Universal Journal of Integral Equations **2 (2014), 1-11**
www.papersciences.com

Three-dimensional Multiphase Flows by Non-linear Singular Integral Equations in Petroleum Engineering

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Abstract
A new method is proposed in the area of three-dimensional multiphase flows for the determination of the properties of reservoir materials, when petroleum reserves together with water are moving through porous media. The above multiphase problem seems to be very important for oil reservoir engineering. Consequently, this petroleum engineering problem is reduced to the solution of a ***non-linear singular integral equation***, which is numerically solved by using the Singular Integral Operators Method (S.I.O.M.). Beyond the above, several properties are analyzed and investigated for the porous medium equation of multiphase flows, defined as a Helmholtz differential equation. So, the estimation of the future oil production from a reservoir could be determined. Finally, an application is given for a well testing to be checked when a heterogeneous oil reservoir together with water in a multiphase flow is moving in a porous medium. Hence, by using the S.I.O.M., then the pressure response from the well test conducted in the above heterogeneous oil and water reservoir, is numerically calculated and investigated.

2010 Mathematics Subject Classification : 65L10, 65R20.
Key Word and Phrases
Non-linear Singular Integral Equations, Singular Integral Operators Method (S.I.O.M.), Oil Reserves, Multiphase Flows, Porous Media, Helmholtz Differential Equation, Petroleum Reservoir.

1. Introduction

The study of the movement of oil reserves through porous media is one of the most important problems on petroleum reservoir engineering. Very often the petroleum reservoir is mixed with water and so by applying a well test analysis, then a history matching process takes place for the determination of the properties of the reservoir materials. Furthermore, the movement of oil reserves through porous media, produces both single-phase and multiphase flows. So, of primary interest is the investigation of the multiphase flows when the oil reservoir is mixed with water. Besides, if a well test is conducted, then the well is subjected to a change of the flow rate and the pressure response can be further measured. Then the estimation of the future oil production from a reservoir can be determined. For the determination of several petroleum reservoir parameters, such as permeability, then numerical calculations should be used, as analytical solutions in most cases are not possible to be derived. Over the past years, several variants of the Boundary Element Method were used for the solution of petroleum engineering problems. As a start at the end of eight's Lafe and Cheng [1] proposed a BEM for the solution of steady flows in heterogeneous solids. At the same period Masukawa and Horne [2] and Numbere and Tiab [3] applied boundary elements for steady state problems of streamline tracking. Also, Kikani and Horne [4] solved transient problems by using a Laplace space boundary element model, for the analysis of well tests in several arbitrarily shaped reservoirs. On the contrary, Koh and Tiab [5] used boundary elements to describe the flow around tortuous horizontal wells, for homogeneous, or piecewise homogeneous reservoirs. Besides, Sato and Horne [6], [7] applied perturbation boundary elements for the study of heterogeneous reservoirs. El Harrouni, Quazar, Wrobel and Cheng [8] proposed the use of a transformed form of Darcy's law combined with dual reciprocity boundary element method to handle heterogeneity. Furthermore, Onyejekwe [9] applied a Green element method to isothermal flows with second order reactions. He [10], [11] used further a combined method of boundary elements together with finite elements for the study of heterogeneous reservoirs. In addition, Taigbenu and Onyejekwe [12] applied a transient one-dimensional transport equation by using a mixed Green element method.

Over the last years, several non-linear singular integral equation methods were used successfully by Ladopoulos [13] - [22] for the solution of applied problems of solid mechanics, elastodynamics, structural analysis, fluid mechanics and aerodynamics. Beyond the above, Ladopoulos [23] - [25] proposed a non-linear singular integral equations method in petroleum reservoir engineering, for the determination of the properties of the reservoir materials, when oil reserves are moving through porous solids. Hence, by the present research, the method of non-linear singular integral equations will be extended in order to determine the properties of the reservoir materials in multiphase flows, when oil reserves mixed with water are moving through porous solids.

Consequently, by using the Singular Integral Operators Method (S.I.O.M.), then the pressure response in multiphase flows from the well test conducted in a heterogeneous reservoir will be computed. Besides, some properties of the porous medium equation, which is a Helmholtz differential equation, are proposed and investigated. Also, basic properties of the fundamental solution will be analyzed and investigated.

Finally, an application is given for a well testing to be investigated when a heterogeneous oil reservoir together with water in multiphase flow is moving in a porous medium. Then this problem will be solved by using the Singular Integral Operators Method and so the pressure response from the well test conducted in this heterogeneous oil reservoir, will be computed. This is very important in petroleum reservoir engineering in order the size of the reservoir to be evaluated.

The proposed petroleum engineering method, as it is a complicated non-linear numerical method can give results for heterogeneous porous media (which of course are the solids in reality) and not only for homogeneous solids as are giving the analytical or numerical existing methods. Thus, the estimation of the properties and the future petroleum production from a new oil reservoir could be done exactly, and not estimated as by the existing methods. From the above mentioned points it can be understood the evidence of the applicability of the new method, as it is based on non-linear software. Also its novelty, as it is based on the theory of non-linear singular integral equations.

So, the non-linear singular integral equation methods which were used with big success for the solution of several engineering problems of fluid mechanics, hydraulics, aerodynamics, solid mechanics, elastodynamics, and structural analysis, are further extended by the current research for the solution of oil reservoir engineering problems in multiphase flows. In such a case the non-linear singular integral equations are used for the solution of one of the most important and interesting problems for petroleum reservoir engineers.

2. Three-dimensional Multiphase Flows of Petroleum Reserves by Well Test Analysis

Petroleum well test analysis is a kind of a very important history matching process for the determination of the properties of reservoir materials. Consequently, during the movement of petroleum reserves through porous media, then both single-phase and multiphase flow occurs. By the current investigation the multiphase flows are studied when the oil reserves are mixed with water. Besides, when a petroleum well test is conducted then the well is subjected to a change of its flow rate and the resulting pressure response is possible to be measured. Beyond the above, this pressure is compared to analytical or numerical models in order to estimate reservoir parameters such as permeability. Then the estimation of the future oil production from the reservoir can be evaluated.

An oil reservoir well test analysis in a multiphase flow is calculated by using the porous medium equation:

$$\nabla \bullet (k_t \nabla p) = 0 \tag{2.1}$$

with: $$k_t = \lambda\left(\frac{\lambda_o}{\varphi_0\xi_0} + \frac{\lambda_w}{\varphi_w\xi_w}\right) \tag{2.2}$$

in which λ denotes the relative permeability, λ_o the permeability of the oil, ϕ_o the porosity of the oil, ξ_o the viscosity of the oil and λ_w, ϕ_w, ξ_w the corresponding values of the water and p the pressure of the reservoir.

By replacing variables as follows:

$$u = (\lambda')^{1/2}\, p \tag{2.3}$$

then (2.1) can be written as:

$$\nabla^2 u + \lambda' u = 0 \tag{2.4}$$

with:

$$\lambda' = -\frac{\nabla^2 (k_t)^{1/2}}{(k_t)^{1/2}} \tag{2.5}$$

So, eqn (2.4) denotes a Helmholtz differential equation.

Besides, consider by $u^*(\mathbf{x},\mathbf{y})$ the fundamental solution of any point $\mathbf{y}$, because of the source point $\mathbf{x}$. Then the fundamental solution can be given by the following relation:

$$\nabla^2 u^*(\mathbf{x},\mathbf{y}) + \lambda' u^*(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x},\mathbf{y}) = 0 \tag{2.6a}$$

which can be further written as:

$$u^*_{,ii}(\mathbf{x},\mathbf{y}) + \lambda' u^*(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x},\mathbf{y}) = 0 \tag{2.6b}$$

So, eqn (2.6) denotes the Helmholtz potential equation governing the fundamental solution.

Consider further by u^* the fundamental solution chosen so that to enforce the Helmholtz equation in terms of the function u, in a weak form. Then the weak form of Helmholtz equation will be written as following:

$$\int_\Omega (\nabla^2 u + \lambda' u) u^* d\Omega = 0 \tag{2.7}$$

in the solution domain Ω.

Beyond the above, by applying the divergence theorem once in (2.7), one obtains a symmetric weak form:

$$\int_{\partial\Omega} n_i u_{,i} u^* dS - \int_\Omega u_{,i} u^*_{,i} d\Omega + \int_\Omega \lambda' u u^* d\Omega = 0 \tag{2.8}$$

where **n** denotes the outward normal vector of the surface S.

Thus, in the symmetric weak form the function u and the fundamental solution u^* are only required to be first - order differentiable. Also, by applying the divergence theorem twice in (2.7) we have:

$$\int_{\partial\Omega} n_i u_{,i} u^* dS - \int_{\partial\Omega} n_i u u^*_{,i} dS + \int_{\Omega} u(u^*_{,ii} + \lambda' u^*) d\Omega = 0 \tag{2.9}$$

Consequently, (2.9) is the asymmetric weak form and the fundamental solution u^* is required to be second - order differentiable. On the other hand, u is not required to be differentiable in the domain Ω.

By combining eqs (2.6) and (2.9), then one obtains:

$$u(\mathbf{x}) = \int_{\partial\Omega} n_i(\mathbf{y}) u_{,i}(\mathbf{y}) u^*(\mathbf{x},\mathbf{y}) dS - \int_{\partial\Omega} n_i(\mathbf{y}) u(\mathbf{y}) u^*_{,i}(\mathbf{x}.,\mathbf{y}) dS \tag{2.10}$$

which can be further written as:

$$u(\mathbf{x}) = \int_{\partial\Omega} q(\mathbf{y}) u^*(\mathbf{x},\mathbf{y}) dS - \int_{\partial\Omega} u(\mathbf{y}) R^*(\mathbf{x},\mathbf{y}) dS \tag{2.11}$$

where $q(\mathbf{y})$ denotes the potential gradient along the outward normal direction of the boundary surface:

$$q(\mathbf{y}) = \frac{\partial u(\mathbf{y})}{\partial n_y} = n_k(\mathbf{y}) u_{,k}(\mathbf{y}) \quad , \quad \mathbf{y} \in \partial\Omega \tag{2.12}$$

and the kernel function:

$$R^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial n_y} = n_k(\mathbf{y}) u^*_{,k}(\mathbf{x},\mathbf{y}) \quad , \quad \mathbf{y} \in \partial\Omega \tag{2.13}$$

By differentiating (2.11) with respect to x_k, we obtain the integral equation for potential gradients $u_{,k}(\mathbf{x})$ by the following formula:

$$\frac{\partial u(\mathbf{x})}{\partial x_k} = \int_{\partial\Omega} q(\mathbf{y}) \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial x_k} dS - \int_{\partial\Omega} u(\mathbf{y}) \frac{\partial R^*(\mathbf{x},\mathbf{y})}{\partial x_k} dS \tag{2.14}$$

3. Basic Properties of Three-dimensional Multiphase Petroleum Flows

We rewrite the weak form of (2.6) governing the fundamental solution, by the following relation:

$$\int_{\Omega}\left[\nabla^{2}u^{*}(\mathbf{x},\mathbf{y})+\lambda' u^{*}(\mathbf{x},\mathbf{y})\right]c\,d\Omega+c=0,\quad \mathbf{x}\in\Omega \tag{3.1}$$

where c denotes a constant, considering as the test function.

In addition, eqn (3.1) can be further written as:

$$\int_{\Omega}\left[u^{*}_{,ii}(\mathbf{x},\mathbf{y})+\lambda' u^{*}(\mathbf{x},\mathbf{y})\right]d\Omega+1=0,\quad \mathbf{x}\in\Omega \tag{3.2}$$

Also, (3.2) takes the form:

$$\int_{\partial\Omega}n_{i}(\mathbf{y})u^{*}_{,i}(\mathbf{x},\mathbf{y})dS+\int_{\Omega}\lambda' u^{*}(\mathbf{x},\mathbf{y})d\Omega+1=0,\quad \mathbf{x}\in\Omega \tag{3.3}$$

By considering further an arbitrary function *u(x)* in *Ω* as the test function, then the weak form of (2.6) may be written as:

$$\int_{\Omega}\left[\nabla^{2}u^{*}(\mathbf{x},\mathbf{y})+\lambda' u^{*}(\mathbf{x},\mathbf{y})+\delta(\mathbf{x},\mathbf{y})\right]u(\mathbf{x})d\Omega=0,\quad \mathbf{x}\in\Omega \tag{3.4}$$

and further as:

$$\int_{\Omega}\left[u^{*}_{,ii}(\mathbf{x},\mathbf{y})+\lambda' u^{*}(\mathbf{x},\mathbf{y})\right]u(\mathbf{x})d\Omega+u(\mathbf{x})=0,\quad \mathbf{x}\in\Omega \tag{3.5}$$

Finally, (3.5) takes the form:

$$\int_{\partial\Omega}\Phi^{*}(\mathbf{x},\mathbf{y})u(\mathbf{x})dS+\int_{\Omega}\lambda' u^{*}(\mathbf{x},\mathbf{y})u(\mathbf{x})d\Omega+u(\mathbf{x})=0,\quad \mathbf{x}\in\Omega \tag{3.6}$$

Besides, if $\mathbf{x}$ approaches the smooth boundary $(\mathbf{x}\in\partial\Omega)$, then the first term in (3.6) may be written as following:

$$\lim_{x\to\partial\Omega}\int_{\partial\Omega}\Phi^{*}(\mathbf{x},\mathbf{y})u(\mathbf{x})dS=\int_{\partial\Omega}^{CPV}\Phi^{*}(\mathbf{x},\mathbf{y})u(\mathbf{x})dS-\frac{1}{2}u(\mathbf{x}) \tag{3.7}$$

in the sense of a Cauchy Principal Value (CPV) integral.

For the understanding of the physical meaning of (3.7), eqs (3.3) and (3.6) can be written as:

$$\int_{\partial\Omega}^{CPV}\Phi^{*}(\mathbf{x},\mathbf{y})dS+\int_{\Omega}\lambda' u^{*}(\mathbf{x},\mathbf{y})d\Omega+\tfrac{1}{2}=0,\quad x\in\partial\Omega \tag{3.8}$$

and:

$$\int_{\partial\Omega}^{CPV} \Phi^{*}(\mathbf{x},\mathbf{y})u(\mathbf{x})dS + \int_{\Omega} \lambda' u^{*}(\mathbf{x},\mathbf{y})u(\mathbf{x})d\Omega + ½u(\mathbf{x}) = 0, \quad x \in \partial\Omega \tag{3.9}$$

From (3.8) follows that only a half of the source function at point **x** is applied to the domain *Ω*, when the point **x** approaches a smooth boundary, $\mathbf{x} \in \partial\Omega$.

Furthermore, let us consider another weak form of eqn (2.6) by supposing the vector functions to be the gradients of an arbitrary function $u(\mathbf{y})$ in *Ω*, chosen in such a way that they have constant values:

$$u_{,k}(\mathbf{y}) = u_{,k}(\mathbf{x}) \ , \quad \text{for } k = 1,2,3 \tag{3.10}$$

Then the weak form of eqn (2.6) will be written as:

$$\int_{\Omega} \left[u^{*}_{,ii}(\mathbf{x},\mathbf{y}) + \lambda'' u^{*}(\mathbf{x},\mathbf{y}) \right] u_{,k}(\mathbf{y}) d\Omega + u_{,k}(\mathbf{x}) = 0 \tag{3.11}$$

By applying further the divergence theorem, then eqn (3.11) takes the following form:

$$\int_{\partial\Omega} \Phi^{*}(\mathbf{x},\mathbf{y}) u_{,k}(\mathbf{x}) dS + \int_{\Omega} \lambda' u^{*}(\mathbf{x},\mathbf{y}) u_{,k}(\mathbf{x}) d\Omega + u_{,k}(\mathbf{x}) = 0 \tag{3.12}$$

In addition, the following property exists:

$$\begin{aligned} &\int_{\partial\Omega} n_i(\mathbf{y}) u_{,i}(\mathbf{x}) u^{*}_{,k}(\mathbf{x},\mathbf{y}) dS - \int_{\partial\Omega} n_k(\mathbf{y}) u_{,i}(\mathbf{x}) u^{*}_{,i}(\mathbf{x},\mathbf{y}) dS \\ &= \int_{\Omega} u_i(\mathbf{x}) u^{*}_{,ki}(\mathbf{x},\mathbf{y}) dS - \int_{\partial\Omega} u_{,i}(\mathbf{x}) u^{*}_{,ik}(\mathbf{x},\mathbf{y}) dS = 0 \end{aligned} \tag{3.13}$$

By adding eqs (3.12) and (3.13) then one obtains:

$$\begin{aligned} &\int_{\partial\Omega} n_i(\mathbf{y}) u_{,i}(\mathbf{x}) u^{*}_{,k}(\mathbf{x},\mathbf{y}) dS - \int_{\partial\Omega} n_k(\mathbf{y}) u_{,i}(\mathbf{x}) u^{*}_{,i}(\mathbf{x},\mathbf{y}) dS \\ &+ \int_{\partial\Omega} \Phi^{*}(\mathbf{x},\mathbf{y}) u^{*}_{,k}(\mathbf{x}) dS + \int_{\Omega} \lambda'' u^{*}(\mathbf{x},\mathbf{y}) u_{,k}(\mathbf{x}) d\Omega + u_{,k}(\mathbf{x}) = 0 \end{aligned} \tag{3.14}$$

which takes finally the form:

$$\int_{\partial\Omega} n_i(\mathbf{y}) u_{,i}(\mathbf{x}) u^*_{,k}(\mathbf{x},\mathbf{y}) dS + \int_{\partial\Omega} e_{ikt} R_t u(\mathbf{x}) u^*_{,i}(\mathbf{x},\mathbf{y}) dS$$
$$+ \int_{\partial\Omega} \lambda' u^*(\mathbf{x},\mathbf{y}) u_{,k}(\mathbf{x}) d\Omega + u_{,k}(\mathbf{x}) = 0 \tag{3.15}$$

4. Three-dimensional Multiphase Flows by Non-linear Singular Integral Equations

The porous medium equation (2.1) will be further written in another form, in order a singular integral equation representation to be applicable:

$$\nabla^2 p + \nabla \ln k_t \bullet \nabla p = 0 \tag{4.1}$$

By applying further the Green Element Method, then eqn (4.1) reduces to the solution of a non-linear singular integral equation:

$$-\frac{\theta}{2\pi} p(r_i) + \int_{\partial\Omega} \left(p \frac{\partial[\ln(r-r_i)]}{\partial n} - \ln(r-r_i) \frac{\partial p}{\partial n} \right) dS +$$
$$\iint_{\Omega} \ln(r-r_i)[-\nabla \ln \Lambda \bullet \nabla p] d\Omega = 0 \tag{4.2}$$

where:
$$\Lambda = \frac{1}{k_t} \tag{4.3}$$

In order the non-linear singular integral equation (4.2) to be numerically solved, then the Singular Integral Operators Method (S.I.O.M.) will be used. So, the non-linear singular integral equation (4.2) is approximated by the formula:

$$-\frac{\theta}{2\pi} p(r_i) + \sum_{i=1}^{M} \left[\int_{\partial\Omega} \left(p \frac{\partial[\ln(r-r_i)]}{\partial n} - \ln(r-r_i) \frac{\partial p}{\partial n} \right) dS + \iint_{\Omega} (-\nabla \ln \Lambda \bullet \nabla p) d\Omega \right] = 0 \tag{4.4}$$

where M denotes the total number of elements.

Besides, we introduce the following functions describing the pressure at any point in an element, in terms of the nodal pressures:

$$p(x,y) = N_j(x,y) p_j \tag{4.5}$$

By replacing (4.5) then eqn (4.4) takes the form:

$$\sum_{i=1}^{M} \left(A^e_{ij} p_i + B^e_{ij} q_i - C^e_{ijl} \ln \Lambda_i p_i \right) = 0, \qquad i,j,l = 1,2,3,4 \tag{4.6}$$

where:

$$A_{ij}^{e} = \int_{\partial\Omega} \frac{\partial\left[\ln(r-r_i)\right]}{\partial n}\Omega_j dS - \delta_{ij}\frac{\theta}{2\pi} \tag{4.7}$$

$$B_{ij}^{e} = -\int_{\partial\Omega} \ln(r-r_i)\Omega_j dS \tag{4.8}$$

$$C_{ijl}^{e} = \iint_{\Omega_j} \ln(r-r_i)\left[\frac{\partial N_j}{\partial x}\frac{\partial N_l}{\partial x} + \frac{\partial N_j}{\partial y}\frac{\partial N_l}{\partial y}\right]d\Omega \tag{4.9}$$

4. Heterogeneous Reservoirs Application of Three-dimensional Petroleum Multiphase Flows

The previous mentioned theory will be applied to the determination of the water production history, in a well known problem where the exact Buckley - Leverett solution is valid [26].

So, by using the Singular Integral Operators Method (S.I.O.M.) as described in the previous paragraphs, then the computation of the water was effected and a comparison was done with exact Buckley - Leverett solution. According to this problem water is injected into one end of a one dimensional oil reservoir and fluids are produced from the other end of the reservoir. In this case the injected water forms a piston - like shock. Consequently, Table 1 shows the watercut with respect to the time.

The computational results of the watercut were compared to the analytical solutions of the same problem. From Table 1 it can be seen that there is very small difference between the computational results and the analytical solutions. Finally same results are shown, in Figure 1 and in three-dimensional form in Figure 1a.

Table 1

Time (days)	Watercut (%) S.I.O.M.	Watercut (%) Analytical
0	0	0
10	0	0
20	0	0
30	0	0
40	0	0
50	0	0
51	6	0
52	20	0
53	40	0
54	60	100
55	98	100
56	100	100

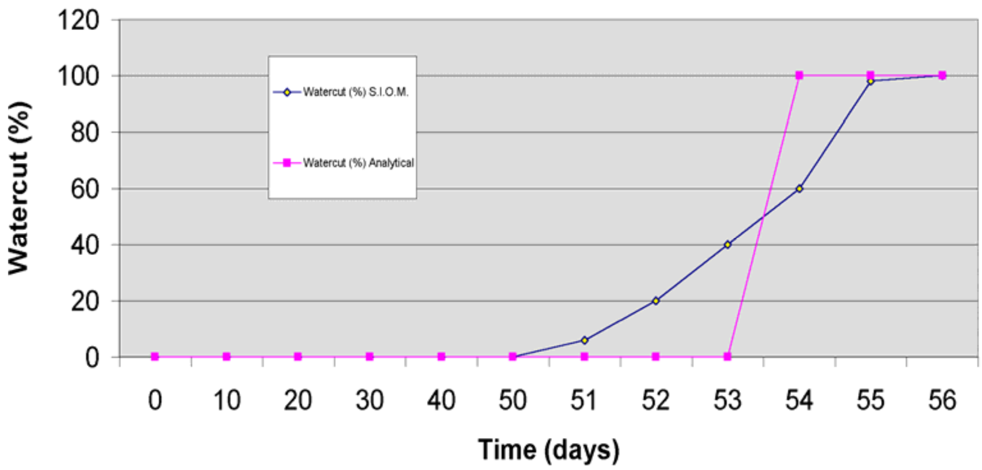

Fig. 1 Multiphase Flow of Oil Reservoir Mixed with Water.

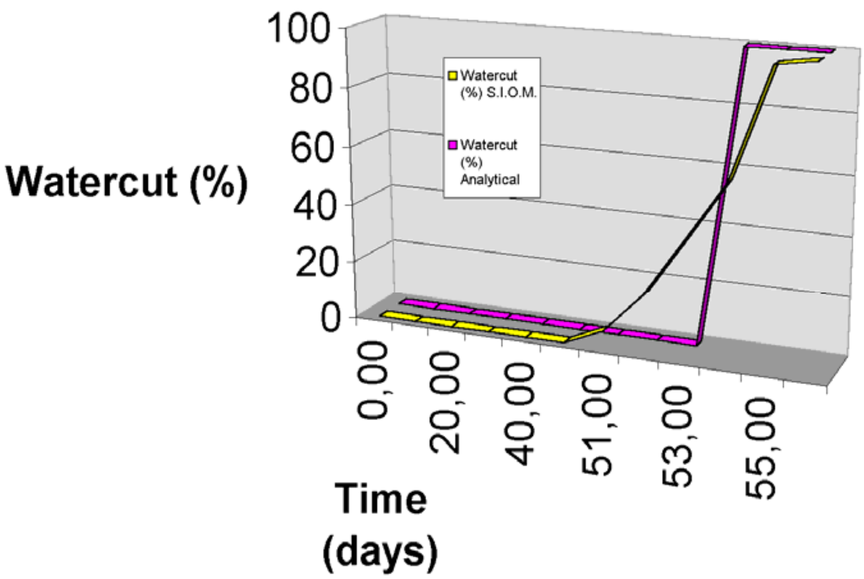

Fig. 2 3-D Distribution of Multiphase Flow of Oil Reservoir Mixed with Water.

5. Conclusions

A non-linear mathematical method has been presented as an attempt to determine the properties of the oil reservoir materials mixed with water in multiphase flow. Hence, the study of the movement of oil reserves through porous media is very important for petroleum reservoir engineers. This problem was reduced to the solution of a non-linear singular integral equation, which was numerically evaluated by using the Singular Integral Operators Method (S.I.O.M.).

Besides, several important properties of the porous medium equation, which is a Helmholtz differential equation, were analyzed and investigated. Hence, the fundamental solution of the porous medium equation was proposed and studied. Also, some basic properties of the fundamental solution were further investigated. The new method, as it is a complicated non-linear numerical method can give results for heterogeneous porous media (which of course are the solids in reality) and not only for homogeneous solids as are giving the analytical or numerical existing methods. Consequently, the estimation of the properties and the future petroleum production from a new oil reservoir could be done exactly, and not estimated as by the existing methods.

An application was finally given to the determination of the water production history, in a well known problem where the exact Buckley - Leverett solution is valid. The above problem was solved by using the Singular Integral Operators Method and thus the watercut, was computed. According to this problem water was injected into one end of a one dimensional oil reservoir and fluids are produced from the other end of the reservoir. In this case the injected water forms a piston - like shock.

During the last years, non-linear singular integral equation methods have been used successfully for the solution of several important engineering problems of structural analysis, elastodynamics, hydraulics, fluid mechanics and aerodynamics. For the numerical evaluation of the non-linear singular integral equations of the above problems, were used several aspects of the Singular Integral Operators Method (S.I.O.M.). Hence, by the present research such methods were extended for the solution of oil reserves problems in multiphase flows of petroleum reservoir engineering.

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