Universal Journal of Integral Equations **6 (2018), 9-19**
www.papersciences.com

New Contributions of Quadrature Approximation Method for Hypersingular Integral Equations defined in Banach Spaces

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Abstract

A sophisticated numerical method is further improved and investigated for the hypersingular integral equations defined in Banach spaces. The hypersingular integral equations belong to a wider class of singular integral equations having much more stronger singularities. The proposed approximation method is an extension beyond the quadrature method. Also, an error estimates theory is proposed and investigated for the hypersingular integral equations by proving some new theorems. Finally, the inequalities valid between the exact solutions of the hypersingular integral equations and the corresponding approximate solutions, are proposed and proved.

2010 Mathematics Subject Classification : 65L10, 65R20.

Key Word and Phrases

Hypersingular Integral Equations, Banach Spaces, Singularity, Quadrature Method, Error Estimates, Finite-Part Singular Integral Equations.

1. Introduction

The hypersingular integral equations consist to a wider class of singular integral equations. In particular the kernel of such integral equations has a stronger singularity as compared to the finite-part singular integral equations. Hence, there is very big interest for the numerical evaluation of the hypersingular integral equations, as closed form solutions are not possible to be determined.

J. Hadamard [1], [2] was the first scientist who introduced the concept of finite - part integrals, and L. Schwartz [3] studied very basic properties of them. Some years later, H.R. Kutt [4] proposed some algorithms for the numerical evaluation of the finite-part singular integrals and studied the difference between a finite - part integral and a "generalized principal value integral".

On the other hand, M.A. Golberg [5] investigated the convergence of several numerical methods for the solution of finite-part integrals. He proposed a method, which was an extension beyond the Galerkin and collocation methods [6]. Besides, A.C. Kaya and F. Erdogan [7], [8] introduced and investigated several problems of elasticity and fracture mechanics, which are reduced to the solution of finite-part singular integral equations.

Beyond the above, by E.G. Ladopoulos [9] - [15] were proposed several numerical methods for the solution of the finite-part singular integral equations of the first and the second kind. He further applied this type of singular integral equations to the solution of very important problems of elasticity, fracture mechanics and aerodynamics. Moreover, E.G. Ladopoulos, V.A. Zisis and D. Kravvaritis [16], [17] used functional analysis for the solution of finite-part singular integral equations. Thus, they studied such type of singular integral equations defined in general Hilbert spaces and L_p spaces and applied them to several crack problems.

By the present paper the hypersingular integral equations are introduced and investigated, which have stronger singularity in comparison to the finite-part singular integral equations. The hypersingular integral equations belong therefore to a wider class of integral equations with kernels of very strong singularities.

A numerical method is proposed for the solution of the hypersingular integral equations, defined in Banach spaces. The proposed approximation method is an extension beyond the quadrature method.

In addition, an error estimates theory is investigated for the hypersingular integral equations, by proving the corresponding theorem. Hence, the inequalities which are valid between the exact solutions of the hypersingular integral equations and the corresponding approximate solutions, are investigated and proved.

In general, the hypersingular integral equations are used for the solution of several important problems of engineering mechanics, and especially in the theories of elasticity, fracture mechanics, fluid mechanics and aerodynamics.

2. New Aspects of Approximation Methods for Hypersingular Integral Equations

Definition 2.1

An equation of the following form is called hypersingular integral equation:

$$\int_a^b \frac{u(x)dx}{|x-t|^{\lambda}} = f(t), \qquad 1<\lambda<3 \tag{2.1}$$

n which *u(x)* is the unknown function and *f(t)* is a known function such as *f(t)* ∈ $C^{\infty}[a,b]$.

Theorem 2.1

Let the hypersingular integral equation (2.1) and suppose that following conditions are satisfied:

$$u(x)=\begin{cases} k_1(x-a)^{(\lambda-1)/2}+k_2(x-a)^{(\lambda+1)/2}+\omega_1(x), & for\ x=a \\ k_1'(b-x)^{(\lambda-1)/2}+k_2'(b-x)^{(\lambda+1)/2}+\omega_2(x), & for\ x=b \end{cases} \tag{2.2}$$

where the functions $\omega_1''(x)$, $\omega_2''(x)$ are Hölder - continuous with exponent ε>o.

Then the hypersingular integral equation (2.1) is approximated by the quadrature formula:

$$R(t_j)=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{u(t_i)}{|x-t_j|^{\lambda}}dx \tag{2.3}$$

and its error function $\Delta(t_j)$ satisfies the estimate:

$$|\Delta(t_j)| \le \begin{cases} D\left(\dfrac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}}+\dfrac{h}{\delta_j^{(\lambda+1)/2}}\right), & for\ \lambda\neq 2 \\ Dh\dfrac{|\ln h|}{\delta_j^{3/2}}, & for\ \lambda=2 \end{cases} \tag{2.4}$$

where $h=\dfrac{b-a}{n}$, D a constant and δ_j the distance of the point t_j from the boundary of the segment $[a, b]$.

Proof

The hypersingular integral in the left hand side of (2.1) is understood in its principal value sense: [5], [15]

$$I(t)=\int_a^b \frac{u(x)}{|x-t|^\lambda}\,dx = \lim_{\varepsilon\to o}\left[\int_a^{t-\varepsilon}\frac{u(x)-u(t)}{|x-t|^\lambda}\,dx+\int_{t+\varepsilon}^{b}\frac{u(x)-u(t)}{|x-t|^\lambda}\,dx+ \\ +u(t)\int_a^{t-\varepsilon}\frac{dx}{|x-t|^\lambda}+u(t)\int_{t+\varepsilon}^{b}\frac{dx}{|x-t|^\lambda}+2u(t)\frac{\varepsilon^{-\lambda+1}}{1-\lambda}\right] \tag{2.5}$$

For the numerical solution of the integral $I(t)$, then the following points are used:

$x_i = a+ih$, $i = 0, 1,..., n$ and $t_j=a + (j+1/2)h$, $j = 0,1,...n-1$

where $h = \dfrac{b-a}{n}$ and the quadrature formula (2.3) is applied.

Besides, consider the point t_j be at the distance δ_j from the boundary of the segment $[a, b]$, where $\delta j \geq 5h$.

If $\Delta(t_j)$ is the error function, then it is valid:

$$\begin{aligned}|\Delta(t_j)| &= |I(t_j)-R(t_j)| - \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\frac{u(x)-u(t_i)}{|x-t_j|^\lambda}\,dx\right| \\ &\leq \left|\sum_{s_i\in[a,b]}\int_{s_i}\frac{u(x)-u(t_i)}{|x-t_j|^\lambda}\,dx\right| + \left|\sum_{s_i\in N_h}\int_{s_i}\frac{u(x)-u(t_i)}{|x-t_j|^\lambda}\,dx\right| = \Delta_1+\Delta_2\end{aligned} \tag{2.6}$$

in which the set N_h consists of segments $s_i= [x_i,x_{i+1}]$.

Moreover, consider the following equality to be valid:

$$\sum_{s_i\in N_h}\int_{s_i}\frac{u'(t_j)(x-t_i)}{|x-t_j|^\lambda}\,dx = \sum_{s_i\in N_h}\int_{s_i}\frac{u'(t_j)(x-t_j)}{|x-t_j|^\lambda}\,dx + \sum_{s_i\in N_h}\int_{s_i}\frac{u'(t_j)(t_j-t_i)}{|x-t_j|^\lambda}\,dx = \Gamma_1+\Gamma_2 \tag{2.7}$$

Since the set N_h is symmetric with respect to t_j, then follows: $\Gamma_1 = \Gamma_2 = 0$.

The following formulas hold:

$$u(x)-u(t_i) - u'(t_j)\,(x-t_i) = u''(z_{ij})\,(z_i-t_j)\,(x-t_i)=a_{ij} \tag{2.8}$$

in which $x\in [x_i, x_{i+1}]$, $t_i\in [x_i, x_{i+1}]$ and $z_{ij} \in [z_i, t_j]$

On the other hand, since $u(x)$ satisfies condition (2.2), then the following inequality is valid:

$$\left| u''(z_{ij}) \right| \leq D\delta_j^{(\lambda-5)/2} \tag{2.9}$$

from which follows:

$$\Delta_2 = \left| \sum_{s_i\in N_h} \int_{s_i} \frac{u'(t_j)(x-t_i)}{\left|x-t_j\right|^{\lambda}} dx + \sum_{s_i\in N_h, i\neq j} \int_{s_i} \frac{a_{ij}(x)}{\left|x-t_j\right|^{\lambda}} dx + \int_{s_i}^{x_{j+1}} \frac{a_{jj}(x)}{\left|x-t_j\right|^{\lambda}} dx \right|$$

$$\leq D\left[\frac{h}{\delta_j^{(5-\lambda)/2}} \int_{t_j+h/2}^{t_j+\delta_j/2-h} \frac{dx}{\left|x-t_j\right|^{\lambda-1}} + \frac{1}{\delta_j^{(5-\lambda)/2}} \int_{t_j-h/2}^{t_j+h/2} \frac{dx}{\left|x-t_j\right|^{\lambda-2}} \right] \tag{2.10}$$

and thus:

$$\Delta_2 \leq \begin{cases} D\left(\dfrac{h^{3-\lambda}}{\delta_j^{(5-\lambda)|2}} + \dfrac{h}{\delta_j^{(\lambda+1)|2}} \right), & for\ \lambda\neq 2 \\ Dh\, \dfrac{\left|\ell n h\right|}{\delta_j^{3|2}} , & for\ \lambda=2 \end{cases} \tag{2.11}$$

By the same way by applying (2.2), then follows inequality:

$$\Delta_1 \leq \max_s \left\{ \left| \sum_{s_i\in[a,b]} \int_{s_i} \frac{g_s(x)-g_s(t_i)}{\left|x-t_j\right|^{\lambda}} dx \right| + \left| \sum_{s_i\in[a,b]} \int_{s_i} \frac{d_s(x)-d_s(t_i)}{\left|x-t_j\right|^{\lambda}} dx \right| \right\} = \Delta_1' + \Delta_1'' \ , \quad s=1,2 \tag{2.12}$$

with:

$$\begin{aligned} g_1(x) &= k_1(x-a)^{(\lambda-1)/2} \\ g_2(x) &= k_1'(b-x)^{(\lambda-1)/2} \\ d_1(x) &= k_2(x-a)^{(\lambda+1)/2}+\omega_1(x) \\ d_2(x) &= k_2'(b-x)^{(\lambda+1)/2}+\omega_2(x) \end{aligned} \tag{2.13}$$

Additionally, following inequality holds for the functions $g_s(x)$:

$$|g_s(x)-g_s(t_i)| \leq Dh, \qquad \text{for } x\in s_i \tag{2.14}$$

and so:

$$\Delta_1'' \leq D_1 h/\delta_j^{(\lambda+1)/2} \tag{2.15}$$

By denoting further by Ω_1 the set of segments $s_i\in[a, \mathrm{b}]$ which are on the left from point t_j and by Ω_2 the set of such segments which are on the right from t_j, then follows:

$$\Delta_1' \le \left|\sum_{s_i\in\Omega_1}\int_{s_i}\frac{g_s(x)-g_s(t_i)}{\left|x-t_j\right|^{\lambda}}dx\right| + \left|\sum_{s_i\in\Omega_2}\int_{s_i}\frac{g_s(x)-g_s(t_i)}{\left|x-t_j\right|^{\lambda}}dx\right| = Z_1 + Z_2 \tag{2.16}$$

On the contrary, by applying the generalized mean-value theorem we have:

$$\begin{aligned} Z_1 &= h^{(\lambda-1)|2}\left|\sum_{s_i\in\Omega_1}\left[(i+\xi_i)^{(\lambda-1)|2} - \left(i+\frac{1}{2}\right)^{(\lambda-1)|2}\right]\int_{s_i}\frac{dx}{\left|x-t_j\right|^{\lambda}}\right| \\ &\le \frac{Dh^{(\lambda-1)|2}}{m}\sum_{i=0}^{m}\frac{1}{(i+1)^{(3-\lambda)|2}}\int_{\alpha}^{t_j-\delta_j|2+h}\frac{dx}{\left|x-t_j\right|^{\lambda}} \end{aligned} \tag{2.17}$$

where m is the number of segments that belong to Ω_1, $mh>\delta_j/2-h$ and $0<\xi_i<1$.

Hence, inequality (2.17) reduces to:

$$Z_1 \le D_1 h/\delta_j^{(\lambda+1)/2} \tag{2.18}$$

By the same way can be proved a similar inequality for Z_2. Hence, from inequalities (2.15) and (2.18) follows:

$$\Delta_1 \le D_1 h/\delta_j^{(\lambda-1)/2} \tag{2.19}$$

and the estimate (2.4) was proved.

3. New Contributions of Hypersingular Integral Equations Error Estimates Analysis

Theorem 3.1

Consider the hypersingular integral equation (2.1) where $f(t)\in C\infty[a,b]$, with an approximate solution $u_h(t_i)$ given by the system:

$$\sum_{i=0}^{n-1}u_h(t_i)\int_{x_i}^{x_i+1}\frac{dx}{\left|x-t_j\right|^{\lambda}} = f(t_j), \quad j=0,1,\ldots,n-1 \tag{3.1}$$

Then the values $u(t_k)$ of an exact solution to (2.1) and the values $u_h(t_k)$ of the approximate solution obtained from (3.1) satisfy the following inequalities:

$$\begin{aligned} &\left|u(t_k) - u_h(t_k)\right| \le Dh^{(\lambda-1)/2}, \quad 1<\lambda<2 \\ &\left|u(t_k) - u_h(t_k)\right| \le Dh\left|lnh\right|^2, \quad \lambda=2 \\ &\left|u(t_k) - u_h(t_k)\right| \le Dh^{(3-\lambda)/2}, \quad 2<\lambda<3 \end{aligned} \tag{3.2}$$

for $k = 0,1,\ldots, n-1$, where $h = \dfrac{b-a}{n}$ and D a constant.

Proof

Let the system of equations :

$$\sum_{i=0}^{n-1} u_h\,(t_i)\; c_{ij} = f\,(t_j) \tag{3.3}$$

$$\sum_{i=0}^{n-1} u\,(t_i)\; c_{ij} = f\,(t_j) + \Delta f \quad , \quad j = 0,1,..., n\text{-}1$$

where Δf is the error of the quadrature formula.

From (3.3) one obtains:

$$\sum_{i=0}^{n-1} [u(t_i) \text{ - } u_h\,(t_i)]\; d_{ij} = \Delta f(t_j) \tag{3.4}$$

and so:

$$\Delta u(t_k) = u(t_k) \text{ - } u_h(t_k) = \sum_{\ell=0}^{n-1} x_{k\ell}\; \Delta f\;(t_\ell) \tag{3.5}$$

The general Fourier operator will be further used:

$$\frac{1}{2\pi}\int_0^{2\pi} H(\varphi)x(\varphi)M_n(y-\varphi)d\varphi = F(y) \tag{3.6}$$

$$H(\varphi) = \sum_{\ell=-\infty}^{\infty} d_{\ell o} e^{i\ell\varphi}$$

in which:

$$x(\varphi) = \sum_{\ell=0}^{n-1} u_h(t_\ell)e^{i\ell\varphi}$$

$$F(y) = \sum_{\ell=0}^{n-1} f(t_\ell)e^{i\ell y} \tag{3.7}$$

$$M_n(y) = \sum_{\ell}^{n-1} e^{ily}$$

$$d_{ij} = \int_{x_i}^{x_{i+1}} \frac{dx}{\left|x - t_j\right|^{\lambda}}$$

Additionally (3.6) reduces to the following equation:

$$\frac{1}{2\pi}\int_{0}^{2\pi} H_n(\varphi)x(\varphi)M_n(y-\varphi)d\varphi = F(y) \tag{3.8}$$

where:

$$H_n(\varphi) = \sum_{\ell=-n}^{n} d_{\ell o}e^{i\ell\varphi} \tag{3.9}$$

Besides, in order to be obtained estimates for the inverse matrix, an approximation formula is necessary for the Fourier coefficients $x_{k\ell}^{(n)}$ of the function $x_k^{(\varphi)} = \sum_{\ell=0}^{n-1} x_{k\ell}(n)e^{i\ell\varphi}$, where x_k denotes a solution of the following equation:

$$\frac{1}{2\pi}\int_{0}^{2\pi} H_n(\varphi)x_k(\varphi)M_n(y-\varphi)d\varphi = e^{iky}, \quad k = 0,1...,n-1 \tag{3.10}$$

Let us consider the equation:

$$\frac{1}{2\pi}\int_{0}^{2\pi} H_n(\varphi)g_k(\varphi)M_n(y-\varphi)d\varphi = e^{iky} \tag{3.11}$$

where: $H_n(\varphi) = \sum_{\ell=-n}^{n} a_\ell e^{i\ell\varphi}$

$$a_\ell = -\, c_{\ell_0}\, h^{\lambda-1} \tag{3.12}$$

and the solutions $x_k(\varphi)$ and $g_k(\varphi)$ are related by the following formula:

$$x_k(\varphi) = -\, g_k(\varphi)h^{\lambda-1} \tag{3.13}$$

Beyond the above, in order to study the properties of the function $H_n(\varphi)$ consider the set of segments $[\pi/(16n), \pi/(8n)]$, $n = 1,2...$ which forms a covering of the half - open interval $(0, \pi/8]$. By choosing an arbitrary $x \in (0, \pi/8]$, then there exists a minimal number N so that $x \in [\pi/(16N), \pi/(8N)]$.

For $\pi/(16N) \le \varphi \le \pi/(8N)$ and $N \le \ell \le 2N$ the inequality $\sin^2(\ell\varphi/2) > \delta > 0$ is obtained, from which follows:

$$H(N,\varphi) = -2\sum_{\ell=N}^{2N} a_\ell \sin^2(\ell\varphi/2) \ge 2\delta\sum_{\ell=N}^{2N} |a_e| \ge D\varphi^{\lambda-1} \tag{3.14}$$

By choosing $D>0$ not depending on N, then from (3.14) follows that inequality:

$$H(\varphi) > D(\varphi)^{\lambda-1} \tag{3.15}$$

holds for $0 \le \varphi \le \pi/8$.

Similarly following inequality:

$$H(\varphi) > D(2\pi-\varphi)^{\lambda-1} \qquad (3.16)$$

holds for $15\pi/8 \le \varphi \le 2\pi$.

Furthermore, consider $H_n(\varphi) = a_0 + \sum_{\ell=1}^{n} a_\ell \cos(\ell\varphi)$. Because of (3.16), inequality $H_n(\varphi) - H_n(0) > D\,\varphi^{\lambda-1}$ holds for $\pi/(16n) \le \varphi \le \pi/8$.

From the inequality $H_n(0) \ge D_1 \sum_{\ell=n+1}^{\infty} \ell^{-\lambda} \ge D_2 (n+1)^{1-\lambda}$ follows that there exists a constant $B_0 > 0$ not depending on n and for which the inequality $H_n(0) \ge B_0\,\varphi^{\lambda-1}$ holds for $0 \le \varphi \le \pi/(16n)$.

Consequently, by setting $B_0^{\,*} = \min(B_0/2, D)$ one has:

$$H_n(\varphi) \ge (1/2)\,H_n(0) + B_0^{\,*}\,\varphi^{\lambda-1} \qquad (3.17)$$

for $0 \le \varphi \le \pi/8$.

Let us further consider the linear space E spanned by the functions $e^{i\ell_\varphi}$, ℓ=0,1..., n-1 and having the following norms :

$$\begin{aligned} \|x\|_1 &= \left(\int_0^{2\pi} H_n(\varphi)\,|x(\varphi)|^2\,d\varphi\right)^{1|2} \\ \|x\|_2 &= \max_{\|y(\varphi)\|=1}\left|\int_0^{2\pi} x(\varphi)\,\overline{y}(\varphi)\,d\varphi\right| \end{aligned} \qquad (3.18)$$

By $\mathrm{E}(\|\bullet\|_1)$ and $\mathrm{E}(\|\bullet\|_2)$ we denote the linear spaces E having the norms $\|\bullet\|_1$ and $\|\bullet\|_2$, respectively.

Besides, the operator $\Psi(x) = (2\pi)^{-1}\int_0^{2\pi} H_n(\varphi)\,x(\varphi)\,B_n(\Psi-\varphi)\,d\varphi$ maps isometrically the space $\mathrm{E}(\|\bullet\|_1)$ into $\mathrm{E}(\|\bullet\|_2)$.

Consequently, following inequality holds:

$$\left\|e^{ik\varphi}\right\|_2 \le D\left(\int_0^{2\pi}\frac{d\varphi}{H_n(\varphi)}\right)^{1|2} \qquad (3.19)$$

From (3.17) and (3.19) follows:

$$\left\| g_k(\varphi) \right\|_1 = \left\| e^{ik\varphi} \right\|_2 \leq \begin{cases} D\,, & \text{for } 1<\lambda<2 \\ D\ \sqrt{|\ell nh|}, & \text{for } \lambda=2 \\ D / h^{(\lambda-2)/2}, & \text{for } 2<\lambda<3 \end{cases} \tag{3.20}$$

In order to calculate the Fourier coefficients $x(\varphi)$ of the functions $x(\varphi)$, $\left\| x(\varphi) \right\|_1 = 1$, one has :

$$|x_\ell| = \frac{1}{2\pi} \left| \int_0^{2\pi} x(\varphi) e^{-i\ell\varphi} d\varphi \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{b(\varphi)}{\sqrt{H_n(\varphi)}} d\varphi \leq D \left(\int_0^{2\pi} \frac{d\varphi}{H_n(\varphi)} \right)^{1/2} \tag{3.21}$$

and thus $|x_\ell|$ satisfy estimates which are similar to (3.20).

Hence, the Fourier coefficients $x_{k\ell}$ of the functions $x_\chi(\varphi)$ which are solutions of the problem (3.10), $x_k(\varphi) = g_k(\varphi)\, h^{\lambda-1}$ satisfy following inequalities :

$$\begin{aligned} &|x_{k\ell}| \leq Dh^{\lambda-1} &&, 1<\lambda<2 \\ &|x_{k\ell}| \leq Dh\ |\ell \text{n} h| &&, \lambda = 2 \\ &|x_{k\ell}| \leq Dh &&, 2<\lambda<3 \end{aligned} \tag{3.22}$$

As $x_{k\ell}$ belong to the kth row of the inverse matrix, then equations (3.22) denote estimates for the elements of the inverse matrix for (3.1).

Finally from eqs (2.4) and (3.22) one obtains:

$$\begin{aligned} |\Delta u(t_k)| &\leq \sum_{j=0}^{n-1} Dh^{\lambda-1} \left[\frac{h^{3-\lambda}}{\delta_j^{(5-\lambda)|2}} + \frac{h}{\delta_j^{(\lambda+1)|2}} \right] \\ &\leq D_1 \left[\sum_{j=0}^{n|2} \frac{h^2}{h^{(5-\lambda)|2} (j+1)^{(5-\lambda)|2}} + \sum_{j=0}^{n|2} \frac{h^\lambda}{h^{(1+\lambda)|2} (j+1)^{(\lambda+1)|2}} \right] \leq D_2 h^{(\lambda-1)/2} \end{aligned} \tag{3.23}$$

Besides, the proof for $\lambda = 2$ and $2 <\lambda <3$ is done by the same way and thus Theorem was proved.

4. Conclusions

A new approximation method has been further improved and investigated for the numerical solution of the hypersingular integral equations defined in Banach spaces. The hypersingular integral equations consist of a very special class of singular integral equations having kernels with very strong singularities, as compared to the finite-part singular integral equations.

The numerical method which was used is an extension beyond the quadrature method. Consequently, it was proved that the quadrature method is a suitable approximation method for the numerical solution of the hypersingular integral equations. Same method has been successfully used in the past for the numerical evaluation of the non-linear singular integral equations [18] - [28].

Also, an error estimates theory was proposed for the hypersingular integral equations, by proving the suitable theorems. Thus, it was shown that same inequalities are valid between the exact solutions of the hypersingular integral equations and the corresponding numerical solutions.

Finally, the hypersingular integral equations are very important for the solution of basic problems of engineering mechanics and mathematical physics, like example problems of elasticity, fracture mechanics, fluid mechanics and aerodynamics. There is therefore a big interest for further research on the numerical evaluation of the hypersingular integral equations.

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