Universal Journal of Integral Equations **3 (2015), 22-32**
www.papersciences.com

Spectral Relationships of Mixed Integral Equation with Potential Kernel in Different Domains

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Abstract
Here, the solution of the mixed integral equation (**MIE**) of the first kind in the space $L_2[\ \Omega\ ]\times C[\ 0,T\ ], T<1,$ is established in the form of spectral relationships (**SRs**), using three different analytic methods, depending on the domain of integration Ω. Also, Ω is considered in the form: $\Omega=\{\ |x|<\infty,\ |y|<a, z>0\},\ \{0<x<\infty, |y|<\infty, z<0\},$ and $\left\{\sqrt{x^2+y^2}\le a, z=0\right\}$. The position kernel is considered in the form of potential function. The importance of **SRs** in mathematical physics and in contact problems, depending on the domain of integration and the harmonic degree of the potential kernel, is explained. Most works of the previous authors in this domain are considered, now as special cases of this work. Moreover, many new **SRs** are established and many special cases are considered.

2010 Mathematics Subject Classification: 45B05, 45R10
Key Word and Phrases
Mixed Integral Equation, System of Integral Equations (**SIEs**), Mathieu Function, Potential Kernel, Orthogonal Polynomial, Krein's Method, Boundary Value problem (**BVP**).

1. Introduction

Singular integral equations of the first kind have received considerable interest in mathematical literatures, because of their many field of applications in different areas of sciences, see [1]-[5]. The solution of these equations can be obtained analytically using Cauchy method [6], potential theory method [7], [8], orthogonal polynomials method [9], Fourier transformation method [10] and Krein's method [11]. While, for the singular **IEs** of the second kind, where the analytic methods often fail, we use numerical methods such as: Galerkin method, fast method, block by block method, Nyström method and Toeplitz matrix method. The interested reader should consult the fine expositions for the numerical methods in [12] - [15]. Many spectral relationships with its applications in the contact problems and mathematical physics, for the **FIE** with discontinuous kernels are obtained and discussed in [16-20].

Consider the **MIE** of the first kind

$$\int_0^t\int_\Omega F(|t-\tau|)k(x-\xi,y-\eta)\Phi(\xi,\eta,\tau)d\xi d\eta d\tau=\pi\theta(\gamma(t)-g(x,y))=f(x,y,t)$$

$$k(x-\xi,y-\eta)=\left[(x-\xi)^2+(y-\eta)^2\right]^{-\frac{1}{2}};\quad \left[\theta=G(1-v)^{-1}\right] \tag{1.1}$$

under the dynamic condition

$$\int_\Omega\Phi(x,y,t)dx\,dy=P(t),\qquad t\in[0,T],T<1. \tag{1.2}$$

The **IE** (1.1), under the condition (1.2), can be investigated from the contact problems of rigid surface (G,v) having an elastic material occupying the domain *Ω,* when the stamp is impressed into an elastic layer surface of equation $g(x,y)$ by a variable known force $P(t)$, $t\in[0,T],T<1$, whose eccentricity of application $e(t)$, that case rigid displacement $\gamma(t)$. Furthermore, G is the displacement magnitude; v is the Poisson's coefficient. The unknown function $\Phi(x,y;t)$

represents the resultant of the normal stresses between the stamp and the elastic layer. The given function $F(|t-\tau|)$ represents the resistance force of the elastic material in the contact domain Ω through the time t.

In order to guarantee the existence of a unique solution of (1.1), under the condition (1.2), we assume the following conditions:

(i) The kernel of position $k(x-\xi, y-\eta)$ satisfies in $L_2(\Omega)$ the discontinuity condition

$$\left\{\int_\Omega k^2(x-\xi, y-\eta)dx\,d\xi\,dy\,d\eta\right\}^{\frac{1}{2}} = A, \qquad (A \text{ is a constant}).$$

(ii) The given function $f(x, y, t) \in L_2(\Omega) \times C[0,T]$; $T < 1$.

(iii) The kernel of time $F(|t-\tau|) \in C([0,T]\times[0,T])$, is positive and continuous with its derivatives and satisfies $F(|t-\tau|) < B$, B is a positive constant, for all values $t, \tau \in [0,T]$. Also, $\gamma(t) \in C[0,T]$.

(iv) The unknown function $\Phi(x, y; t)$ satisfies Hölder condition with respect to time and Lipcshitz condition with respect to position.

To obtain **SIEs**, we divide the interval [*0,T*], as $0 = t_0 < t_1 < \ldots < t_N = T$, where $t = t_\ell$, $\ell = 0,1,2,\ldots,N$, then the **MIE** (1.1) takes the form

$$\sum_{j=0}^{\ell} u_j F_{j,l} \int_\Omega \frac{\Phi_j(\xi,\eta)d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} = f_\ell(x, y). \tag{1.3}$$

So, we used the following notations $\Phi(x, y; t_\ell) = \Phi_\ell(x, y)$, $F(|t_\ell - t_j|) = F_{\ell,j}$, $F(0) \neq 0$, $f(x, y; t_\ell) = f_\ell(x, y)$. Also we neglected the error term of $0(h_l^{p+l})$, where $h_l = \max_{0\le j\le l} h_j$, $(h_j = t_{j+1} - t_j)$. The values of the characteristic numbers u_j and the constant *p* depend on the number of the derivatives of $F(|t-\tau|)$ with respect to the time *t*, see [12, 13].

Besides, the condition (1.2) yields:

$$\int_\Omega \Phi_\ell(x, y)dx\,dy = P_\ell \tag{1.4}$$

The formula (1.3) represents **SIEs** of the first kind with potential kernel and its solution depends on the domain of integration *Ω* and the condition (1.4).

Additionally, in the remainder part of current research, we establish and discuss three main theorems for **SRs** of the **SIEs** of the first kind with potential kernel, using suitable methods depending on the domain of integration. These methods depend on the domain of integration Ω. In addition, in certain domain of integration, the Weber-Sonien integral formula can be established. Moreover, many important special and new cases especially, when the kernel takes the logarithmic form, Carleman function and elliptic function form are established and derived from this work.

The importance of this work, will lead directly to obtain the solution of any singular **IE** of the second kind, using the suitable theorem.

2. Potential Theory Method

Theorem 1

In terms of Mathieu functions and Macdonald kernel $K_0|.|$ of zero order, where the domain of integration is $\Omega_1 = \{(x, y, z) \in \Omega_1 : -\infty < x < \infty,\ |y| < a,\ z > 0\}$; the **SRs** of **SIEs** (1.3), under the condition (1.4) are:

$$\sum_{j=0}^{l} u_j F_{j,l} \int_{-a}^{a} \frac{Ce_{n_j}\left[\cos^{-1}\frac{\xi}{a}, -q\right]}{\sqrt{a^2-\xi^2}} K_0\left(\alpha\left|\xi - x\right|\right) d\xi = \pi \sum_{j=0}^{l} \frac{u_j F_{j,l} F_{e_j} k_{n_j}(0,-q)}{F_{e_j} k'_{n_j}(0,-q)} Ce_{n_j}\left[\cos^{-1}\frac{x}{a}, -q\right]$$

Proof

By using the potential theory method [7, 8], to discuss the solution of (1.3), under the condition (1.4) in the domain Ω_1, we introduce the simple layers potential of densities $\Phi_j(x,y)$ distributed over the domain *Ω*:

$$U_\ell(x,y,z) = \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{\Omega_1} \frac{\Phi_j(\xi,\eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}. \tag{2.1}$$

The functions $U_\ell(x,y,z)$, $\ell = 0,1,2,\dots,N$ are harmonic everywhere except in the domain *Ω* and its vanish at infinity as $N_\ell R^{-1}\left(N_\ell \text{ are constants}, \ \ell = 1,2,\dots,N\ ;\ R = \sqrt{x^2+y^2+z^2}\right)$. Moreover the functions $U_\ell(x,y,z)$ are continuous in all space including the domain Ω_1 and its normal derivative takes the form:

$$\left(\frac{\partial U_\ell}{\partial z}\right)_{z\to 0} = \begin{cases} \mp\pi\Phi_\ell(x,y), & (x,y)\in\Omega_1, \\ 0 & (x,y)\notin\Omega_1. \end{cases} \tag{2.2}$$

By forming the two formulas (1.3) and (2.1), we can set the following boundary condition:

$$U_\ell(x,y;0) = f_\ell(x,y), \quad (x,y)\in\Omega_1 \tag{2.3}$$

From the above we deduce that the solution of the **SIEs** (1.3) is equivalent to the **BVP** (2.1)-(2.3) of determining the harmonic functions$U_\ell(x,y,z)$. Therefore, we use the following Fourier integral transformations, see [21]:

$$\Phi_\ell(\alpha,x) = \int_{-\infty}^{\infty} \Phi_\ell(x,y) e^{i\alpha y}\, dy; \quad f_\ell(\alpha,x) = \int_{-\infty}^{\infty} f_\ell(x,y) e^{i\alpha y}\, dy, \tag{2.4}$$

and the Macdonald kernel of zero order, see[22]

$$K_0\left(|\alpha||x-\xi|\right) = \int_0^\infty \frac{\cos\alpha y \cdot dy}{\sqrt{(x-\xi)^2+y^2}} \tag{2.5}$$

where *α* is the Fourier parameter, and $K_0\left(\left|\cdot\right|\right)$ is the Macdonald kernel, to get:

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} K_0\left(\alpha|x-\xi|\right)\Phi_j(\alpha,\xi)\, d\xi = f_\ell(\alpha,x) \tag{2.6}$$

The formula (2.6) represents **SFIEs** that can be reduced, by using potential theory method, to the following **BVP:**

$$\frac{\partial^2 U_\ell}{\partial x^2} + \frac{\partial^2 U_\ell}{\partial z^2} - \alpha U_\ell(\alpha,x,z) = 0, \ (x\notin\Omega_1, z\neq 0); \ \ U_\ell(\alpha,x,0) = f_\ell(\alpha,x)$$

$$\frac{\partial U_{\ell}\left(\alpha,x,\pm 0\right)}{\partial z}=\begin{cases}\mp\pi\sum_{j=0}^{\ell}u_j F_{j,\ell}\Phi_j\left(\alpha,x\right), & \left|x\right|<a,\ \ell=0,1,2,\ldots,N\\ 0 & \left|x\right|>a\end{cases}$$

(2.7)

and

$$U_l\left(\alpha,x,z\right)=\sum_{j=0}^{l}u_j F_{j,l}\int_{-a}^{a}K_0\left[\alpha\sqrt{\left(x-\xi\right)^2+z^2}\right]\Phi_j\left(\lambda,\xi\right)d\xi$$

By using the following local coordinate system x, z in the elliptic cylindrical coordinates of (2.7):

$$x=a\cosh\xi\cos\eta,\ z=a\sinh\xi\sin\eta,\quad\left(0\le\eta\le 2\pi,0\le\xi<\infty\right),\qquad(2.8)$$

we obtain the following **BVP:**

$$\frac{\partial^2\tilde{U}_{\ell}}{\partial\xi^2}+\frac{\partial^2\tilde{U}_{\ell}}{\partial\eta^2}-\frac{a\alpha^2}{2}\left(\cosh 2\xi-\cos 2\eta\right)\tilde{U}_{\ell}=0,\quad\ell=0,1,2,\ldots,N$$

$$\left(U_{\ell}\left(\alpha,x,z\right)=U_{\ell}\left(\alpha,a\cosh\xi\cos\eta,a\sinh\xi\sin\eta\right)=\tilde{U}_{\ell}\left(\alpha,\xi,\eta\right)\right)$$

(2.9)

and: $\tilde{U}_{\ell}\left(\alpha,0,\eta\right)=f_{\ell}\left(\alpha,a\cos\eta\right),\ f_{\ell}\left(\alpha,x\right)=f_{\ell}\left(\alpha,a\cos\eta.\cosh\xi\right);\ \ \tilde{U}_{\ell}\left(\alpha,\xi,0\right)=0.$

while the second formula of (2.7) becomes:

$$\sum_{j=0}^{\ell}u_j F_{j,\ell}\Phi_j\left(\alpha,a\cos\eta\right)=\frac{1}{a\left|\sin\eta\right|}\frac{\partial\tilde{U}_{\ell}}{\partial\xi}\Big|_{\xi=0}\quad\left(\text{for all values of }\ \eta\in\left[0,2\pi\right]\right)$$

$$\left(\Phi_{\ell}\left(\alpha,x\right)=\Phi_{\ell}\left(\alpha,a\cosh\xi\cos\eta\right)\right)\qquad(2.10)$$

The formula (2.10) is called the equivalent condition between the differential system and the integral system in the potential theory method, see [18].

To solve the **BVP** (2.9), we expand the function $f_{\ell}\left(\alpha,a\cos\eta\right)$ to a uniform convergent series of periodic Mathieu function, see [23]:

$$f_{\ell}\left(\alpha,a\cos\eta\right)=\sum_{i=0}^{\infty}\gamma_{i\ell}C_{e_{i\ell}}\left(\eta,-q\right),\qquad\left[0\le\eta\le 2\pi;\ q=\frac{\alpha^2a^2}{4}\right].\qquad(2.11)$$

Then, assume the solution of (2.9) in the form $\tilde{U}_{\ell}\left(\alpha,\xi,\eta\right)=V_{\ell}\left(\xi\right)W_{\ell}\left(\eta\right)$. By following the general method of theory of Mathieu function [22], the solution of the **BVP** (2.9) takes the following form:

$$\tilde{U}\left(\alpha,\xi,\eta\right)=\sum_{n=0}^{\infty}r_{\ell n}F_{e_{\ell}}k_{n_{\ell}}\left(\xi,-q\right)C_{e_{n_{\ell}}}\left(\eta,-q\right)\qquad(2.12)$$

The coefficients $r_{\ell n}$ are found from the set of infinite system:

$$r_{i\ell}+\sum_{s\ne i}^{n}\sum_{n=0}^{\infty}r_{sn}T_{\ell n}\left(s,i\right)=\frac{\gamma_{i\ell}}{F_{e_{\ell}}k_n\left(\xi,-q\right)},\quad\left(T_{\ell n}\left(s,i\right)=\frac{Q_{n\ell}\left(s,i\right)Ce_{\ell}\left(0,-q\right)}{F_{e_{\ell}}k_{\ell}\left(0,-q\right)}\right).\qquad(2.13)$$

where $\gamma_{i\ell}$ are the coefficients of expansion of the function (2.11).

Differentiating (2.12) and using the result in (2.10), we obtain:

$$\sum_{j=0}^{l} u_j F_{j,l} \Phi_j\left(\alpha, a\cos\eta\right) = \frac{1}{a\left|\sin\eta\right|}\sum_{n=0}^{\infty}\gamma_{jn}\frac{F_{e_j}k'_{n_j}\left(0,q\right)}{F_{e_j}k_{n_j}\left(0,-q\right)}Ce_{nj}\left(\eta,-q\right) \tag{2.14}$$

By using (2.14) in (2.6), we arrive to the theorem1◘

3. Orthogonal Polynomials Method

Consider the following domain $\left\{\left(x,y,z\right)\in\Omega_2 : 0 < x < \infty, -\infty < y < \infty, z < 0\right\}$, then the **SIEs** (1.3), under the condition (1.4), yield:

$$\sum_{j=0}^{\ell} u_j F_{j,\ell}\int_0^{\infty}\int_0^{\infty}\frac{\Phi_j\left(\xi,\eta\right)d\xi\, d\eta}{\sqrt{\left(x-\xi\right)^2+\left(y-\eta\right)^2}} = f_\ell\left(x,y\right) \tag{3.1}$$

under the condition

$$\int_0^{\infty}\int_0^{\infty}\Phi_\ell\left(x,y\right)dx\, dy = P_\ell \tag{3.2}$$

Theorem 2

In terms of Chebyshev- Laguerre polynomials $L_{nl}^{\frac{1}{2}}\left(.\right)$, Macdonald function $K_0\left|.\right|$ of order zero, and the domain Ω_2; the **SRs** of **SFIEs** (3.1) under the condition (3.2) are given in the form:

$$\sum_{j=0}^{\ell} u_j F_{j,\ell}\int_0^{\infty} K_0\left(\alpha\left|u-v\right|\right)\frac{L_{n_j}^{-\frac{1}{2}}\left(2u\right)}{\sqrt{u_j e^{u}}}du = \sum_{j=0}^{\ell} u_j F_{j,\ell}\lambda_{n_j} e^{-v} L_{n_j}^{-\frac{1}{2}}\left(2v\right); \left(\lambda_{n_\ell} = \frac{\pi\Gamma\left(n_l+\frac{1}{2}\right)}{\sqrt{2}\left(n_l\right)!};\right)$$

$$\left(n = 0,1,2,\ \ell = 0,1,\ldots,N\right).$$

Proof

By using the potential theory method is very difficult and may fail. Thus, our aim is constructing the general solution of the **SIEs** (3.1), under the condition (3.2), using orthogonal polynomials method. For this, we use the following Fourier integral transformation:

$$G\left(\alpha,x\right) = \int_{-\infty}^{\infty} G\left(x,y\right)e^{i\alpha y}\, dy\ . \tag{3.3}$$

So, the **SIEs** (3.1) and the condition (3.2) yield:

$$\sum_{j=0}^{l} u_j F_{j,l}\int_0^{\infty} K_0\left(\left|u-v\right|\right)\Psi_\ell\left(v\right)dv = g_\ell\left(u\right), \quad \left(0<u<\infty;\ \ell = 0,1,2,\ldots,N\right); \tag{3.4}$$

$$\int_0^{\infty}\Psi_l\left(u\right)du = M_l \tag{3.5}$$

Here, K_0 is the Macdonald kernel. In addition, using the following notations $\left|\alpha\right|x = u$, $\left|\alpha\right|\xi = v$, $\Psi_\ell\left(x\right) = \Phi_\ell\left(\frac{u}{\left|\alpha\right|}\right)$, $g_\ell\left(u\right) = f_\ell\left(\frac{u}{\left|\alpha\right|}\right)$, $P_\ell\left(\alpha\right) = M_\ell$, and the following relations:

(i) Expansion formulas, see [23]

$$\Psi_\ell(u)=\sqrt{u}\,e^{-u}\sum_{n_\ell=0}^{\infty}B_{n_\ell}L_{n_\ell}^{\frac{1}{2}}(2u),\quad g_\ell(u)=\sqrt{u}\,e^{-u}\sum_{n_\ell=0}^{\infty}g_{n_\ell}L_{n_\ell}^{\frac{1}{2}}(2u),$$

$$K_0(|u-v|)=\frac{\sqrt{\pi}}{e^{u+v}}\sum_{n=0}^{\infty}L_n^{-\frac{1}{2}}(2u)L_n^{-\frac{1}{2}}(2v) \tag{3.6}$$

where $\{B_{n_\ell}\}$ are unknown coefficients to be determined later, while $\{g_{n_\ell}\}$ are known coefficients that can be known from the orthogonal relation (3.7).

(ii) Orthogonal relation, see [22]

$$\int_0^\infty L_{n_\ell}^{\frac{1}{2}}(v)L_{m_\ell}^{\frac{1}{2}}(v)\sqrt{v}\,e^{-v}\,dv=\begin{cases}0 & m_\ell\neq n_\ell\\ \frac{\Gamma(n_\ell+3/2)}{(n_\ell)!} & m_\ell=n_\ell\end{cases} \tag{3.7}$$

(iii) Algebraic relations

$$(1)\ \sqrt{u}L_n^{\frac{1}{2}}(2u)=\frac{1}{2\sqrt{u}}\left[\left(n+\tfrac{1}{2}\right)L_n^{-\frac{1}{2}}(2u)-(n+1)L_{n+1}^{-\frac{1}{2}}(2u)\right],\left(see\ Eq.23of\ [23,p.190]\right)$$

$$(2)\quad L_n^{m-1}(u)=L_n^m(u)-L_{n-1}^m(u),\qquad \left(u\in R^+\right) \tag{3.8}$$

(iv) Integral formulas (Eq. 4 of [22.p.208] for $\alpha=\frac{1}{2}=-\beta$)

$$G_{m,n}=\int_0^\infty e^{-2v}\,L_m^{\frac{1}{2}}(2v)L_n^{-\frac{1}{2}}(2v)\,dv,\quad m,n=0,1,2,\ldots \tag{3.9}$$

or $$G_{m,n}=\frac{1}{2\pi}\frac{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2\pi\,\Gamma(m+1)\Gamma(n+1)\Gamma\left(m-n+\frac{1}{2}\right)}$$

where $\Gamma(\cdot)$ is the gamma function.
By using the above relations (i) – (v), then Theorem2 is proved.

4. Krein's Method

In the theory of elasticity and contact problems, Krein's method is considered as one of the best method for solving the singular integral equation, where the singularity disappears and the integral equations can be solved directly without singularity, see [11]. Consider the **SIEs:**

$$\sum_{j=0}^{\ell}u_jF_{j,\ell}\int_{\Omega_3}\int\frac{\Phi_j(\xi,\eta)d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}=f_\ell(x,y),\Omega_3=\left\{(x,y,z)\in\Omega_3:\sqrt{x^2+y^2}\le a,z=0\right\}. \tag{4.1}$$

under the condition:

$$\int_{\Omega_3}\int\Phi_\ell(x,y)dx\,dy=P_3 \tag{4.2}$$

Principal of Krein, see [11]: The unique solution of **FIE** of the first kind with singular kernel takes the form:

$$\Phi(x)=\frac{1}{2M'(a)}\left[\frac{d}{du}\int_{-a}^{a}q(y,a)f(y)dy\right]q(x,a)-\frac{1}{2}\int_{|x|}^{a}q(x,u)\frac{d}{du}\left[\frac{1}{M'(u)}\frac{d}{du}\int_{-a}^{a}q(y,u)f(y)dy\right]du$$

$$-\frac{1}{2}\int_{|x|}^{a}\frac{q(x,u)}{M'(u)}\left[\int_{-a}^{a}q(y,u)df(y)\right]du,\quad (|x|<a). \tag{4.3}$$

where:

$$M(u) = \int_0^a q(y,u)\,dy, \quad M'(u) = \frac{d}{du} M(u),$$

and q(y, a) is the solution of the **IE:**

$$\int_{-a}^{a} k(x,y)\,q(y,a)\,dy = 1 \tag{4.4}$$

Theorem 3

SRs of **SFIEs** (4.1) under the condition (4.2) in the domain of integration $\Omega_3 = \left\{(x,y,z) \in \Omega_3 : \sqrt{x^2 + y^2} \le a, z = 0\right\}$ are given by the form:

$$\sum_{j=0}^{\ell} u_j F_{j,l} \int_0^1 \frac{k_n(r,\rho) P_{m_j}^{\left(n,-\frac{1}{2}\right)}\left(1-2\rho^2\right) d\rho}{\sqrt{1-\rho^2}} = r^n \sum_{j=0}^{\ell} \lambda_{m_j} u_j F_{j,\ell} P_{m_j}^{\left(n,-\frac{1}{2}\right)}\left(1-2r^2\right)$$

where, $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials, and $k_n(r,\rho)$ is Weber- Sonien integral kernel.

Proof

In this aim, we use in (4.1) and (4.2) the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$; $\xi = \rho\cos\psi$, $\eta = \rho\sin\psi$, to have:

$$\sum_{j=0}^{l} u_j F_{j,l} \int_0^a \int_{-\pi}^{\pi} \frac{\rho \breve{\Phi}_l(\rho,\psi)\,d\rho\,d\psi}{\sqrt{r^2 + \rho^2 - 2r\rho\cos(\theta-\psi)}} = \tilde{f}_l(r,\theta), \tag{4.5}$$

$$\int_0^a \int_{-\pi}^{\pi} \rho \breve{\Phi}_\ell(\rho,\theta)\,d\rho\,\theta = P_\ell \tag{4.6}$$

Here, we used the general notation, $Z_\ell(x,y) = Z_\ell(r\cos\theta, r\sin\theta) = \tilde{Z}_\ell(r,\theta)$.

In order to separate the variable, we assume $\breve{\Phi}_\ell(r,\theta) = \Psi_{m\ell}(r)\cos m\theta, \breve{f}_\ell(r,\theta) = f_{m_\ell}(r)\cos m\theta$, then after following the same way of [11, 19], we arrive to the following relations:

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_0^a k_{m_\ell}(r,\rho) \Psi_{m_\ell}(\rho)\,d\rho = f_{m_\ell}(r) \tag{4.7}$$

$$\int_0^a \rho \Psi_\ell(\rho)\,d\rho = \frac{1}{2\pi} P_0; \tag{4.8}$$

The Weber-Sonien integral formula takes the form, see [16]:

$$k_m(r,\rho) = 2\pi\sqrt{r\rho} \int_0^\infty J_m(t\rho) J_m(tr)\,dt \tag{4.9}$$

where $J_m(\cdot)$ is the Bessel function of the first of order *m and* m is called the harmonic oscillator of the kernel.

The formula (4.7) represents **SFIEs** of the first kind, under the condition (4.8).

By using the following relations, see [22, 23],

1- $P_{2n}(x) = C_{2n}^{\frac{1}{2}}(x) = P_n^{\left(0,-\frac{1}{2}\right)}\left(2x^2 - 1\right)$

2- $$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1} P_n^{(\alpha,\beta)}(1-\gamma t)dt = \frac{\Gamma(n+\alpha+1)\Gamma(\lambda)\Gamma(\mu)}{\Gamma(1+\alpha)\Gamma(\lambda+\mu)n!}$$

$$X\ {}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \alpha+1, 1+\mu; \tfrac{\gamma}{2}\right); \left(\text{see Eq. } (7.392), \text{ pp. 856 of } [23]\right)$$

3- $$\frac{dP_n^{(\alpha,\beta)}(x)}{dx} = \left(\frac{n+\alpha+\beta+1}{2}\right) P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

where, $P_n(x), C_n^{\alpha}(x), P_n^{(\alpha,\beta)}(x), \Gamma(x)$ and ${}_3F_2(a,b,c;\alpha;\beta;z)$ are, respectively Legendre polynomials, Gegenbaur polynomials, Jacobi polynomials, gamma function and generalized hypergeometric series. And assuming $f_{m_j}(s) = P_{2m_j}\left(\sqrt{1-s^2}\right)$ in (4.7), we can directly proof the theorem. Where $P_{2m_l}(y)$ are the Legendre polynomials .

5. Special Cases

We are in a position to consider many special and new cases. In Figs. (1-4) we consider some special cases for the Weber-Sonien integral formula (4.9) when the harmonic m takes different values:

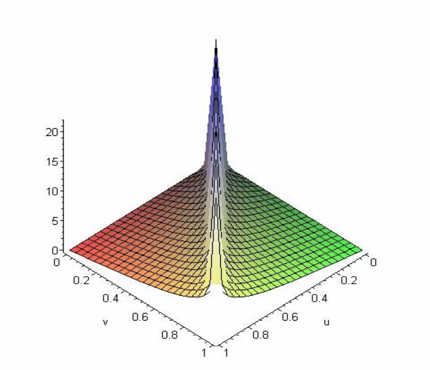

Fig.1, m=0.5

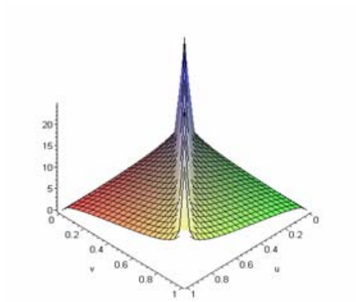

Fig. 2, m=0

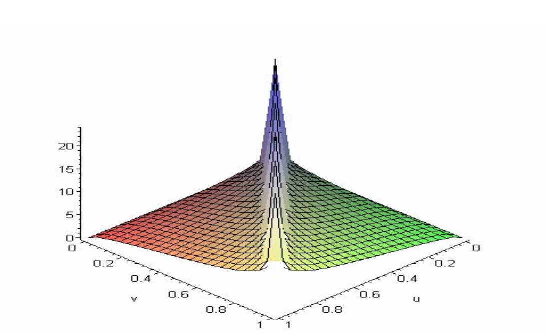

Fig.3, m=0.1

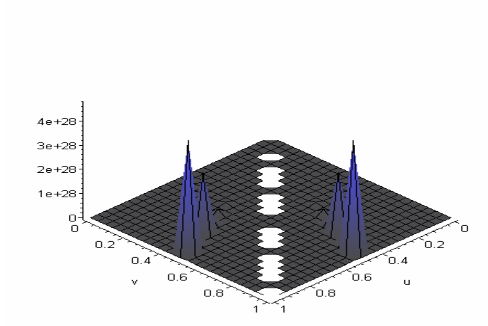

Fig.4, m=150

Case (1): When $m = 0$, the kernel (4.9) is called the complete elliptic integral. The importance of this kernel came from the work of Covalence [24], who developed the **FIE** of the first kind for the mechanics mixed problems of continuous media and obtained an approximate solution of **FIF** with a complete elliptic kernel.

The corresponding **SIEs** of theorem 3, for m=0, are:

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_0^l \frac{vE\left(\frac{2\sqrt{uv}}{u+v}\right) P_{2m_j}\left(\sqrt{1-u^2}\right) du}{\sqrt{1-u^2}} = \sum_{j=0}^{\ell} u_j F_{j,l} \left[\frac{\left(2m_j-1\right)!!}{\left(2m_j\right)!!}\right]^2 P_{2m_j}\left(\sqrt{1-v^2}\right) \tag{5.1}$$

Here, $E\left(\frac{2\sqrt{uv}}{u+v}\right)$ is the elliptic function, while $P_m(x)$ is the Legendre polynomial.

Case (2): Let, in the potential kernel (4.9) $m=\pm\frac{1}{2}$, we have **SFIEs** of the first kind with the logarithmic kernels. In this case, the spectral relationships take the following form:

$$\sum_{j=0}^{l} u_j F_{j,l} \int_{-1}^{1} \ln\frac{1}{|u-v|} T_{m_j}(u) du = \begin{cases} \pi \ln 2 \sum_{j=0}^{l} u_j F_{j,l} P_j(0) & m_j = 0 \\ \pi \sum_{j=0}^{l} \frac{u_j F_{j,l}}{m_j} T_{m_j}(v) & \end{cases} \tag{5.2}$$

where, $T_m(x)$ is the Chebyshev polynomial of the first kind of order m .

Case (3): By using the following relations, (see [23, Eq. 398 P. 1044]):

(i) $\lim_{v\to 0} \Gamma\left(\frac{v}{2}\right) C_m^{\frac{v}{2}}(x) = \frac{2}{m} T_m(x)$

(ii) $\ln\frac{1}{|x-y|} = \lim_{v\to 0}\left[\frac{1}{|x-y|^v} - 1\right]^v \quad , \quad 0 \le v < 1,$

we obtain the following **SRs** for the Carleman function:

$$\sum_{j=0}^{l} u_j F_{j,l} \int_{-1}^{1} \frac{C_{2m_j}^{\frac{v}{2}}(u) du}{|u-v|^v (1-u)^{\frac{1-v}{2}}} = \sum_{j=0}^{l} u_j F_{j,l} \lambda_{2m_j} C_{2m_j}^{\frac{v}{2}}(v) \quad ,\left(m_j \ge 0\right), \tag{5.3}$$

where , $\lambda_{2m_j} = \pi \Gamma\left(2m_j + v\right)\left[\Gamma\left(2m_j+1\right)\Gamma(v)\cos\frac{\pi v}{2}\right]^{-1} \quad (n \ge 0),$

Here, $\Gamma(\cdot)$ is defined as a gamma function and $C_n^v(x)$ are Gegenbauer polynomials.

The kernel $|x-y|^{-v}, 0 \le v < 1$ is called the Carleman function. The important of Carleman function came from the work of Arutiunion [25] who has shown that the plane contact problem of the nonlinear theory of plasticity, in its first approximation reduces to **FIE** of the first kind with Carleman kernel.

6. Conclusion and Results

From the above results and discussion, the following may be concluded:

(1) The quadratic numerical method transforms the **MIE** in position and time to **SIEs** in position. The **SIEs** depends on the number of derivatives of $F(t,\tau)$ with respect to time $t, t \in [0, \mathrm{T}], \mathrm{T} \le 1$.

(2) The contact problems of rigid surfaces of elastic materials, when stamps of different lengths are impressed into elastic layer surfaces of strips by variable forces see [17], become special cases of this work.

(3) The displacement problems of anti plane deformation of an infinite rigid strip with width $2a$, putting on an elastic layer of thickness h is considered as a special case of this work when $t=1$, $F(t,\tau)=1$, $f(x,t)=\mathrm{H}$ and $\varphi(x,1)=\psi(x)$. Here, H represents the displacement magnitude and $\psi(x)$ the unknown function represents the displacement stress, see [8,10].

(4) The problems of infinite rigid strip with width $2a$ impressed in a viscous liquid layer of thickness h, when the strip has a velocity resulting from the impulsive force $V = V_0 e^{-iwt}, i = \sqrt{-1}$, where V_0 is the constant velocity, w is the angular velocity resulting rotating the strip about z-axis are considered as special case of this work, when $F(t,\tau)=$ constant and $t = 1$, see [5,10,26].

(5) In the discussion of (4) and (5), when $h \to \infty$, i.e. the depth of the liquid (fluid mechanics) or the thickness of elastic material (contact problem) becomes an infinite, see [10,26] .

(6) The three kinds of the displacement problem, in the theory of elasticity and mixed contact problems, which discussed in [5,17,18] ,are considered special cases of this work

(7) The potential kernel (4.9) represents a Weber-Sonein integral formula and it represents a non homogeneous wave equation

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \rho^2}\right) k_m(r,\rho) = (a(r) - a(\rho)) k_m(r,\rho), \; a(r) = \left(m^2 - \frac{1}{4}\right) r^{-2}, \; (m \neq \pm \frac{1}{2})$$

(8) The potential method is suitable to be used in solving the mathematical physics problems in the domain $\Omega = \{ |x| < \infty, \; |y| < a, z > 0\}$, while for the domain $\Omega = \{ 0 < x < \infty, |y| < \infty, z < 0\}$, the orthogonal polynomials are suitable to solve the contact and mixed problems. Finally, Krain's method is suitable in the domain $\Omega = \left\{ \sqrt{x^2 + y^2} \le a, z = 0 \right\}$ to solve the mathematical physics problems.

(9) The **SFIEs** of the second kind can be solved, directly using the **SRs**, see [20]

(10) Most the problems of contact and mixed problems in [4, 26] can be solved directly by using our SRs in suitable domain of integration.

Acknowledgments: I thank Prof. M.A. Abdou for his helpful guidance and advice through this work.

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