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New Aspects for Non-linear Semigroups in L1 Applied to Heat Equation's Analysis

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Abstract
Non-linear semigroups are studied and investigated in order to prove the existence and uniqueness of solutions for the non-linear partial differential equation defined in L1. The above partial differential equation is derived from the general heat equation's analysis at high temperature. The above differential equation has many applications in potential flow problems. In addition, the existence and uniqueness of solutions for the non-linear heat equation is proved, by presenting some general boundary conditions. Besides, some properties of the solutions for the above non-linear partial differential equation are finally proved.

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Non-linear Partial Differential Equations, Non-linear Semigroups, L1 Spaces, Heat Equation, Dissipative Operator.

1. Introduction

Over the past years of sufficiently Increasing interest was the investigation of non-linear semigroups in general Banach spaces and their application to the existence and uniqueness theory for differential equations associated with non-linear operators. Consequently, the fundamental results on non-linear semigroups are applied to the solution of several types of partial differential equations arising in mathematical physics and engineering.

Besides, the study of non-linear semigroups was derived from the examination of non-linear parabolic equations and from various non-linear boundary value problems. Generally, the theory of non-linear semigroups is a generalization of the Hille-Yosida theory for one-parameter semigroups of linear operators and is further closely related to the theory of non-linear monotone operators.

The first work on semigroups was published by A.V.Balakrishnan [1], when studying fractional powers of closed operators. Some years later T.Kato [2] studied non-linear semigroups in connection with evolution equations, while Y.Komura [3], [4] studied non-linear semigroups defined in Hilbert spaces.

Beyond the above, K.Sato [5] investigated non-negative contraction semigroups in Banach spaces, while a general theory of non-linear semigroups was investigated by M.G. Crandall et al. [6] - [8]. Also, J.Watanabe [9], [10] studied semigroups of non-linear operators on closed convex sets and H.Brezis et al. [11], [12] introduced a general semigroups formulation.

At the same time, M.Iannelli [13] proposed non-linear semigroups on cones of a non-reflexive Banach space, while J.Mermin [14] and S.Oharu [15] investigated general theories of non-linear semigroups. Additionally, I.Miyadera [16] studied semigroups of non-linear operators and B.K.Quinn [17] investigated semigroups in L1 spaces.

On the contrary, Y.Konishi [18] studied non-linear semigroups associated with some partial differential equations and U.Westphal [19] and S.Aizawa [20] investigated some formulations for non-linear semigroups. Moreover, T.Kurtz [21] studied semigroups of non-linear operators applied to gas kinetics, while R.Bruck [22] investigated asymptotic convergence of non-linear contraction semigroups in Hilbert spaces.

The theory of non-linear semigroups was generated by Y.Kobayashi [23], [24] and a monograph on the above subject was written by V. Barbu [25]. Moreover, J.M. Ball [26] studied strongly continuous semigroups, while B.C.Burch [27] investigated a semigroup treatment of the Hamilton - Jacobi equations in several space variables.

At the same time, J.H.Lightbourne and R.H.Martin [28] investigated relatively continuous perturbations of analytic semigroups, when A.T.Plant [29] studied non-linear semigroups of translations in Banach spaces generated by functional differential equations.

The theory of non-linear semigroups on general Banach spaces was also investigated by J.B.Baillon [30] and A.Pazy [31], [32], while J.A.Goldstein [33] wrote a monograph on semigroups of linear operators with some general applications. Also, a monograph on non-linear evolution operators and semigroups was written by N.H.Pavel [34].

Thus, over the past years there has been an increasing use of semigroups techniques in solving problems related to partial differential equations, defined in infinite dimensional Banach spaces. The method of semigroups has considerably simplified the proofs and has unified the treatment of several different classes of differential equations. These differential equations are solved successfully by using semigroup techniques in dealing with discontinuous data and regularity.

By the current research, the non-linear semigroups are used in order to prove the existence and uniqueness of solutions for the non-linear partial differential equation defined in L1 spaces. This differential equation is derived from the general theory of heat equations analysis. The above theory is a part of potential flows analysis as was investigated and analyzed by E.G.Ladopoulos [35] - [49] over the last two decades.

So, a general theory is presented in order to prove the existence and uniqueness of solutions for the above non-linear partial differential equation defined in L1 spaces. This theory consists to the use of non-linear semigroups, by applying them to the existence and uniqueness theorems.

2. Heat Equation's Investigation at High Temperature

Theorem 2.1

Consider by B a bounded domain in R^3 with smooth boundary Γ and by B_s an arbitrary subdomain of B with smooth boundary Γ_s (Figure 1). Suppose that in B there a gas with temperature $u=u(x,t)$ at the point $x=(x_1,x_2,x_3)\in B$ at time t.

Then, in the case of high temperature, the heat equation is equal to:

$$\frac{\partial u(x,t)}{\partial t}=\lambda^*\nabla^2\left[u(t,x)\right]^m \tag{2.1}$$

in which ∇^2 denotes the Laplace operator:

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\nabla x_3^2} \tag{2.2}$$

and λ^* is equal to:

$$\lambda^*=\lambda/c(a+1) \tag{2.3}$$

with $\lambda>0$ the thermal conduction constant, c the specific heat constant and $a\in]4,5;5,5[$, $m=a+1>1$.

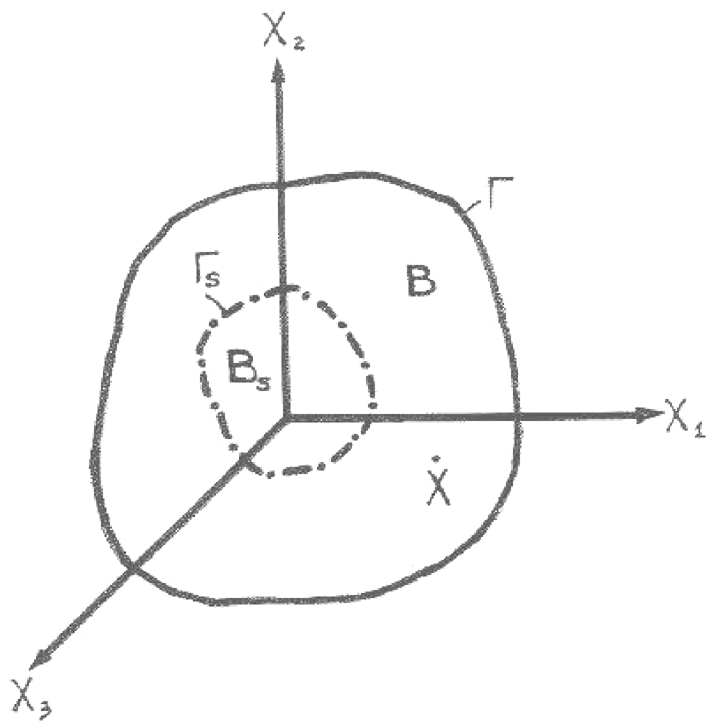

Fig. 1 A bounded domain B in R^3 with smooth boundary Γ inside which there is a gas with temperature u=$u(x,t)$ at the point $x = (x_1, x_2, x_3) \in D$.

Proof

Let the amount of heat in B_s at time *t*:

$$Q_I(t) = \int_{D_s} c(t,x,u)dx \tag{2.4}$$

and the amount of heat outflowing from B_s per unit of time:

$$Q_{II}(t) = -\int_{D_s} < \mathbf{R}(t,x,u,\nabla u),\mathbf{n} > ds \tag{2.5}$$

where **n** denotes the outward normal to Γ_s and ds the element of area of B_s.

Besides, consider the amount of heat from sources *F* in B_s per unit of time:

$$Q_{III}(t) = \int_{D_s} F(t,x,u)dx \tag{2.6}$$

Consequently, the balance of heat is equal to:

$$\frac{d}{dt}Q_I(t) = -Q_{II}(t) + Q_{III}(t) \tag{2.7}$$

In addition, by using the formula of Gauss then (2.5) can be written as:

$$Q_{II}(t) = -\int_{D_S} < \mathbf{R}, \mathbf{n} > ds = -\int_{D_S} \nabla \cdot \mathbf{R} dx \tag{2.8}$$

Then, by substituting eqs (2.4), (2.6) and (2.8) into (2.7) one obtains:

$$\frac{\partial}{\partial t} c(t,x,u) = \nabla \cdot \mathbf{R}(t,u,\nabla u) + F(t,x,u) \tag{2.9}$$

Furthermore, we assume the following formula to be valid:

$$c(t,x,u) = c\,u \tag{2.10}$$

in which c denotes the specific heat constant.

In the case of high temperature, then the radiation of heat is equal to:

$$\mathbf{R} = \lambda u^a \nabla u \tag{2.11}$$

where $\lambda > 0$ denotes the thermal conduction constant and $a \in]4,5;5,5[$.

So, from (2.11) we obtain:

$$\nabla \cdot \mathbf{R} = \lambda \nabla \cdot u^a \nabla u \tag{2.12}$$

which is further equal to:

$$\nabla \cdot \mathbf{R} = \frac{\lambda}{a+1} \nabla \cdot \nabla u^{a+1} \tag{2.13}$$

and finally to:

$$\nabla \cdot \mathbf{R} = \frac{\lambda}{a+1} \nabla^2 u^{a+1} \tag{2.14}$$

in which ∇^2 denotes the Laplace operator.

Hence, from eqs (2.9) and (2.14) by neglecting the source term *F*, we obtain the required heat equation (2.1).

3. The use of Non-linear Semigroups for the Existence and Uniqueness Theorems of Non-linear Partial Differential Equations in L1

Theorem 3.1

Let *u=u(x,t)* the temperature function at the point $x = (x_1, x_2, x_3) \in B$ at time *t.* (Fig.1). Then the following operator *E*:

$$D(E) = \{u \in L_1(B)\ ;\ u^m \in W_0^{1,1}(B),\ \nabla^2 u^m \in L_1(B)\} \tag{3.1}$$

$$Eu = \nabla^2 u^m \quad , \text{ for } \ u \in D(E) \tag{3.1a}$$

is m - dissipative in $L_1(B)$, with $\overline{D(E)} = L_1(B)$.

Proof

In order to prove the dissipativity of *E* in $L_1(B)$, we have to prove the following inequality:

$$\|u - v\|_{L_1} \leq \|u - v - t(Eu - Ev)\|_{L_1} \ , \ \forall t > 0 \ , \ u, v \in D(E) \tag{3.2}$$

Consequently, we choose the following sequence $h_n \in C_1(R)$:

$$h_n(s) = \frac{ns}{n|s| + 1} \ , \ s \in R \tag{3.3}$$

with the properties:

$$\begin{aligned} & h_n(0) = 0 \\ & |h_n(s)| \leq 1 \\ & h_n'(s) \geq 0 \\ & \lim_{n \to \infty} h_n(s) = sign\ s, \ \forall s \in R \end{aligned} \tag{3.4}$$

In addition, by choosing $u, v \in D(E)$, one has:

$$\nabla h_n[n^m - v^m] = h_n'[n^m - v^m] \nabla [u^m - v^m] \tag{3.5}$$

So, by using Green's formula we obtain:

$$\int_B (Eu - Ev) h_n [u^m - v^m] dx = -\int_B h_n'[u^m - v^m] \, \left|\nabla[u^m - v^m]\right|^2 dx \leq 0 \tag{3.6}$$

Thus, one obtains for every *t*>0:

$$\begin{aligned} \int_B (u - v) h_n [u^m - v^m] dx &\leq \int_B [u - v - t(Eu - Ev)] \, h_n \, [u^m - v^m] dx \\ &\leq \int_B |u - v - t(Eu - Ev)| \, dx = \|u - v - t(Eu - Ev)\|_{L_1} \end{aligned} \tag{3.7}$$

Beyond the above, one has:

$$sign[u^m - v^m] = sign[u - v] \tag{3.8}$$

and thus, by letting $n \to \infty$ in (3.7), we obtain the required formula (3.2).

Also, we have to prove that for each $h \in L_1(B)$ there exists a unique $u \in D(E)$, such that:

$$u - \nabla^2 u^m = h \tag{3.9}$$

So, with $v = u^m$, then eqn (3.9) is equivalent to the following equation:

$$v^{-m} - \nabla^2 v = h \tag{3.10}$$

Consequently, the Laplace operator:

$$D(\nabla^2) = \{v \in W_0^{1,1}(B),\ \nabla^2 v \in L_1(B)\} \tag{3.11}$$

is dissipative in $L_1(B)$. Furthermore, eqn (3.10) has a unique solution $v \in D(\nabla^2)$ and thus, $u = v^{-m}$ is the solution of (3.9).

Thus, $D(E) \supset \{u \in L_1(B)\ ;\ u^m \in D(\nabla^2)\}$, which is dense in $L_1(B)$ and thus $\overline{D(E)} = L_1(B)$, which completes the proof.

Theorem 3.2

Consider by $u=u(x,t)$ the temperature function at the point $x = (x_1, x_2, x_3) \in B$ at time *t* (Fig. 1). Also, for every $h \in L_1(0,T\ ;\ L_1(B))$, $T>0$, and $u_0 \in L_1(B)$ let the heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \lambda^* \nabla^2 \left[u(t,x)\right]^m + h \qquad in\,]0,T[\ , x \in B \tag{3.12}$$

with the boundary conditions:

$$u(t,x) = 0, \quad on\ ]0,T[\ ,\ x \in \Gamma \tag{3.13}$$

$$u(0,x) = u_0(x), \quad in\ \ B \tag{3.14}$$

in which Γ denotes the boundary of B and λ^* is given by (2.3) and $m > 1$.

Then, the heat equation (3.12) has a unique integral solution $u \in C\{[0,T]\ ;\ L_1(B)\}$.

Proof

Let *E* be the infinitesimal generator of a C_0 contraction semigroup *S,* while the operator *E* is given by (3.2).

Then, the following function:

$$u(t,x) = (S(t)u_0)(x) + \int_0^t S(t-s)h(s)ds, \;\; 0 \le t \le T \tag{3.15}$$

is said to be the solution of the non-linear partial differential equation (3.12).

In addition, consider the sequence $h_n \in C_1\,(0,T,L_1(B))$ be such that:

$$\lim_{n \to \infty} h_n = h \tag{3.16}$$

By choosing $u_{0n}(x) \in D(E)$, with $u_{0n}(x) \to u_0(x)$ as $n \to \infty$, then the following problem:

$$\frac{\partial u(t,x)}{\partial t} = \lambda^* \nabla^2 [u_n(t,x)]^m + h_n \quad in \;\;]0,T[\;,\; x \in B \tag{3.17}$$

with the conditions:

$$u_n(t,x) = 0, \quad on \;\;]0,T[\;,\;\; x \in \Gamma \tag{3.18}$$

$$u_n(0,x) = u_{0n}(x), \quad in \;\; B \tag{3.19}$$

has a unique strong solution $u_n(t,x)$ equal to:

$$u_n(t,x) = (S(t)u_{0n})(x) + \int_0^t S(t-s)h_n(s)ds\;, \quad 0 \le t \le T \tag{3.20}$$

where the last term of eqn (3.20) is continuously differentiable.

Furthermore, by setting:

$$w(t) = \int_0^t S(t-s)h_n(s)ds \tag{3.21}$$

then it is easily proved that:

$$\frac{S(\xi)w(t) - w(t)}{\xi} - \frac{w(t+\xi) - w(t)}{\xi} - \frac{1}{\xi}\int_t^{t+\xi} S(t+\xi-s)h_n(s)ds \tag{3.22}$$

for all $\xi > 0$ sufficiently small and $0 \le t \le T$.

Hence, since:

$$u_n(t,x) = (S(t)u_{0n})(x) + w(t) \tag{3.23}$$

follows that (3.20) satisfies (3.17).

Finally, the solution $u_n(t,x)$ is the unique solution to eqn (3.17). Hence, passing to the limit, follows that the solution (3.15) is the unique solution to the non-linear partial differential equation (3.12).

4. Conclusions

A general heat equation's analysis was presented, by applying a bounded domain in R^3, inside which there is a gas with temperature $u = u(x,t)$ at some point *x*. This problem is reduced to the solution of a non-linear partial differential equation.

Consequently, a non-linear semigroup technique was introduced in order to prove the existence and uniqueness of solutions for the non-linear partial differential heat equation, when some general boundary conditions are presented.

Hence, the fundamental results and properties of the non-linear semigroups were presented in order to unify the treatment of the nonlinear partial differential heat equation, defined in L_1, by using some general boundary conditions.

Finally, the method of non-linear semigroups has clearly simplified the proofs of the existence and uniqueness of solutions for the non-linear partial differential equation, which are of main interest for the solution of potential flow problems.

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