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Non-linear Structures by Three-dimensional Dynamic Analysis

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Abstract
An innovative and groundbreaking dynamic model is introduced and investigated for the solution of the three-dimensional dynamic analysis problem of a non-linear (non-symmetrical) structure, subjected under seismic forces. Such dynamic analysis problem is reduced to the solution of a system of ordinary differential equations of the second kind and the above system is numerically evaluated by using a special kind of finite elements and by solving the corresponding eigenvalues-eigenvectors problem. Finally, an application of dynamic analysis is given to the determination of the eigenvalues and eigenvectors of a 15-floor building consisting of reinforced concrete and subjected to an horizontal seismic vibration.

Key Word and Phrases
3-D Dynamic Analysis, Seismic Forces, Structural Analysis, Ordinary Differential Equations, Non-Linear Structures, Finite Elements, Multistory Frame Structures. Eigenvalues-Eigenvectors.

1. Introduction

A big variety of engineering and applied mechanics problems, have the time dimension as very important factor to be considered. So, such typical problems of engineering applications are wave transmission in fluids, transient heat conduction and dynamic behavior of structures. The latter category of applied mechanics problems, is probably the most important for practical applications, and of leading interest for dynamic analysis theory.

Consequently, because of the seismic motion of the soil, a corresponding motion is created to the frame structure. Such a seismic motion affects in many different ways the various structures. The first problem therefore to be determined is the action of an earthquake over a structure. As a second problem, the reaction of the structure over the above earthquake has to be determined. The above problems are finally reduced to the calculation of the motion for each time, for every point of the structure under study.

In addition, the study of a structural analysis problem needs a series of several steps. First of all the geometry of the structure should be extensively described and then there is a necessity for the description of the solid and section properties of the members. Also, the dynamic load conditions for which the structure needs to be analyzed, should be described.

Over the past years, E.G.Ladopoulos introduced and studied several linear [1] - [6] and non-linear singular integro-differential equations methods [7] - [10] for the solution of many important problems of structural analysis and fluid mechanics. On the other hand, an ordinary differential equations analysis is proposed by the present investigation, for the solution of some generalized dynamic analysis problems for non-linear (non-symmetrical) structures. Such problems are reduced to the solution of a system of differential equations of the second kind and such a system is finally numerically solved by using a kind of finite elements.

According to the finite element method, the continuum is divided into a finite number of elements and its behavior is specified by a finite number of parameters. The solution further of the complete system as an assembly of its elements results basically the same rules as those applicable to standard discrete problems. During the past years, many studies have been published on the application of finite elements for the numerical evaluation of problems in structural analysis and many other fields of engineering applications. Among the survey investigations on finite-element methods, are mentioned those by J.T.Oden [11] - [13], E.L.Wilson [14], K.J.Bathe and E.L.Wilson et al. [15]-[17], O.C.Zienkiewicz [18], [19], O.C.Zienkiewicz et al. [20]-[23], H.Tottenham and C.A.Brebbia [24], J.H.Argyris et al. [25],[26], C.C.Fu [27], R.D.Kreig and S.W.Key [28], T.Belytschko, R.L.Chiapetta and H.D.Bartel [29], G.L.Goudreau and R.L.Taylor [30], T.K.Hellen [31], I.Fried [32], [33], M.Ziamal [34], L.S.D.Morley [35], J.C.Nactegaal, D.M.Parks and J.R.Rice [36], J.L.Batoz, A.Chattopadhyay and G.Dhatt [37] and T.Matsui and O.Matsuoka [38].

By the present investigation a new dynamic model is proposed for the solution of the three-dimensional structural analysis problem of a non-linear (non-symmetrical) structure under seismic forces. Such problem is reduced to the solution of a system of ordinary differential equations of the second kind and such a system is numerically evaluated by using finite elements and solving the corresponding eigenvalues-eigenvectors problem.

So, the proposed finite element method will be much smaller in degree of freedom size than commercial software, because classical linear stiffness matrices of 3-dimensional beam element have six degrees of freedom per node. The new model is therefore only applied for buildings and thus some special beam elements are used, which differs from the usual 3-D elements for structures.

An application is finally given, to the determination of the eigenvalues and eigenvectors of a 15-floor building consisting of reinforced concrete and subjected under horizontal seismic vibration. For the calculation of the above eigenvalues and eigenvectors, the Jacobi transformation method of a real symmetric matrix is used. This method consists of a sequence of orthogonal similarity transformations, known as the Jacobi rotations.

2. Three-Dimensional Non-linear Dynamic Analysis

Consider the multistory frame structure of Figure 1, under the following assumptions:

I. The masses of this structure are concentrated in the levels of plates.
II. The material of columns performs Hooke's law of elasticity.
III. Simple harmonic seismic forces are applied to the frame structure.

If the above mentioned assumptions are fulfilled, then the multistory frame structure is reduced to the solution of a system of n masses $m_1, m_2, ...m_i, ...m_n$ which is concentrated in the levels of girders and connected together and with the ground, by the elastic joints $k_1, k_2, ...k_i, ...k_n$.

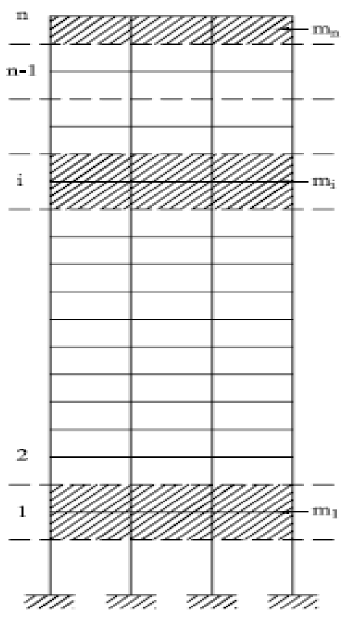

Fig.1 Multistory frame structure subjected to seismic forces

Beyond the above, consider by w_{ot} the deflection of the foundation from its initial position at time t , and w_{it} (i=1,2,........n) the corresponding deflection of mass m_{it} (i=1,2,........n), at the same time. Also, let by u_{it} the displacement of mass m_{it} at this time t, too (Figure 2).

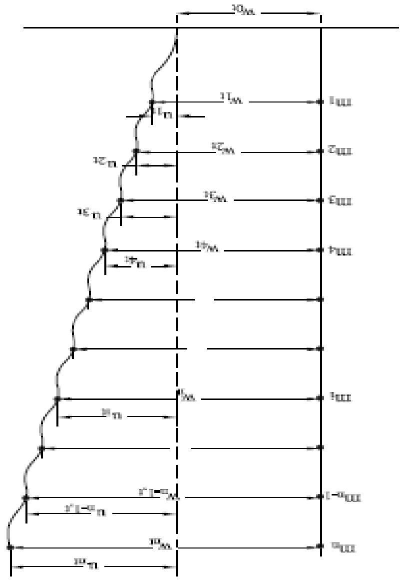

Fig. 2 The multistory frame structure is reduced to a system of masses m_i concentrated in the levels of girders

Consequently, from Figure 2 it is well seen, that following relation between the deflections and the displacements is valid:

$$w_{it} = w_{0t} + u_{it} \quad , \; i = 1,2,......n \tag{2.1}$$

Moreover, equations of motion for the masses m_i, $i = 1,2,......n$ are considered. So, in the mass m_i, of the i-floor are effected the elastic force of restoring R_{it}, the force of damping A_{it} and the force of inertia $m_i \ddot{w}_{it}$. (Figure 3)

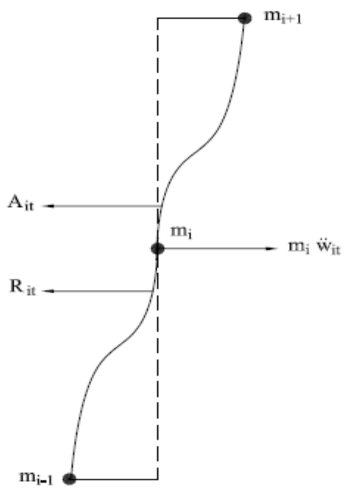

Fig. 3 Dynamic model. of multistory frame structure

The following equality is further valid between the forces on the mass m_i :

$$A_{it} + R_{it} = m_i \ddot{w}_{it} \tag{2.2}$$

As the behavior of the multistory frame structure is elastic, then the forces A_{it} and R_{it} are linear functions of the displacements u_{it} and the velocities $\dot{u}_{it}$, correspondingly, of the following forms:

$$R_{it} = -\left[k_{i1}u_{1t} + k_{i2}u_{2t} + ...+ k_{in}u_{nt}\right] \tag{2.3}$$

$$A_{it} = -\left[c_{i1}\dot{u}_{1t} + c_{i2}\dot{u}_{2t} + ...+ c_{in}\dot{u}_{nt}\right] \tag{2.4}$$

where k_{ij} and c_{ij} $(j = 1,2,.......n)$ denote the stiffness and coefficients of damping, correspondingly. The negative sign in eqs (2.3) and (2.4) is due to the opposite direction of the forces from the positive displacements and velocities.

Hence, by replacing (2.3) and (2.4) in (2.2), then the equation of dynamic motion for the mass m_i becomes as following:

$$m_i \ddot{w}_{it} + \sum_{j=1}^{n} c_{ij}\dot{u}_{jt} + \sum_{j=1}^{n} k_{ij}u_{jt} = 0 \quad , \quad i = 1,2,........,n \tag{2.5}$$

while for the totality of masses, the following system of n differential equations of the second kind is valid:

$$\begin{aligned} m_1 \ddot{w}_{1t} + \sum_{j=1}^{n} c_{1j}\dot{u}_{jt} + \sum_{j=1}^{n} k_{1j}u_{jt} &= 0 \\ m_2 \ddot{w}_{2t} + \sum_{j=1}^{n} c_{2j}\dot{u}_{jt} + \sum_{j=1}^{n} k_{2j}u_{jt} &= 0 \\ &\cdots\cdots\cdots \\ m_n \ddot{w}_{nt} + \sum_{j=1}^{n} c_{nj}\dot{u}_{jt} + \sum_{j=1}^{n} k_{nj}u_{jt} &= 0 \end{aligned} \tag{2.6}$$

So, the above mentioned system of ordinary differential equations can be written in matrix form, by introducing following vectors:

$$\mathbf{u}_t = \left[u_{1t}, u_{2t}, ..., u_{nt}\right]^{\mathrm{T}} \tag{2.7a}$$

$$\mathbf{w}_t = \left[w_{1t}, w_{2t}, ..., w_{nt}\right]^{\mathrm{T}} = \mathbf{u}_t + \boldsymbol{\delta} w_{0t} \tag{2.7b}$$

$$\boldsymbol{\delta} = \left[1,1,.......,1\right]^{\mathrm{T}} \tag{2.7c}$$

Furthermore, by using eqs (2.7a) to (2.7c), then the system of differential equations (2.6) reduces to:

$$\mathbf{M}\ddot{\mathbf{w}}_t + \mathbf{C}\dot{\mathbf{u}}_t + \mathbf{K}\mathbf{u}_t = 0 \tag{2.8}$$

and by replacing $\ddot{\mathbf{w}}_t = \ddot{\mathbf{u}}_t + \boldsymbol{\delta}\ddot{w}_{0t}$ takes the form:

$$\mathbf{M}\ddot{\mathbf{u}}_t + \mathbf{C}\dot{\mathbf{u}}_t + \mathbf{K}\mathbf{u}_t = -(\mathbf{M}\cdot\boldsymbol{\delta})\ddot{w}_{0t} \tag{2.9}$$

where:

$$\mathbf{M} = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & .... & c_{1n} \\ c_{21} & c_{22} & .... & c_{2n} \\ .... & .... & .... & .... \\ c_{n1} & c_{n2} & .... & c_{nn} \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & .... & k_{1n} \\ k_{21} & k_{22} & .... & k_{2n} \\ .... & .... & .... & .... \\ k_{n1} & k_{n2} & .... & k_{nn} \end{bmatrix} \tag{2.10}$$

with $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ the mass, damping and stiffness matrices, correspondingly.

The stiffness will be determined by following next method. Hence, consider that the displacements u_{it} are due to the external loads $P_{it} = -R_{it}$. Then because of (2.3) we obtain: (Fig. 4)

$$P_{it} = k_{i1}u_{1t} + k_{i2}u_{2t} + ... + k_{ii}u_{it} + ... + k_{ij}u_{jt} + ...k_{in}u_{nt} \tag{2.11}$$

So, from (2.11) follows that if by the use of girders all floors of the multistory frame structure are fixed, then k_{ii} will be the proper external force for the unity displacement $u_{it} = 1$ of the floor i and k_{ij} will be the corresponding reaction in the girder of the floor i for the unity displacement $u_j = 1$ of the floor j. (Fig. 4)

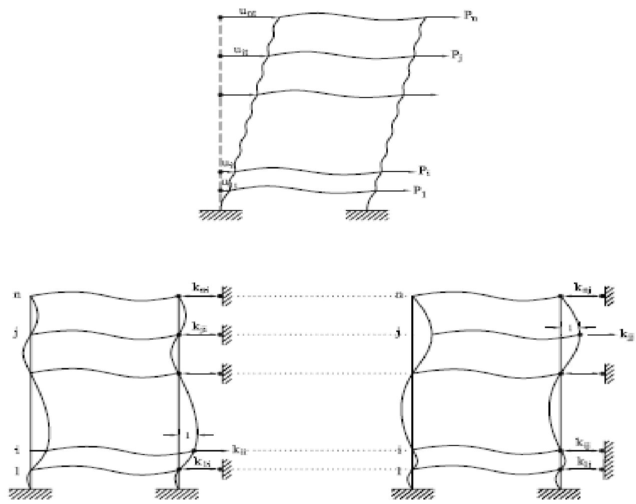

Fig. 4 Calculation of the stiffness matrix.

Consequently, by the former unity displacement of the floor i are calculated the stiffnesses $k_{\lambda i}$ $(\lambda = 1,2,.......,n)$ of the column i of the stiffness matrix $\mathbf{K}$, and by the latter by the same way are calculated the stiffnesses $k_{\lambda j}$ $(j = 1,2,.......,n)$ of column j. By successive therefore unity displacement of the total number of floors of the multistory frame structure are calculated all the columns of the stiffness matrix. On the other hand, according to Betti's reciprocal theorem the equalities $k_{ij} = k_{ji}$ are equal, and thus the stiffness matrix is always symmetric.

3. Natural Vibration and Eigenvalues for Dynamic Analysis

In order the system of ordinary differential equations (2.9) to be solved, there is a necessity for the study of the natural vibration, according to which some special solutions are calculated, which are independent from the external seismic excitation. By the above special solutions are simultaneously determined all the dynamic characteristics of the mechanical system and thus the final solution of the problem is obtained.

Hence, in order to determine the above special solutions, then the following assumptions should be considered:

I. The damping of the system is equal to zero ($\mathbf{c}$ = 0).

II. The vibration is due to known displacements u_{i0} and velocities $\dot{u}_{i0}$ when beginning the calculation of the time (t = 0).

If the above mentioned assumptions are fulfilled, then (2.9) takes the following form:

$$\mathbf{M}\ddot{\mathbf{u}}_t + \mathbf{K}\mathbf{u}_t = 0 \tag{3.1}$$

Some solutions of the following form are investigated for the system of ordinary differential equations (3.1):

$$\mathbf{u}_t = [u_{1t}, u_{2t}, ..., u_{nt}]^{\mathrm{T}} = [\varphi_1, \varphi_2, ..., \varphi_n]^{\mathrm{T}} f_t = \boldsymbol{\varphi} f \tag{3.2}$$

By substituting (3.2) in (3.1), follows:

$$\begin{bmatrix} m_{11} & m_{12} & .... & m_{1n} \\ m_{21} & m_{22} & .... & m_{2n} \\ .... & .... & .... & .... \\ m_{n1} & m_{n2} & .... & m_{nn} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ .... \\ \varphi_n \end{bmatrix} \ddot{f}_t + \begin{bmatrix} k_{11} & k_{12} & .... & k_{1n} \\ k_{21} & k_{22} & .... & k_{2n} \\ .... & .... & .... & .... \\ k_{n1} & k_{n2} & .... & k_{nn} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ .... \\ \varphi_n \end{bmatrix} f_t = \begin{bmatrix} 0 \\ 0 \\ .... \\ 0 \end{bmatrix} \tag{3.3}$$

and the i - equation of the above system of differential equations has the following form:

$$\left(\sum_{j=1}^{n} m_{ij}\varphi_j \right) \ddot{f}_t + \left(\sum_{j=1}^{n} k_{ij}\varphi_j \right) f_t = 0 \tag{3.4}$$

or further the form:

$$\frac{\sum_{j=1}^{n} k_{ij}\varphi_j}{\sum_{j=1}^{n} m_{ij}\varphi_j} = -\frac{\ddot{f}_t}{f_t} = \omega^2 \tag{3.5}$$

where ω denotes the cyclic frequency.

Thus, from (3.5) follows:

$$\sum_{j=1}^{n} k_{ij}\varphi_j = \omega^2 \sum_{j=1}^{n} m_{ij}\varphi_j \quad , \quad i = 1,2,.....,n \tag{3.6}$$

which in matrix form can be also written as:

$$\mathbf{K}\boldsymbol{\varphi} = \omega^2 \mathbf{M}\boldsymbol{\varphi} \tag{3.7}$$

or of the form:

$$[\mathbf{K} - \omega^2 \mathbf{M}]\boldsymbol{\varphi} = 0 \tag{3.8}$$

where (3.8) corresponds to a generalized eigenvalue - eigenvectors problem.

The generalized eigenvalue problem (3.8) may be further written as following:

$$\begin{bmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & .... & k_{1n} - \omega^2 m_{1n} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} & .... & k_{2n} - \omega^2 m_{2n} \\ .... & .... & .... & .... \\ k_{n1} - \omega^2 m_{n1} & k_{n2} - \omega^2 m_{n2} & .... & k_{nn} - \omega^2 m_{nn} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ .... \\ \varphi_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .... \\ 0 \end{bmatrix} \tag{3.9}$$

which is an homogeneous system of n linear equations with unknowns the displacements φ_i and the parameter ω^2. In order this system to have non-zero solutions, except the obvious solution $\varphi_i = 0$, its determinant should be zero:

$$\begin{vmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & .... & k_{1n} - \omega^2 m_{1n} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} & .... & k_{2n} - \omega^2 m_{2n} \\ .... & .... & .... & .... \\ k_{n1} - \omega^2 m_{n1} & k_{n2} - \omega^2 m_{n2} & .... & k_{nn} - \omega^2 m_{nn} \end{vmatrix} = 0 \tag{3.10}$$

This determinant will have an expansion of an algebraic equation of n - degree, with unknowns the ω^2, from the solutions of which are calculated the following n - eigenvalues ω_j^2:

$$\omega^2 = \omega_j^2 \quad , \quad j = 1,2,.....,n \tag{3.11}$$

So, in each cyclic frequency ω_j corresponds a natural period:

$$T_j = \frac{2\pi}{\omega_j} \tag{3.12}$$

Beyond the above, the homogeneous system (3.9) has n - solutions which are the eigenvectors φ_j :

$$\boldsymbol{\varphi}_j = [\varphi_{1j}, \varphi_{2j}, ..., \varphi_{nj}]^{\mathrm{T}} \quad , \quad j = 1,2,.....,n \tag{3.13}$$

in correspondence to the n eigenvalues ω_j^2.

The above homogeneous system is solved by an arbitrary selection of one component of the eigenvector φ_j (like the first one $\varphi_{1j} = c$ or the last one $\varphi_{nj} = c$) and hence the system (3.9) becomes non-homogeneous and from the solution of any subsystem with (n-1) equations are further calculated the rest (n-1) components of the φ_j.

Thus, the system (3.1) has n-independent solutions of the form (3.2):

$$\mathbf{u}_{jt} = [\varphi_{1j}, \varphi_{2j}, ..., \varphi_{nj}]^{\mathrm{T}} f_{jt} = \boldsymbol{\varphi}_j f_{jt} \quad , \quad j = 1,2,.....,n \tag{3.14}$$

which corresponds to simple harmonic vibrations with cyclic frequencies ω_j (or periods $T_j = 2\pi / \omega_j$) and these are the natural vibrations of the mechanical system.

The above eigenvalues can be classified with increasing order of cyclic frequencies. Consequently, the eigenvalue which belongs to the lowest cyclic frequency ω_1 and correspondingly to the biggest natural period (fundamental natural period) is known as the fundamental eigenvalue.

4. Dynamic Analysis Application of a Multistory Frame Structure

Let us consider a 15-floor building consisting of reinforced concrete and which has six similar frames and hence a level of symmetry parallel to the level of the frames (Fig. 5). The dynamic characteristics of this building will be calculated for an horizontal seismic vibration.

For the above frame structure following data are valid:
$L = 5.0m,\ L_1 = 10.0m,\ L_2 = 7.0m,\ h_1 = 4.0m,\ h_2 = h_3 = ... = h_{10} = 3.0m,$ moment of inertia of all the columns $I = 0.02m^4$ $(70 \times 70),$ moment of inertia of T - beams $J = 0.022m^4$ (beams 30×70,

plus effective width of plate $b = b_0 + \frac{0.85l}{5} = 0.30 + \frac{0.85 \cdot 5}{5} = 1.15m$, plate thickness = 0.16$m$), modulus of elasticity of concrete E = 29GPa, total load = 15 KN/m^2.

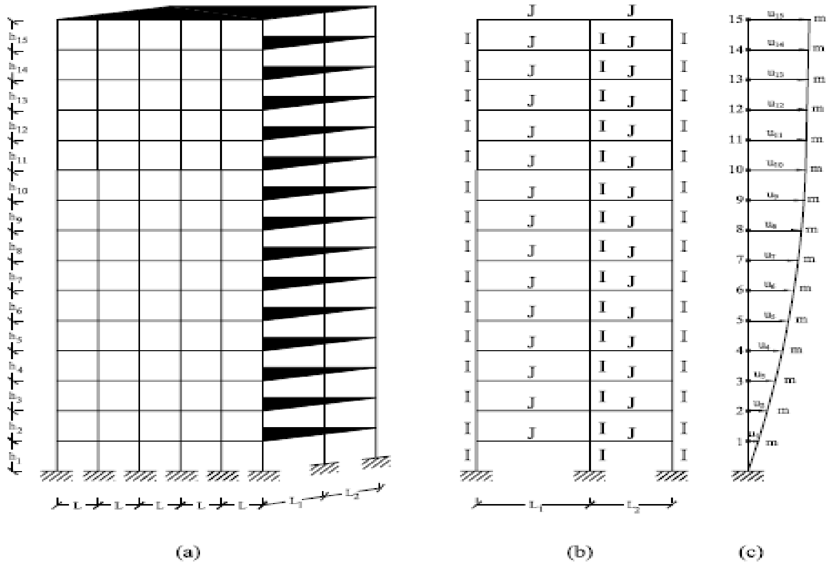

Fig. 5 15-floor building, 6-bay frame

Furthermore, the surface of a typical floor is $S = 5L(L_1 + L_2) = 510m^2$ and so the mass of this floor will be (where Q denotes the weight of the floor):

$$m = \frac{Q}{g} = \frac{510 \cdot 15}{9.81} = 780{,}000\,kg$$

Hence, Figure 5b shows the static system of a frame, while Figure 5c shows the corresponding dynamic model of the 15-floor building. In addition, the mass matrix will have the diagonal form (2.10):

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & & & & & & & & & & \\ & m_2 & & & & & & & & & & & & & \\ & & m_3 & & & & & & & & & & & & \\ & & & m_4 & & & & & & & & & & & \\ & & & & m_5 & & & & & & & & & & \\ & & & & & m_6 & & & & & & & & & \\ & & & & & & m_7 & & & & & & & & \\ & & & & & & & m_8 & & & & & & & \\ & & & & & & & & m_9 & & & & & & \\ & & & & & & & & & m_{10} & & & & & \\ & & & & & & & & & & m_{11} & & & & \\ & & & & & & & & & & & m_{12} & & & \\ & & & & & & & & & & & & m_{13} & & \\ & & & & & & & & & & & & & m_{14} & \\ & & & & & & & & & & & & & & m_{15} \end{bmatrix} = m \begin{bmatrix} 1 & & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & & \\ & & 1 & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & \\ & & & & & 1 & & & & & & & & & \\ & & & & & & 1 & & & & & & & & \\ & & & & & & & 1 & & & & & & & \\ & & & & & & & & 1 & & & & & & \\ & & & & & & & & & 1 & & & & & \\ & & & & & & & & & & 1 & & & & \\ & & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & & 1 & & \\ & & & & & & & & & & & & & 1 & \\ & & & & & & & & & & & & & & 1 \end{bmatrix} = m\mathbf{M}_0 \tag{4.1}$$

Because of symmetry the stiffness matrix of the building follows by a simple sum of the stiffness matrices $\mathbf{K}_0$ of the six frames. Hence, Fig. 6 shows the direct calculation of the horizontal stiffness matrix of a frame which corresponds to three freedoms of movement of its floors.

For example by enforcing the unity condition:

$$u_1 = 1, \quad u_2 = u_3 = ... = u_{15} = 0 \tag{4.2}$$

and applying any static method, by using one kind of finite-elements [Falter: 39], then k_{11} is calculated as the requested external force for the unity displacement $u_1 = 1.0m$ of the first floor and k_{i1}, $i = 1,2,......,15$ are calculated as the reactions in the node i of the floor i, i = 1,2,.......,15 because of the unity displacement $u_1 = 1.0m$ of the first floor.

So, the results of the static calculations are:

$$\begin{aligned}
k_{11} &= 695{,}471kN \\
k_{21} &= -458{,}457kN \\
k_{31} &= 22{,}107kN \\
k_{41} &= 5599kN \\
k_{51} &= 131kN \\
k_{61} &= -219kN \\
k_{71} &= -14kN \\
k_{81} &= 7kN \\
k_{91} &= 1.5kN \\
k_{101} &= 0.4kN \\
k_{111} &= 0.5kN \\
k_{121} &= 0.4kN \\
k_{131} &= -4kN \\
k_{141} &= -11kN \\
k_{151} &= 96kN
\end{aligned} \tag{4.3}$$

By the same way and by enforcing the unity conditions $u_2 = 1,\ u_3 = 1, ..., u_{15} = 1$ are calculated the values of the coefficients of stiffness $(k_{12}, k_{22}, ..., k_{152})$, $(k_{13}, k_{23}, ..., k_{153})$.......$(k_{110,}\ k_{210}, ..., k_{1515})$.

Hence, the stiffness matrix of the building takes the following form:

K=

$$
\begin{bmatrix}
695{,}471 & -458{,}457 & 22{,}107 & 5599 & 131 & -219 & -14 & 7 & 1.5 & 0.4 & 0.5 & 0.4 & -4 & -11 & 96 \\
-458{,}457 & 860{,}111 & -457{,}404 & 22{,}020 & 5458 & 134 & -215 & -14 & -6 & 1.3 & 0.4 & 0.3 & -3 & -10 & 83 \\
22{,}107 & -457{,}404 & 860{,}168 & -457{,}521 & 22{,}033 & 5{,}462 & 134 & -215 & -14 & 6 & 1.3 & 0.2 & -3 & -10 & 81 \\
5599 & 22{,}020 & -457{,}521 & 860{,}161 & -457{,}520 & 22{,}034 & 5462 & 133 & -215 & -14 & 6 & 1.1 & -3 & -12 & 82 \\
131 & 5458 & 22{,}033 & -457{,}520 & 860{,}164 & -457{,}523 & 22{,}035 & 5462 & 133 & -215 & -14 & 6 & -2 & -10 & 82 \\
-219 & 134 & 5462 & 22{,}034 & -457{,}523 & 860{,}165 & -457{,}525 & 22{,}036 & 5462 & 133 & -215 & -14 & 3 & -10 & 83 \\
-14 & -215 & 134 & 5462 & 22{,}035 & -457{,}525 & 860{,}167 & -457{,}527 & 22{,}036 & 5462 & 133 & -215 & -17 & -4.2 & 84 \\
7 & -14 & -215 & 133 & 5462 & 22{,}036 & -457{,}527 & 860{,}168 & -457{,}529 & 22{,}037 & 5462 & 133 & -219 & -25 & 91 \\
1.5 & 6 & -14 & -215 & 133 & 5462 & 22{,}036 & -457{,}529 & 860{,}169 & -457{,}560 & 22{,}038 & 5463 & 127 & -230 & 80 \\
0.4 & 1.3 & 6 & -14 & -215 & 133 & 5462 & 22{,}037 & -457{,}560 & 860{,}166 & -457{,}530 & 22{,}040 & 5465 & 113 & -132 \\
0.5 & 0.4 & 1.3 & 6 & -14 & -215 & 133 & 5462 & 22{,}038 & -457{,}530 & 860{,}166 & -457{,}531 & 22{,}084 & 5541 & -49 \\
0.4 & 0.3 & 0.2 & 1.1 & 6 & -14 & -215 & 133 & 5463 & 22{,}040 & -457{,}531 & 860{,}146 & -457{,}474 & 22397 & 5304 \\
-4 & -3 & -3 & -3 & -2 & 3 & -17 & -219 & 127 & 5465 & 22{,}084 & -457{,}474 & 859{,}260 & -458{,}714 & 28{,}075 \\
-11 & -10 & -10 & -12 & -10 & -10 & -4.2 & -25 & -230 & 113 & 5541 & 22{,}397 & -458{,}714 & 844{,}912 & -420{,}702 \\
96 & 83 & 81 & 82 & 82 & 83 & 84 & 91 & 80 & -132 & -49 & 5304 & 28{,}075 & -420{,}702 & 394.538
\end{bmatrix}
\tag{4.4}
$$

From eqn. (4.4) can be seen that the stiffness matrix is symmetrical.

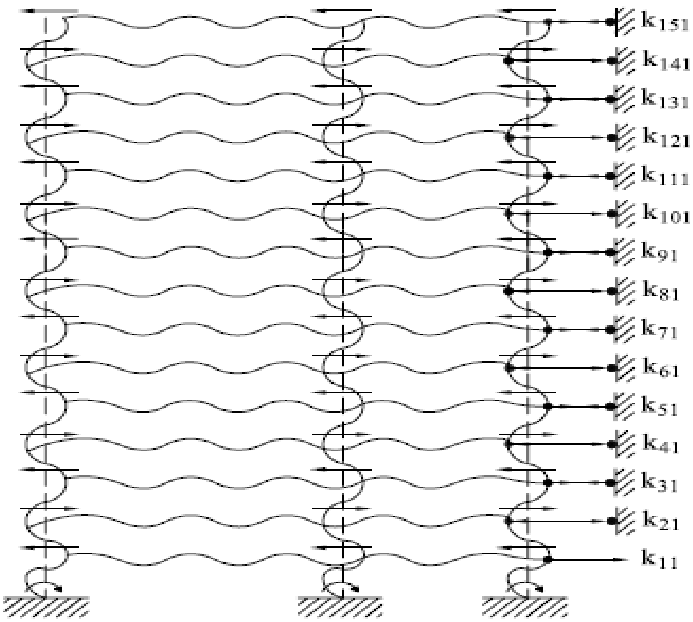

Fig. 6 Direct calculation of the stiffness matrix.

Beyond the above, by using the expressions for the mass and the stiffness matrices given by eqs (4.1) and (4.4), then the characteristic equation (3.10) of the eigenvalue-eigenvectors problem (3.8) reduces to the following form:

$$\left|\mathbf{K} - m\omega^2 \mathbf{M}_0\right| =$$

$$\begin{bmatrix}
695,471-\lambda & -458,457 & 22,107 & 5599 & 131 & -219 & -14 & 7 & 1.5 & 0.4 & 0.5 & 0.4 & -4 & -11 & 96 \\
-458,457 & 860,111-\lambda & -457,404 & 22,020 & 5458 & 134 & -215 & -14 & -6 & 1.3 & 0.4 & 0.3 & -3 & -10 & 83 \\
22,107 & -457,404 & 860,168-\lambda & -457,521 & 22,033 & 5,462 & 134 & -215 & -14 & 6 & 1.3 & 0.2 & -3 & -10 & 81 \\
5599 & 22,020 & -457,521 & 860,161-\lambda & -457,520 & 22,034 & 5462 & 133 & -215 & -14 & 6 & 1.1 & -3 & -12 & 82 \\
131 & 5458 & 22,033 & -457,520 & 860,164-\lambda & -457,523 & 22,035 & 5462 & 133 & -215 & -14 & 6 & -2 & -10 & 82 \\
-219 & 134 & 5462 & 22,034 & -457,523 & 860,165-\lambda & -457,525 & 22,036 & 5462 & 133 & -215 & -14 & 3 & -10 & 83 \\
-14 & -215 & 134 & 5462 & 22,035 & -457,525 & 860,167-\lambda & -457,527 & 22,036 & 5462 & 133 & -215 & -17 & -4.2 & 84 \\
7 & -14 & -215 & 133 & 5462 & 22,036 & -457,527 & 860,168-\lambda & -457,529 & 22,037 & 5462 & 133 & -219 & -25 & 91 \\
1.5 & 6 & -14 & -215 & 133 & 5462 & 22,036 & -457,529 & 860,169-\lambda & -457,560 & 22,038 & 5463 & 127 & -230 & 80 \\
0.4 & 1.3 & 6 & -14 & -215 & 133 & 5462 & 22,037 & -457,560 & 860,166-\lambda & -457,530 & 22,040 & 5465 & 113 & -132 \\
0.5 & 0.4 & 1.3 & 6 & -14 & -215 & 133 & 5462 & 22,038 & -457,530 & 860,166-\lambda & -457,531 & 22,084 & 5541 & -49 \\
0.4 & 0.3 & 0.2 & 1.1 & 6 & -14 & -215 & 133 & 5463 & 22,040 & -457,531 & 860,146-\lambda & -457,474 & 22397 & 5304 \\
-4 & -3 & -3 & -3 & -2 & 3 & -17 & -219 & 127 & 5465 & 22,084 & -457,474 & 859,260-\lambda & -458,714 & 28,075 \\
-11 & -10 & -10 & -12 & -10 & -10 & -4.2 & -25 & -230 & 113 & 5541 & 22,397 & -458,714 & 844,912-\lambda & -420,702 \\
96 & 83 & 81 & 82 & 82 & 83 & 84 & 91 & 80 & -132 & -49 & 5304 & 28,075 & -420,702 & 394.538-\lambda
\end{bmatrix}$$

(4.5)

where:

$$\lambda = m\omega^2 \qquad (4.6)$$

Moreover, by using the Jacobi transformations method of a real symmetrical matrix [40], then all eigenvalues are computed. The above method consists of a sequence of orthogonal similarity transformations. Each transformation known as a Jacobi rotation, is only a plane rotation designed to annihilate one of the off-diagonal matrix elements. The successive transformations undo previously set zeros, but the off-diagonal elements are getting smaller and smaller, until the matrix finally becomes diagonal to the computer precision.

Thus, the eigenvalues which are computed are:

$\lambda_1 = 3046\ KN/m^2$, $\lambda_2 = 27,872\ KN/m^2$, $\lambda_3 = 79,188\ KN/m^2$,
$\lambda_4 = 158,143\, KN/m^2$, $\lambda_5 = 266,042\, KN/m^2$, $\lambda_6 = 402,322\ KN/m^2$,
$\lambda_7 = 564,379\ KN/m^2$, $\lambda_8 = 746,338\ KN/m^2$, $\lambda_9 = 939,734\ KN/m^2$,
$\lambda_{10} = 1,134,090\ KN/m^2$, $\lambda_{11} = 1,318,720\ KN/m^2$, $\lambda_{12} = 1,483,780\ KN/m^2$
$\lambda_{13} = 1,620,980\ KN/m^2$, $\lambda_{14} = 1,723,810\ KN/m^2$, $\lambda_{15} = 1,787,480\ KN/m^2$

For the eigenvalues we have:

$$\omega_1^2 = \frac{\lambda_1}{m} = \frac{3046}{780} \Rightarrow \omega_1 = 1.98\ rad/\sec,\ T_1 = 3.18\sec$$
$$\omega_2^2 = \frac{\lambda_2}{m} = \frac{27,872}{780} \Rightarrow \omega_2 = 5.98\ rad/\sec,\ T_2 = 1.05\sec$$
$$\omega_3^2 = \frac{\lambda_3}{m} = \frac{79,188}{780} \Rightarrow \omega_3 = 10.08\ rad/\sec,\ T_3 = 0.62\sec$$
$$\omega_4^2 = \frac{\lambda_4}{m} = \frac{158,143}{780} \Rightarrow \omega_4 = 14.24\ rad/\sec,\ T_4 = 0.44\sec$$
$$\omega_5^2 = \frac{\lambda_5}{m} = \frac{266,042}{780} \Rightarrow \omega_5 = 18.47\ rad/\sec,\ T_5 = 0.34\sec$$
$$\omega_6^2 = \frac{\lambda_6}{m} = \frac{402,322}{780} \Rightarrow \omega_6 = 22.71\ rad/\sec,\ T_6 = 0.28\sec$$
$$\omega_7^2 = \frac{\lambda_7}{m} = \frac{564,379}{780} \Rightarrow \omega_7 = 26.90\ rad/\sec,\ T_7 = 0.23\sec$$
$$\omega_8^2 = \frac{\lambda_8}{m} = \frac{746,338}{780} \Rightarrow \omega_8 = 30.93\ rad/\sec,\ T_8 = 0.20\sec$$

$$\omega_9^2 = \frac{\lambda_9}{m} = \frac{939{,}734}{780} \Rightarrow \omega_9 = 34.71\ rad/\sec,\ T_9 = 0.18\sec$$

$$\omega_{10}^2 = \frac{\lambda_{10}}{m} = \frac{1{,}134{,}090}{780} \Rightarrow \omega_{10} = 38.13\ rad/\sec,\ T_{10} = 0.16\sec$$

$$\omega_{11}^2 = \frac{\lambda_{11}}{m} = \frac{1{,}318{,}720}{780} \Rightarrow \omega_{10} = 41.12\ rad/\sec,\ T_{11} = 0.15\sec$$

$$\omega_{12}^2 = \frac{\lambda_{12}}{m} = \frac{1{,}483{,}780}{780} \Rightarrow \omega_{10} = 43.62\ rad/\sec,\ T_{12} = 0.14\sec$$

$$\omega_{13}^2 = \frac{\lambda_{13}}{m} = \frac{1{,}620{,}980}{780} \Rightarrow \omega_{10} = 45.59\ rad/\sec,\ T_{13} = 0.14\sec$$

$$\omega_{14}^2 = \frac{\lambda_{14}}{m} = \frac{1{,}723{,}810}{780} \Rightarrow \omega_{10} = 47.01\ rad/\sec,\ T_{14} = 0.13\sec$$

$$\omega_{15}^2 = \frac{\lambda_{15}}{m} = \frac{1{,}787{,}480}{780} \Rightarrow \omega_{10} = 47.87\ rad/\sec,\ T_{15} = 0.13\sec$$

The eigenvectors are further calculated from the homogeneous system of linear equations (3.9), by using the Jacobi transformations method [40]:

$$(\mathbf{K} - \lambda_i \mathbf{M}_0)\boldsymbol{\varphi}_i =$$

$$\begin{bmatrix}
695{,}471-\lambda_i & -458{,}457 & 22{,}107 & 5599 & 131 & -219 & -14 & 7 & 1.5 & 0.4 & 0.5 & 0.4 & -4 & -11 & 96 \\
-458{,}457 & 860{,}111-\lambda_i & -457{,}404 & 22{,}020 & 5458 & 134 & -215 & -14 & -6 & 1.3 & 0.4 & 0.3 & -3 & -10 & 83 \\
22{,}107 & -457{,}404 & 860{,}168-\lambda_i & -457{,}521 & 22{,}033 & 5{,}462 & 134 & -215 & -14 & 6 & 1.3 & 0.2 & -3 & -10 & 81 \\
5599 & 22{,}020 & -457{,}521 & 860{,}161-\lambda_i & -457{,}520 & 22{,}034 & 5462 & 133 & -215 & -14 & 6 & 1.1 & -3 & -12 & 82 \\
131 & 5458 & 22{,}033 & -457{,}520 & 860{,}164-\lambda_i & -457{,}523 & 22{,}035 & 5462 & 133 & -215 & -14 & 6 & -2 & -10 & 82 \\
-219 & 134 & 5462 & 22{,}034 & -457{,}523 & 860{,}165-\lambda_i & -457{,}525 & 22{,}036 & 5462 & 133 & -215 & -14 & 3 & -10 & 83 \\
-14 & -215 & 134 & 5462 & 22{,}035 & -457{,}525 & 860{,}167-\lambda_i & -457{,}527 & 22{,}036 & 5462 & 133 & -215 & -17 & -4.2 & 84 \\
7 & -14 & -215 & 133 & 5462 & 22{,}036 & -457{,}527 & 860{,}168-\lambda_i & -457{,}529 & 22{,}037 & 5462 & 133 & -219 & -25 & 91 \\
1.5 & 6 & -14 & -215 & 133 & 5462 & 22{,}036 & -457{,}529 & 860{,}169-\lambda_i & -457{,}560 & 22{,}038 & 5463 & 127 & -230 & 80 \\
0.4 & 1.3 & 6 & -14 & -215 & 133 & 5462 & 22{,}037 & -457{,}560 & 860{,}166-\lambda_i & -457{,}530 & 22{,}040 & 5465 & 113 & -132 \\
0.5 & 0.4 & 1.3 & 6 & -14 & -215 & 133 & 5462 & 22{,}038 & -457{,}530 & 860{,}166-\lambda_i & -457{,}531 & 22{,}084 & 5541 & -49 \\
0.4 & 0.3 & 0.2 & 1.1 & 6 & -14 & -215 & 133 & 5463 & 22{,}040 & -457{,}531 & 860{,}146-\lambda_i & -457{,}474 & 22397 & 5304 \\
-4 & -3 & -3 & -3 & -2 & 3 & -17 & -219 & 127 & 5465 & 22{,}084 & -457{,}474 & 859{,}260-\lambda_i & -458{,}714 & 28{,}075 \\
-11 & -10 & -10 & -12 & -10 & -10 & -4.2 & -25 & -230 & 113 & 5541 & 22{,}397 & -458{,}714 & 844{,}912-\lambda_i & -420{,}702 \\
96 & 83 & 81 & 82 & 82 & 83 & 84 & 91 & 80 & -132 & -49 & 5304 & 28{,}075 & -420{,}702 & 394.538-\lambda_i
\end{bmatrix}$$

$$\times \begin{bmatrix} \varphi_{1i} \\ \varphi_{2i} \\ \varphi_{3i} \\ \varphi_{4i} \\ \varphi_{5i} \\ \varphi_{6i} \\ \varphi_{7i} \\ \varphi_{8i} \\ \varphi_{9i} \\ \varphi_{10i} \\ \varphi_{11i} \\ \varphi_{12i} \\ \varphi_{13i} \\ \varphi_{14i} \\ \varphi_{15i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.7)$$

for $\varphi_{15i} = 1.0,\quad i = 1,2,.......,15$

Consequently, by solving the above homogeneous system, then next eigenvalues are obtained:

$$\boldsymbol{\varphi}_1=\begin{bmatrix}\varphi_{11}\\ \varphi_{21}\\ \varphi_{31}\\ \varphi_{41}\\ \varphi_{51}\\ \varphi_{61}\\ \varphi_{71}\\ \varphi_{81}\\ \varphi_{91}\\ \varphi_{101}\\ \varphi_{111}\\ \varphi_{121}\\ \varphi_{131}\\ \varphi_{141}\\ \varphi_{151}\end{bmatrix}=\begin{bmatrix}0.153\\ 0.254\\ 0.349\\ 0.441\\ 0.528\\ 0.611\\ 0.688\\ 0.758\\ 0.822\\ 0.877\\ 0.925\\ 0.964\\ 0.994\\ 1.010\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_2=\begin{bmatrix}\varphi_{12}\\ \varphi_{22}\\ \varphi_{32}\\ \varphi_{42}\\ \varphi_{52}\\ \varphi_{62}\\ \varphi_{72}\\ \varphi_{82}\\ \varphi_{92}\\ \varphi_{102}\\ \varphi_{112}\\ \varphi_{122}\\ \varphi_{132}\\ \varphi_{142}\\ \varphi_{152}\end{bmatrix}=\begin{bmatrix}-0.442\\ -0.698\\ -0.887\\ -0.997\\ -1.022\\ -0.960\\ -0.816\\ -0.602\\ -0.336\\ -0.041\\ 0.257\\ 0.534\\ 0.756\\ 0.928\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_3=\begin{bmatrix}\varphi_{13}\\ \varphi_{23}\\ \varphi_{33}\\ \varphi_{43}\\ \varphi_{53}\\ \varphi_{63}\\ \varphi_{73}\\ \varphi_{83}\\ \varphi_{93}\\ \varphi_{103}\\ \varphi_{113}\\ \varphi_{123}\\ \varphi_{133}\\ \varphi_{143}\\ \varphi_{153}\end{bmatrix}=\begin{bmatrix}0.678\\ 0.971\\ 1.024\\ 0.831\\ 0.441\\ -0.053\\ -0.535\\ -0.889\\ -1.033\\ -0.932\\ -0.610\\ -0.143\\ 0.358\\ 0.772\\ 1.000\end{bmatrix}$$

$$\boldsymbol{\varphi}_4=\begin{bmatrix}\varphi_{14}\\ \varphi_{24}\\ \varphi_{34}\\ \varphi_{44}\\ \varphi_{54}\\ \varphi_{64}\\ \varphi_{74}\\ \varphi_{84}\\ \varphi_{94}\\ \varphi_{104}\\ \varphi_{114}\\ \varphi_{124}\\ \varphi_{134}\\ \varphi_{144}\\ \varphi_{154}\end{bmatrix}=\begin{bmatrix}-0.856\\ -1.040\\ -0.735\\ -0.088\\ 0.600\\ 1.013\\ 0.962\\ 0.470\\ -0.237\\ -0.836\\ -1.050\\ -0.783\\ -0.157\\ 0.542\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_5=\begin{bmatrix}\varphi_{15}\\ \varphi_{25}\\ \varphi_{35}\\ \varphi_{45}\\ \varphi_{55}\\ \varphi_{65}\\ \varphi_{75}\\ \varphi_{85}\\ \varphi_{95}\\ \varphi_{105}\\ \varphi_{115}\\ \varphi_{125}\\ \varphi_{135}\\ \varphi_{145}\\ \varphi_{155}\end{bmatrix}=\begin{bmatrix}0.994\\ 0.930\\ 0.164\\ -0.729\\ -1.080\\ -0.628\\ 0.291\\ 0.994\\ 0.958\\ 0.211\\ -0.694\\ -1.082\\ -0.666\\ 0.248\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_6=\begin{bmatrix}\varphi_{16}\\ \varphi_{26}\\ \varphi_{36}\\ \varphi_{46}\\ \varphi_{56}\\ \varphi_{66}\\ \varphi_{76}\\ \varphi_{86}\\ \varphi_{96}\\ \varphi_{106}\\ \varphi_{116}\\ \varphi_{126}\\ \varphi_{136}\\ \varphi_{146}\\ \varphi_{156}\end{bmatrix}=\begin{bmatrix}-1.107\\ -0.669\\ 0.502\\ 1.136\\ 0.545\\ -0.634\\ -1.129\\ -0.405\\ 0.756\\ 1.101\\ 0.258\\ -0.864\\ -1.053\\ -0.100\\ 1.000\end{bmatrix}$$

(4.8)

$$\boldsymbol{\varphi}_7=\begin{bmatrix}\varphi_{17}\\ \varphi_{27}\\ \varphi_{37}\\ \varphi_{47}\\ \varphi_{57}\\ \varphi_{67}\\ \varphi_{77}\\ \varphi_{87}\\ \varphi_{97}\\ \varphi_{107}\\ \varphi_{117}\\ \varphi_{127}\\ \varphi_{137}\\ \varphi_{147}\\ \varphi_{157}\end{bmatrix}=\begin{bmatrix}1.209\\ 0.284\\ -1.062\\ -0.869\\ 0.587\\ 1.190\\ 0.063\\ -1.155\\ -0.693\\ 0.777\\ 1.118\\ -0.166\\ -1.209\\ -0.487\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_8=\begin{bmatrix}\varphi_{18}\\ \varphi_{28}\\ \varphi_{38}\\ \varphi_{48}\\ \varphi_{58}\\ \varphi_{68}\\ \varphi_{78}\\ \varphi_{88}\\ \varphi_{98}\\ \varphi_{108}\\ \varphi_{118}\\ \varphi_{128}\\ \varphi_{138}\\ \varphi_{148}\\ \varphi_{158}\end{bmatrix}=\begin{bmatrix}-1.303\\ 0.209\\ 1.342\\ -0.005\\ -1.341\\ -0.194\\ 1.313\\ 0.389\\ -1.255\\ -0.575\\ 1.169\\ 0.749\\ -1.059\\ -0.898\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_9=\begin{bmatrix}\varphi_{19}\\ \varphi_{29}\\ \varphi_{39}\\ \varphi_{49}\\ \varphi_{59}\\ \varphi_{69}\\ \varphi_{79}\\ \varphi_{89}\\ \varphi_{99}\\ \varphi_{109}\\ \varphi_{119}\\ \varphi_{129}\\ \varphi_{139}\\ \varphi_{149}\\ \varphi_{159}\end{bmatrix}=\begin{bmatrix}1.392\\ -0.785\\ -1.195\\ 1.087\\ 0.915\\ -1.322\\ -0.575\\ 1.470\\ 0.198\\ -1.520\\ 0.193\\ 1.471\\ -0.572\\ -1.315\\ 1.000\end{bmatrix}$$

$$\boldsymbol{\varphi}_{10}=\begin{bmatrix}\varphi_{110}\\ \varphi_{210}\\ \varphi_{310}\\ \varphi_{410}\\ \varphi_{510}\\ \varphi_{610}\\ \varphi_{710}\\ \varphi_{810}\\ \varphi_{910}\\ \varphi_{1010}\\ \varphi_{1110}\\ \varphi_{1210}\\ \varphi_{1310}\\ \varphi_{1410}\\ \varphi_{1510}\end{bmatrix}=\begin{bmatrix}-1.471\\ 1.412\\ 0.552\\ -1.768\\ 0.604\\ 1.373\\ -1.501\\ -0.392\\ 1.758\\ -0.756\\ -1.264\\ 1.581\\ 0.229\\ -1.722\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_{11}=\begin{bmatrix}\varphi_{111}\\ \varphi_{211}\\ \varphi_{311}\\ \varphi_{411}\\ \varphi_{511}\\ \varphi_{611}\\ \varphi_{711}\\ \varphi_{811}\\ \varphi_{911}\\ \varphi_{1011}\\ \varphi_{1111}\\ \varphi_{1211}\\ \varphi_{1311}\\ \varphi_{1411}\\ \varphi_{1511}\end{bmatrix}=\begin{bmatrix}1.540\\ -2.049\\ 0.557\\ 1.474\\ -2.067\\ 0.644\\ 1.408\\ -2.086\\ 0.728\\ 1.340\\ -2.100\\ 0.811\\ 1.268\\ -2.102\\ 1.000\end{bmatrix}\qquad \boldsymbol{\varphi}_{12}=\begin{bmatrix}\varphi_{112}\\ \varphi_{212}\\ \varphi_{312}\\ \varphi_{412}\\ \varphi_{512}\\ \varphi_{612}\\ \varphi_{712}\\ \varphi_{812}\\ \varphi_{912}\\ \varphi_{1012}\\ \varphi_{1112}\\ \varphi_{1212}\\ \varphi_{1312}\\ \varphi_{1412}\\ \varphi_{1512}\end{bmatrix}=\begin{bmatrix}-1.596\\ 2.651\\ -1.991\\ 0.048\\ 1.927\\ -2.656\\ 1.668\\ 0.399\\ -2.208\\ 2.589\\ -1.295\\ -0.836\\ 2.425\\ -2.439\\ 1.000\end{bmatrix}$$

$$\boldsymbol{\varphi}_{13} = \begin{bmatrix} \varphi_{113} \\ \varphi_{213} \\ \varphi_{313} \\ \varphi_{413} \\ \varphi_{513} \\ \varphi_{613} \\ \varphi_{713} \\ \varphi_{813} \\ \varphi_{913} \\ \varphi_{1013} \\ \varphi_{1113} \\ \varphi_{1213} \\ \varphi_{1313} \\ \varphi_{1413} \\ \varphi_{1513} \end{bmatrix} = \begin{bmatrix} 1.641 \\ -3.174 \\ 3.522 \\ -2.557 \\ 0.637 \\ 1.522 \\ -3.112 \\ 3.539 \\ -2.643 \\ 0.760 \\ 1.408 \\ -3.049 \\ 3.549 \\ -2.717 \\ 1.000 \end{bmatrix} \quad \boldsymbol{\varphi}_{14} = \begin{bmatrix} \varphi_{114} \\ \varphi_{214} \\ \varphi_{314} \\ \varphi_{414} \\ \varphi_{514} \\ \varphi_{614} \\ \varphi_{714} \\ \varphi_{814} \\ \varphi_{914} \\ \varphi_{1014} \\ \varphi_{1114} \\ \varphi_{1214} \\ \varphi_{1314} \\ \varphi_{1414} \\ \varphi_{1514} \end{bmatrix} = \begin{bmatrix} -1.673 \\ 3.581 \\ -4.883 \\ 5.362 \\ -4.993 \\ 3.668 \\ -1.783 \\ -0.405 \\ 2.524 \\ -4.215 \\ 5.192 \\ -5.290 \\ 4.491 \\ -2.926 \\ 1.000 \end{bmatrix} \quad \boldsymbol{\varphi}_{15} = \begin{bmatrix} \varphi_{115} \\ \varphi_{215} \\ \varphi_{315} \\ \varphi_{415} \\ \varphi_{515} \\ \varphi_{615} \\ \varphi_{715} \\ \varphi_{815} \\ \varphi_{915} \\ \varphi_{1015} \\ \varphi_{1115} \\ \varphi_{1215} \\ \varphi_{1315} \\ \varphi_{1415} \\ \varphi_{1515} \end{bmatrix} = \begin{bmatrix} 1.692 \\ -3.834 \\ 5.815 \\ -7.550 \\ 8.961 \\ -9.990 \\ 10.590 \\ -10.737 \\ 10.425 \\ -9.665 \\ 8.491 \\ -6.953 \\ 5.115 \\ -3.054 \\ 1.000 \end{bmatrix}$$

The above eigenvalues are giving the final solution of the eigenvalue-eigenvectors problem (3.8). The big practical value of the modal analysis consists on the possibility of expressing of every dynamic behavior of a building as a linear superposition of its eigenvalues.

5. Conclusions

In general, there is very little known about the procedure which makes earthquakes. Several forces of much different kind try continuously to change the present state of earth. These forces are external, i.e. these which get energy from sources outside the earth (like solar energy, total energy, etc.) and internal, i.e. these which their energy rises from the interior of the earth (for example gravity, the mechanical forces due to the rotation of the earth, the loss of heat, the nuclear energy, etc.).

Hence, the action of these forces leads to the accumulation of elastic stresses between the masses of the rocks. As the accumulation of these stresses is continuously effected, then the strength of the rocks is overdrawed and hence its equilibrium is disturbed and some sudden changes are created. So, a big part of the dynamic energy due to the stresses is changed to kinetic under the form of seismic wavings.

The most important problem of dynamic analysis is therefore to calculate the motion for each time and for every point of the structure. Thus, by the current research a new dynamic model has been proposed for the solution of the three-dimensional structural analysis problem of a non-linear structure subjected under seismic forces. This problem of dynamic analysis was finally reduced to the solution of a system of ordinary differential equations of the second kind.

Some kind of finite elements were used for the solution of this system of ordinary differential equations and solving the corresponding eigenvalues-eigenvectors problem. An application was finally given to the determination of the eigenvalues and eigenvectors of a 15-floor building consisting of reinforced concrete and subjected under horizontal seismic vibration. For the calculation of the above eigenvalues and eigenvectors, the Jacobi transformation method of a real symmetric matrix was used. This method consists of a sequence of orthogonal similarity transformations, known as the Jacobi rotations.

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