral equations', *Numer. Algor., **4** (2014),.8-16.*
25. *Ghasemi M., Kajani M.T., Davari A.*, 'Numerical solution of two-dimensional nonlinear differential equation by homotopy perturbation method', *Appl. Math. Comput., **189** (2007),341-345.*
26. *Dong C., Chen Z., JiangW.*, 'A modified homotopy perturbation method for solving the nonlinear mixed Volterra-Fredholm integral equation', *J. Comput. Appl. Math., **239** (2013), 359-366.*
27. *Abdou M. A.*, 'On the solution of linear and nonlinear integral equation', *Appl. Math. Comput., **146** (2003), 857 – 871.*
28. *Abdou M. A., El-Sayed W. G., Deebs E. I.*, 'A solution of nonlinear integral equation', *Appl. Math. Comput., **160** (2005), 1-14.*
29. *Ladopoulos E.G.*, 'Non-linear integro-differential equations in sandwich plates stress analysis', *Mech. Res. Commun., **21** (1994), 95 – 102.*
30. *Ladopoulos E.G.*, 'Non-linear singular integral representation for unsteady inviscid flowfields of 2-D airfoils', *Mech. Res. Commun., **22** (1995), 25 – 34.*
31. *Ladopoulos E.G.*, 'Non-linear multidimensional singular integral equations in 2-dimensional fluid mechanics analysis', *Int. J.Non-Lin. Mech., **35** (2000), 701 – 708.*
32. *Ladopoulos E.G. and Zisis V.A.*,'Existence and uniqueness for non-linear singular integral equations used in fluid mechanics', *Appl. Math., **42** (1997), 345 – 367.*
33. *Delves L. M., Mohamed J. L., 'Computational Methods for Integral Equations'*, Cambridge Univ. Press, Cambridge, *1985*.
34. *Golberg M. A.. ed*, 'Numerical Solution for Integral Equations', Plenum Press, New York , *1990.*
35. *Kanwal R. P., Liu K. C.*, 'A Taylor expansion approach for solving integral equations', *Int. J. Math. Educ. Sci. Technol., **20** (1989), 411-420.*
36. *Sezer M.*, 'Taylor polynomial solution of Volterra integral equations', *Int. J. Math. Educ. Sci. Technol. **25** (1994), 625-632.*
37. *Sezer M.*, 'A method for the approximate solution of the second-order linear differential equations in terms of Taylor polynomials', *Int. J. Math. Educ. Sci. Technol., **27** (1996), 821-829.*