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Non-linear Multidimensional Singular Integral Equations for Vertical Axis Wind Turbines

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Abstract
A new two-dimensional fluid mechanics representation analysis is introduced for the investigation of inviscid flowfields of unsteady airfoils. So, the velocity and pressure coefficient field around a NACA airfoil is determined, while the above problem is reduced to the solution of a non-linear multidimensional singular integral equation, when the form of the source and vortex strength distribution is dependent on the history of the vorticity and source distribution on the NACA airfoil surface. An application is finally given to the determination of the pressure coefficient and the velocity field around the blades of a vertical axis wind turbine, by assuming linear vortex distribution.

Key Word and Phrases
Two-dimensional Fluid Mechanics, NACA airfoil, Non-linear Multidimensional Singular Integral Equations, Unsteady Flow, Vertical Axis Wind Turbine, Linear Vortex Distribution.

1. Introduction

Over the recent years, the non-linear singular integral equations developed an increasing interest, because of their application to the solution of general problems of fluid dynamics, referring to unsteady flows. Such fluid mechanics and aerodynamic problems, are reduced to the solution of a non-linear singular integral equation, connected with the design and evaluation of the aerodynamic characteristics of an airfoil section.

Such aerodynamic characteristic of the NACA airfoils were always an important element in several engineering designs, like aircraft wings, turbomachinery vanes, helicopter rotors, propellers and prop fans. Recently, the continuous improvement of wind turbine structures, for the minimization of the cost per unit produced electric energy, has concentrated the attention on aerodynamic applications in connection with wind energy methods.

As a beginning A.M.O.Smith and J.L.Hess [1], were the first scientists who investigated aerodynamic panel methods for studying airfoils with zero lift, while they modeled the airfoil either with distributed potential source panels for nonlifting flows, or vortex panels for flow with lift. Their method was extended by R.H.Djojodihardjo and S.E.Widnall [2], P.E.Robert and G.R.Saaris [3], J.M.Summa [4], T.Sarpkaya and R.L.Schoaf [5], D.R.Bristow [6], D.R.Bristow and J.D.Hawk [7] and R.J.Lewis [8], for studying three-dimensional steady and unsteady flows, by combining source and vortex singularities.

On the other hand, N.D.Ham [9], F.D.Deffenbaugh and F.J.Marschall [10], M.Kiya and M.Arie [11] and T.Sarpkaya and H.K.Kline [12] investigated some potential flow models, while the separating boundary layers were represented by some discrete vortices, emanating from a known separation point location on the airfoil surface.

In addition, several scientists made extensive calculations by using unsteady turbulent boundary layer methods during the last years, like R.E.Singleton and J.F.Nash [13], J.F.Nash, L.W.Carr and R.E.Singleton [14], A.A.Lyrio, J.H.Ferzinger and S.J.Kline [15], J.Kim, S.J.Kline and J.P.Johnston [16] and W.J.McCroskey and S.I.Pucci [16].

Recently, E.G.Ladopoulos [18] - [21] proposed non-linear singular integral equation methods for the solution of fluid mechanics problems. Beyond the above, over the past years, wind turbines have merited special attention because of their many important energy applications. Vertical axis wind turbines have a continuously increasing aspect, for the minimization of the cost per unit produced electric energy. R.J.Muraca, M.V.Stephens and J.R.Dagenhart [22], J.H.Strickland [23], P.N.Shankar [24] and R.E.Wilson, P.B.S.Lissaman and S.N.Walker [25] introduced a momentum model, consisting in the use of multiple streamtubes, for studying vertical axis wind turbines.

On the other hand, J.Paraschivoiu [26] and J.Paraschivoiu et al. [27] improved the above model, by separating the flow into two different parts. Also, J.L.Loth and H.McCoy [28] proposed an upwind and downwind momentum model for the optimization of vertical axis wind turbines.

Apart from these, J.B.Fanucci and R.E.Walters [29] and J.H.Strickland, B.T.Webster and T.Nguyen [30], [31] introduced a vortex method for two- and three-dimensional flows for the investigation of vertical axis wind turbines.

In the present research by using the field theory of Green, the problem of the unsteady flow of a two-dimensional NACA airfoil is reduced to the solution of a non-linear multidimensional singular integral equation. This nonlinearity is because the source and vortex strength distribution are dependent on the history of the vorticity and source distribution on the NACA airfoil surface.

Application is finally given to the determination of the velocity and pressure coefficient field presented in a vertical axis wind turbine by assuming linear vortex distribution.

2. Analysis of Non-linear Unsteady Inviscid Flowfields

A general non-linear unsteady fluid mechanics representation is further investigated, for the unsteady flow of a two-dimensional NACA airfoil. The method presented consists in the generalization of all past methods, by reducing the problem to the solution of a non-linear multidimensional singular integral equation. This nonlinearity results because of the general form given to the source and vortex strength distribution, while these are dependent on the history of the vorticity and source distribution on the NACA airfoil surface. [18] – [22]

Consequently, consider a two-dimensional airfoil moving in an homogeneous and inviscid fluid. (Fig.1).

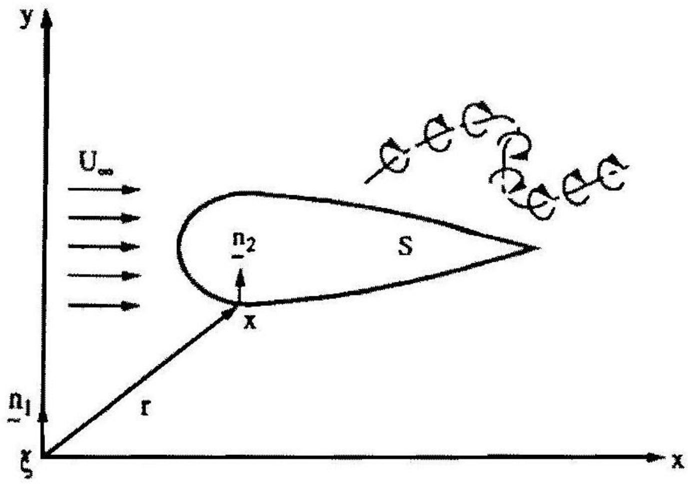

Fig. 1 A two-dimensional airfoil of surface S in an homogeneous, incompressible and inviscid fluid.

The airfoil with the wake comprise s complete lifting system in an irrotational flow through the ideal fluid. Because of the existence of such an irrotationality, then for the local fluid velocity **U** we have:

$$\nabla \times \mathbf{U} = 0 \tag{2.1}$$

Also, by replacing the fluid velocity with the total velocity potential H one has:

$$\mathbf{U} = \nabla H \tag{2.2}$$

while (2.2) can be further written as:

$$\mathbf{U} = \mathbf{U}_\infty + \nabla h \tag{2.3}$$

with $\mathbf{U}_\infty$ the outward velocity (Fig. 1) and h the potential due to the presence of the airfoil.

Because of the conservation of mass for an incompressible fluid, the vector field doesn't diverge:

$$\nabla \cdot \mathbf{U} = 0 \tag{2.4}$$

Thus, by using (2.2), (2.3) and (2.4) we obtain equation of Laplace which is the governing equation:

$$\nabla^2 h = 0 \tag{2.5}$$

In addition, by using Green's theorem [32] follows a basic relation for the velocity potential $h(\mathbf{x},t)$, with t the time, at any point $\mathbf{x}$ in continuous, acyclic irrotational flow:

$$h(\mathbf{x},t) = -1/2\pi \int_S \frac{g[\xi,t,h]}{r} \,\mathrm{d}\,S + 1/2\pi \int_{S+W} \delta[\xi,t,h] \frac{\partial}{\partial n_1}\left(\frac{1}{r}\right) \mathrm{d}\,S \tag{2.6}$$

where S is the surface of the airfoil (Fig. 1), W the surface of the wake, $\mathbf{n_1}$ the surface normal at the source point ξ (Fig. 1), $g[\xi,t,h]$ the source strength distribution, $\delta[\xi,t,h]$ the vortex strength distribution and r the distance equal to:

$$r = |\mathbf{x} - \xi| \tag{2.7}$$

The velocity potential (2.4) can be also written as following, which denotes a two-dimensional non-linear singular integral equation:

$$h(\mathbf{x},t) = -1/2\pi \int_S \frac{g[\xi,t,h]}{r} \mathrm{d}S + 1/2\pi \int_{S+W} \frac{\delta[\xi,t,h]}{r^2} \mathrm{d}S \tag{2.8}$$

The kinematical surface tangency condition on the surface of the airfoil can be written as following: [33]

$$\left(1/\left|\nabla S(\mathbf{x},t)\right|\right) \frac{\partial S(\mathbf{x},t)}{\partial t} + \frac{\partial h}{\partial n_2} + \mathbf{U}_\infty \cdot \mathbf{n}_2 = 0 \tag{2.9}$$

where $\mathbf{n}_2$ denotes the surface normal at the field point $\mathbf{x}$ (Fig. 1).

The above condition can be further written as following, for a body fixed coordinate system:

$$\left(1/\left|\nabla S(\mathbf{x},t)\right|\right) \frac{\partial S(\mathbf{x},t)}{\partial t} = -\left(\mathbf{U}_A + \boldsymbol{\omega}_\mathbf{A} \times \mathbf{x}\right) \cdot \mathbf{n}_2 \tag{2.10}$$

in which $\mathbf{U}_A$ denotes the airfoil translation velocity and $\boldsymbol{\omega}_A$ the airfoil angular rotation.

From eqs (2.9) and (2.10) follows:

$$\frac{\partial h}{\partial n_2} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \mathbf{n}_2 = 0 \tag{2.11}$$

In addition, by inserting (2.11) into (2.8) results the following two-dimensional non-linear singular integral equation:

$$\begin{gathered} 1/2\pi \int_S g[\xi,t,h] \frac{\partial}{\partial n_2}\left(\frac{1}{r}\right) \mathrm{d}S + 1/2\pi \int_{S+W} \delta[\xi,t,h] \frac{\partial}{\partial n_2}\left(\frac{1}{r^2}\right) \mathrm{d}S = \\ -\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \mathbf{n}_2 \end{gathered} \tag{2.12}$$

The non-linear singular integral equation (2.12) can be further written as:

$$\begin{gathered} 1/2\pi \int_S \frac{g[\xi,t,h]}{r^2} \mathrm{d}S + 1/\pi \int_{S+W} \frac{\delta[\xi,t,h]}{r^3} \mathrm{d}S = \\ \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_\mathbf{A} \times \mathbf{x}\right) \cdot \mathbf{n}_\mathbf{2} \end{gathered} \tag{2.13}$$

Hence, by solving the non-linear integral equation (2.13) with the corresponding boundary conditions, then the velocity at any field point will be determined through (2.9).

3. Non-linear Airloads Analysis

The pressure distribution on the airfoil may be obtained by the unsteady Bernoulli equation, valid at any point in an irrotational, ideal flow:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + 1/2\left(\nabla H\right)^2\right] \tag{3.1}$$

where ρ denotes the fluid density.

Furthermore, by using the derivation of the previous section, then (3.1) will be written as:

$$P = P_\infty - \rho\left[\frac{\partial h}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla h + 1/2\left(\nabla h\right)^2\right] \tag{3.2}$$

Also, (3.2) reduces to the following form:

$$\begin{aligned} P = P_\infty - \rho\Bigg[&\frac{\partial H}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla_S H + \\ &+ \frac{\partial H}{\partial n_1}\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \mathbf{n}_1 + 1/2\left(\nabla_S H\right)^2 + 1/2\left(\frac{\partial H}{\partial n_1}\right)^2\Bigg] \end{aligned} \tag{3.3}$$

if we replace the ∇f , by the surface gradient $\nabla_S h$:

$$\nabla h = \nabla_S h + \frac{\partial h}{\partial n_1}\varepsilon_{n_1} \tag{3.4}$$

Thus, because of (2.9), then (3.3) can be written as:

$$\begin{aligned} P = P_\infty - \rho\Bigg[&\frac{\partial H}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla_S H - \\ &- 1/2\left\{\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \mathbf{n}_1\right\}^2 + 1/2\left(\nabla_S H\right)^2\Bigg] \end{aligned} \tag{3.5}$$

which will be used for the computations.

The basic object of the current research was to develop a general non-linear model for the determination of the velocity field around a NACA airfoil in two-dimensional unsteady flow. This problem was reduced to the solution of a two-dimensional non-linear singular integral equation, while this form of nonlinearity was obtained because of the form of the general type of the source and vortex strength distribution.

4. Velocity and Pressure Coefficient Field for Linear Vortex Distribution

Suppose the special case of a linear vortex distribution δ. In this case the general non-linear problem presented previously, will be much more simplified and will be solved as a linear problem. The geometrical representation of this problem is shown in Fig. 3.

For a linear vortex distribution δ, then the fluid velocity U is given as following:

$$\mathbf{U} = \int_{-A/2}^{A/2} \frac{\delta \, \mathrm{d}r}{2\pi r} \left(-\sin\varphi \mathbf{i} + \cos\varphi \mathbf{j}\right) \tag{4.1}$$

where **i**, **j** are the unit vectors on the x and y axes, respectively, and A denotes the separating wake (Fig. 2).

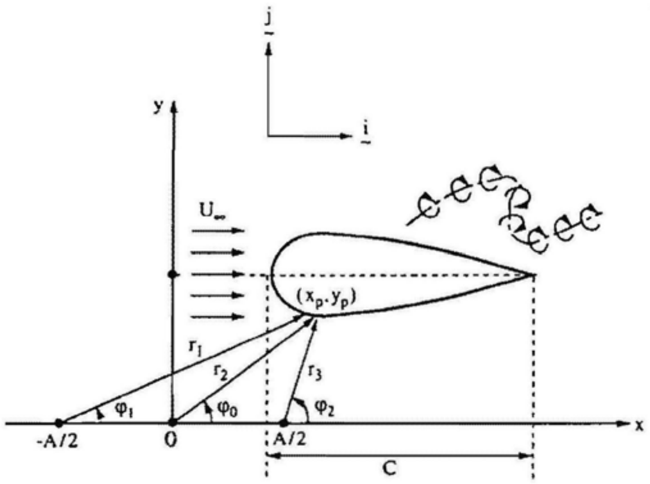

Fig. 2 Coordinate system for the 2D airfoil

So, when $y_p \neq 0$ and $y_p = 0$, then the fluid velocity **U** will be computed by the following formulas:

$$\mathbf{U} = \begin{cases} a/2\pi \left[x_p(\varphi_1 - \varphi_2) + y_p \ln\left|\frac{r_1}{r_2}\right| \right]\mathbf{i} + a/2\pi \left[x_p \ln\left|\frac{r_1}{r_2}\right| - A - y_p(\varphi_1 - \varphi_2) \right]\mathbf{j}, \; y_p \neq 0 \\ a/2\pi \left[x_p \ln\left|\frac{r_1}{r_2}\right| - A \right]\mathbf{j}, \quad y_p = 0 \end{cases} \tag{4.2}$$

where a is the angle of attack.

Moreover, we consider the pressure coefficient C_p:

$$C_p = (P - P_\infty)\big/\left(1/2\,\rho U_\infty^2\right) \tag{4.3}$$

where ρ denotes the fluid density and P_∞ the stream pressure.

Beyond the above, by using the unsteady Bernoulli's equation, then the pressure coefficient will be simplified as following:

$$C_p = -U^2\big/U_\infty^2 \tag{4.4}$$

which will be used for the computations.

5. Vertical Axis Wind Turbine Application

The previous mentioned theory of 2D unsteady inviscid flowfields will be applied for the computation of the velocity and pressure coefficient field presented in a vertical axis wind turbine. Such types of wind turbines are of continuously increasing interest the last years, because of their big advantage to the minimization of the cost per unit produced electric energy.

Two big advantages of the vertical axis wind turbines, are their efficiency which is continuously improved and their independence of their operation on the orientation of the wind. In the present application the vertical axis wind turbine considered, has the following geometrical sizes: head diameter: $D = 1.60m$, length: $H = 1.10m$, 2 blades, blade chord: $c = 0.12m$ and blade airfoil section NACA 0018 (Fig. 3)

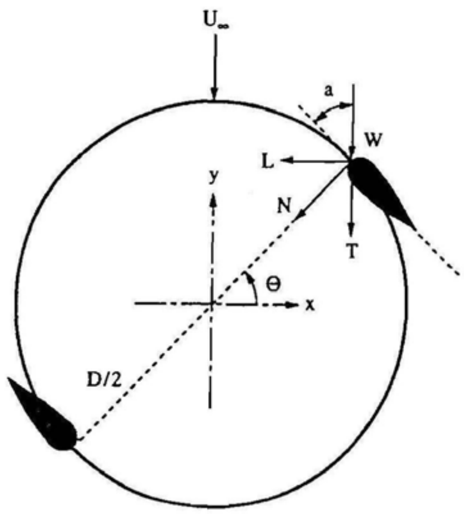

Fig. 3 Vertical axis wind turbine

Moreover, it was supposed linear vortex distribution and thus, the velocity field on the boundary and around of the airfoil was computed by (4.2) for azimuthal angle $\Theta = 90°$. Also, the pressure coefficients C_p were calculated by (4.4) for several wind velocities U_∞ and for angle of attack $a = 20°$.

Consequently, Figs. 4 - 7 show the pressure distribution on the vertical axis wind turbine considered, for wind speed $U_\infty = 10,15,20,25 m/s$, respectively.

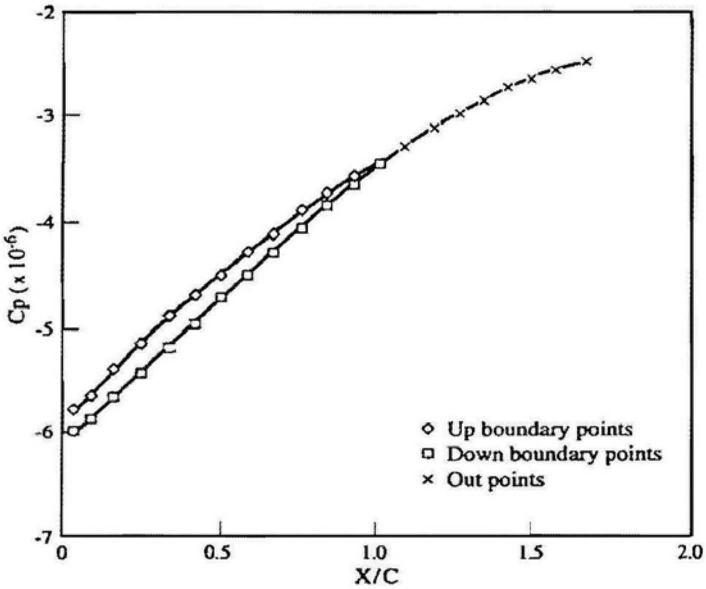

Fig. 4 Pressure distribution on the vertical axis wind turbine of Fig. 3 for wind speed 10 m/sec.

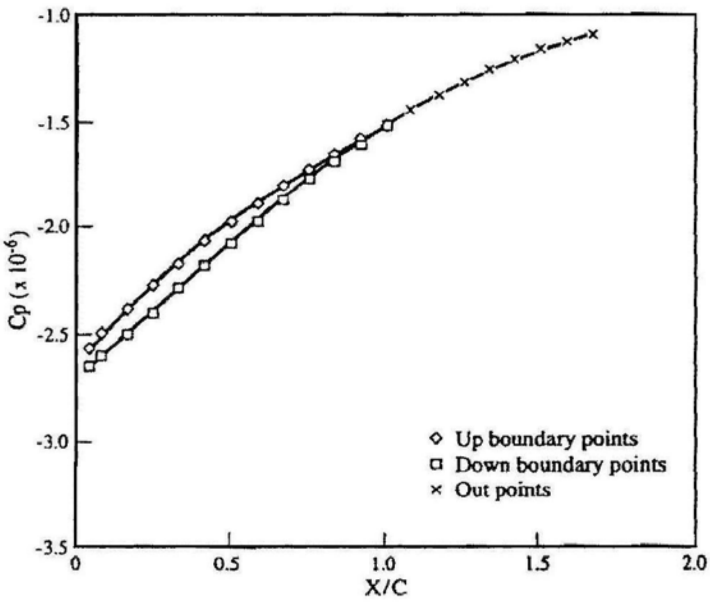

Fig. 5 Pressure distribution on the vertical axis wind turbine of Fig. 3 for wind speed 15 m/sec.

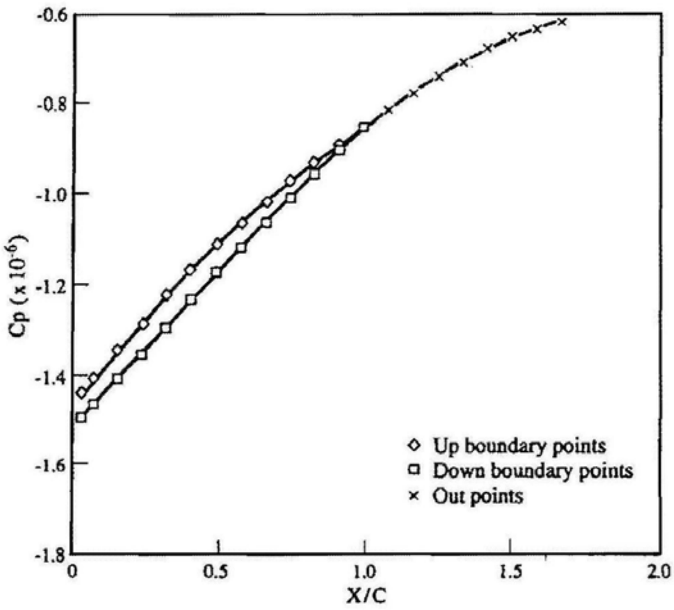

Fig. 6 Pressure distribution on the vertical axis wind turbine of Fig. 3 for wind speed 20 m/sec.

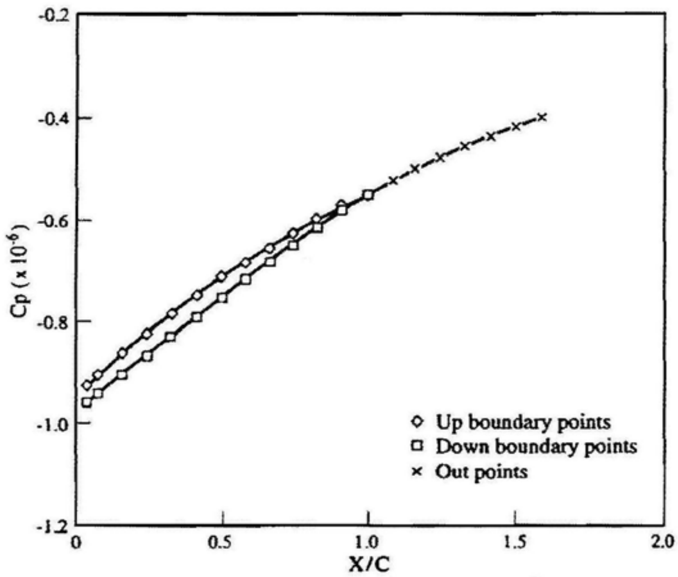

Fig. 7 Pressure distribution on the vertical axis wind turbine of Fig. 3 for wind speed 25 m/sec.

From the above Figures, it is shown that the values for both up and down points on the boundary of the airfoil, are continuously increasing when beginning from $x/c = 0$ up to $x/c = 1$.

7. Conclusions

A general non-linear model has been proposed for the determination of the velocity and pressure coefficient field around a NACA airfoil in two-dimensional inviscid and unsteady flow. Such a problem was reduced to the solution of a 2-D non-linear singular integral equation by applying the field theory of Green.

Consequently, in future non-linear singular integral equation methods will be of continuously interest, because these are very important for the determination of generalized solid mechanics and fluid mechanics problems. So, modern problems of solid and fluid mechanics will be very much simplified, when solved by general non-linear singular integral equation methods.

In the above analysis, special attention was given to the investigation of the vertical axis wind turbines, which are of continuously increasing interest for the minimization of the cost per unit produced electric energy. The special application presented, was for the determination of the pressure coefficient field around the blades of a vertical axis wind turbine, by assuming linear vortex distribution.

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