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# Non-linear Singular Integro-differential Equations in Banach Spaces by Collocation Evaluation Methods

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**Abstract**
An "innovative" and "groundbreaking" numerical evaluation method is proposed for the approximation of the non-linear singular integro-differential equations defined in Banach spaces. Consequently, the collocation numerical evaluation method is applied for the approximation of such type of non-linear equations, by using a system of Chebyshev functions. In addition, through the application of the collocation numerical method is investigated the existence of solutions for the system of non-linear equations used for the approximation of the non-linear singular integro-differential equations, which are defined in a complete normed space, i.e. a Banach space.

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Non-linear Singular Integro-differential Equations, Banach Spaces, Collocation Evaluation Methods, Chebyshev Functions, Existence Theorems.

## 1. Introduction

A big number of problems of engineering mechanics, like structural analysis, fluid mechanics and aerodynamics, reduce to the solution of non-linear singular integral and integro-differential equations. Hence, there is an increasing interest for the solution of such type of non-linear integral equations, since these are connected with a wide range of problems of an applied character. The theory of non-linear singular integral and integro-differential equations seems to be particularly complicated if closely linked with applied mechanics problems.

Having in mind the implications for different problems of engineering mechanics, E.G.Ladopoulos [1]-[9] and E.G.Ladopoulos and V.A.Zisis [10]-[12] introduced and investigated non-linear singular integral equations and non-linear finite-part singular integral equations. This type of non-linear equations has been applied to many problems of structural analysis, fluid mechanics and aerodynamics.

On the other hand, some studies have been published, investigating non-linear integral equations of simpler form, without any singularities. Among those who studied non-linear theories used in applied mechanics, we shall mention the following: J.Andrews and J.M.Ball [13], S.S.Antman [14], [15], S.S.Antman and E.R.Carbone [16], J.M.Ball [17] - [19], H.Brezis [20], P.G.Ciarlet and P.Destuynder [21], P.G.Ciarlet and J.Necas [22], [23], J.E.Dendy [24], Guo Zhong-Heng [25], H.Hattori [26], D.Hoff and J.Smoller [27], W.J.Hrusa [28], R.C.MacCamy [29] - [31], B.Neta [32], [33], R.W.Ogden [34], R.L.Pego [35], M.Slemrod [36], and O.J.Staffans [37].

By the current research a new approximation method is proposed, for the numerical evaluation of the non-linear singular integro-differential equations defined in Banach spaces. Consequently, a new form of the collocation approximation method is investigated for the numerical solution of the non-linear singular integro-differential equations, by studying the existence and uniqueness for their solution.

For the numerical evaluation of the non-linear singular integro-differential equations which are defined in the Banach spaces, is used a system of Chebyshev functions continuous on $[-\pi,\pi]$. Hence, through application of the collocation method the existence of solutions for the system of non-linear equations used for the approximation of the non-linear singular integro-differential equations is investigated.

## 2. Existence Theory of Non-linear Singular Integrodifferential Equations

*Definition 2.1*

Consider the non-linear singular integro-differential equation:

$$F[\lambda, t, u(t), S(u,t)] = u'(t) \tag{2.1}$$

with:

$$S(u,t) = \frac{1}{2\pi} g(t) \int_{-\pi}^{\pi} \frac{u(x)}{\tan\left[\frac{1}{2}(x-t)\right]} \,\mathrm{d}\,x \tag{2.2}$$

in which $u(x)$ is the unknown function, $g(t)$ a known function, $(-\pi < t < \pi)$ and $F[\lambda, t, u(t), S(u,t)]$ is the non-linear kernel.

*Definition 2.2*

Let $C_p^1([-\pi,\pi], t_0)$ denote the set of functions $u(t)$ satisfying a Lipschitz condition on the interval $[-\pi,\pi]$, which satisfy equation $u(t_0) = 0$ and for which the period $p(t)u'(t)$ is continuous on the above interval, where $p(t)$ is some nonnegative function defined on $[-\pi,\pi]$ such that the integrals:

$$a(t) = \left| \int_{t_0}^{t} (1/p(\xi))\,\mathrm{d}\,\xi \right| \tag{2.3}$$

and:

$$b(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{p(\xi)} \left| g(t) \ln \left| \cos\frac{1}{2} t \Big/ \sin\frac{1}{2}(\xi - t) \right| \right| \mathrm{d}\,\xi \tag{2.4}$$

exist.

By introducing the norm $\|u(t)\| = \max|p(t)u'(t)|$ into $C_p^1([-\pi,\pi], t_0)$, it becomes a complete normed space, and thus a Banach space.

*Theorem 2.1*

Consider the nonlinear singular integro-differential equation (2.1). In addition, concerning $F[\lambda, t, u, v]$ we assume that the function $p(t)F[\lambda, t, u(t), S(u,t)]$ is continuous on $[-\pi,\pi]$ for $u(t) \in C_p^1([-\pi,\pi], t_0)$, has continuous partial derivatives with respect to $u$ and $v$ with the other arguments fixed, and satisfies the inequalities:

$$
\begin{aligned}
p(t)a(t)\left|F_u\left[\lambda,t,a(t)\xi,b(t)x\right]\right| &\le f_1\left(\lambda,|\xi|,|x|\right) \\
p(t)b(t)\left|F_v\left[\lambda,t,a(t)\xi,b(t)x\right]\right| &\le f_2\left(\lambda,|\xi|,|x|\right)
\end{aligned}
\tag{2.5}
$$

in the region $\{-r \le u, v \le r, -\pi < t < \pi\}(0, r \le \infty)$, where $f_1(\lambda,|\xi|,|x|)$ and $f_2(\lambda,|\xi|,|x|)$ are nondecreasing functions of $|\xi|$ and $|x|$.

Consider further that $f(\lambda,r) \le q$ with $0 < q < 1, 0 < r < \infty$. Then, for any initial function $u_0(t) \in U_r = \{u(t): \|u(t)\| \le r\}$, the sequence:

$$
u_{n+1}(t) = S_1(\lambda, u_n, t) \ (n = 0,1,2,...) \tag{2.6}
$$

converges to the unique solution $u_*(t) \in C_p^1([-\pi,\pi], t_0)$ of the non-linear singular integro-differential equation (2.1).

*Proof.* We replace $f(\lambda,t)$ by two terms as follows:

$$
f(\lambda,t) = f_1(\lambda,t,t) + f_2(\lambda,t,t) \tag{2.7}
$$

and consider the following equation:

$$
f(\lambda,r)r + h^2(\lambda) = r \tag{2.8}
$$

where:

$$
h^2(\lambda) = \max_{-\pi \le t \le \pi} \left| p(t)F(\lambda,t,0,0) \right|
$$

Also, the operator $S_1(\lambda,u_n,t), n = 0,1,2,...$ in (2.6) shall be of the form:

$$
S(\lambda,u,t) = \int_{t_0}^{t} F\left[\lambda,\xi,u(\xi),I(u,\xi)\right] \mathrm{d}\,\xi \tag{2.9}
$$

and $u_0(t)$ is a given function. Consequently, on the basis of the study in [10] the theorem can be proved.

If $f(\lambda,r_0) \le q$, where $r_0$ is a root of (2.8), then for any initial approximation $u_0(t) \mathrm{in} U_{r_0}$, the sequence (2.6) converges to the unique solution $u_*(t) \in C_p^1([\pi,-\pi], t_0)$ of (2.1). Furthermore:

$$\left|u_n(t)-u_*(t)\right| \le \frac{q^n}{1-q}\,\delta(t) \tag{2.10}$$

on $[-\pi, \pi]$, where:

$$\delta(t) = a(t)\max\left|p(t)\left\{F\left[\lambda, t, u_0(t), I(u_0, t)\right] - u_0'(t)\right\}\right| (-\pi \le t \le \pi)\,.$$

**3. Collocation Approximation Method for Non-linear Singular Integro-differential Equations**

*Theorem 3.1*

Consider the non-linear singular integro-differential equation (2.1). Suppose that the function $F[\lambda, t, u, v]$ has continuous partial derivatives with respect to $u$ and $v$, which for $p(t) > 0$ on $[-\pi, \pi]$ satisfies (2.5) in the square $u, v \in [-r, r]$ and suppose that $f(\lambda, r) \le q < 1$ with $f(\lambda, r)$ given by (2.7).

In addition, consider the system of non-linear equations:

$$H(v_m, \xi_i) = 0 \ (i = 1,2,..., m) \tag{3.1}$$

in which $H(v_m, \xi_i) = v_m'(\xi_i) - F\left[\lambda, \xi_i, v_m(\xi_i), I(v_m, \xi_i)\right]$, $\xi_i = \xi_i^m$ are fixed distinct points in $[-\pi, \pi]$ with:

$$V_m(t) = \sum_{k=1}^{m} a_{mk} \int_{t_0}^{t} \frac{\varphi_k(x)}{p(x)}\,dx, \ m = 1,2,... \tag{3.2}$$

where $\{\varphi_k(x)\}$ is a given system of Chebyshev functions which are continuous on $[-\pi, \pi]$. Then the system of non-linear equations (3.1) has a solution $u_m^*(t)$ in $U_r = \{u(t) : \|u(t)\| \le r\}, 0 < r < \infty$ which approaches the unique solution $u^*(t)$ of (2.1) as $m \to \infty$.

*Proof.* According to (3.1) consider the equation:

$$H_m(u_m, t) = 0 \tag{3.3}$$

where $H_m(u_m, t)$ is a function of the form:

$$\begin{gathered} H_m(u_m, t) = p(t)u_m'(t) - \sum_{i=1}^{m} \psi_i(t) p(\xi_i) F\left[\lambda, \xi_i, u_m(\xi_i), I(u_m, \xi_i)\right] \\ u_m(t) = 0, \ \psi_i(\xi_i) = \delta_{ij} \end{gathered} \tag{3.4}$$

and $\delta_{ij}$ is the Kronecker delta.

Beyond the above, consider the existence of a solution of (3.3). Consequently, we use an interpolation process of the form (2.6), which in connection with this equation, may be written as:

$$u_m^{(n+1)}(t) = S_m(\lambda, u_m^{(n)}, t) \ (n = 0,1,2,...) \tag{3.5}$$

where:

$$S_m(\lambda, u_m^{(n)}, t) = \sum_{i=1}^{m} \int_{t_0}^{t} \psi_i(x) \frac{\mathrm{d}\, x}{p(x)}\, p(\xi_i) F\left[\lambda, \xi_i, u_m^{(n)}(\xi_i), I(u_m^{(n)}, \xi_i)\right] \tag{3.6}$$

and $u_m^{(0)}(t)$ is a given initial function.

It can be therefore shown that, under the same assumptions under which we proved Theorem 2.1, for any two functions $u_1(t)$ and $u_2(t)$ belonging to the ball $U_r = \{u(t) : \|u(t)\| \leq r\}$, the following inequality norm of $C_p^1([-\pi, \pi], t_0)$ is fulfilled:

$$\|S_m(\lambda, u_1, t) - S_m(\lambda, u_2, t)\| \leq \lambda_m f(\lambda, r) \|u_1(t) - u_2(t)\| \tag{3.7}$$

where:

$$\lambda_m = \sup \sum_{i=1}^{m} |\psi_i(t)|, \ -\pi \leq t \leq \pi \tag{3.8}$$

and:

$$\begin{aligned} & F[\lambda, t, u_2(t), I(u_2, t)] - F[\lambda, t, u_1(t), I(u_1, t)] \\ & = \frac{1}{2}(u_2(t) - u_1(t)) \int_{-1}^{1} F_u[\lambda, t, v_1(\xi, t), I(u_2, t)] \mathrm{d}\, \xi + \\ & + \frac{1}{2}[I(u_2, t) - I(u_1, t)] \int_{-1}^{1} F_v[\lambda, t, u_1(t), v_2(\xi, t)] \mathrm{d}\, \xi \end{aligned} \tag{3.9}$$

with:

$$v_1(\xi, t) = \frac{1}{2}(1 + \xi) u_2(t) + \frac{1}{2}(1 - \xi) u_1(t) \tag{3.10a}$$

$$v_2(\xi,t)=\frac{1}{2}(1+\xi)S(u_2,t)+\frac{1}{2}(1-\xi)S(u_1,t) \tag{3.10b}$$

As the derivatives $F_u[\lambda,t,x,I(u_2,t)]$ and $F_v[\lambda,t,u_1(t),x]$ are continuous in $x$, then (3.9) can be written as:

$$\begin{aligned}&F[\lambda,t,u_2(t),I(u_2,t)]-F[\lambda,t,u_1(t),I(u_1,t)]\\&=F_u[\lambda,t,v_1(\xi_1,t),I(u_2,t)][u_2(t)-u_1(t)]\\&+F_v[\lambda,t,u_1(t),v_2(\xi_2,t)][I(u_2,t)-I(u_1,t)]\end{aligned} \tag{3.11}$$

where $-1<\xi_1,\xi_2<1, \xi_1=\xi_1(\lambda,t),\xi_2=\xi_2(\lambda,t)$.

Then, as $|u(t)|\le a(t)\|u\|$ and $|I(u,t)|\le b(t)\|u\|$, where $a(t)$ and $b(t)$ are determined by the given function $p(t)$ and because (2.5) is true, (3.7) is true, too.

In addition, if $\lambda_m \le 1$, then the operator (3.6), acting from $C_p^1([-\pi,\pi],t_0)$ into the same space, is a contraction operator, and therefore (3.3) will have a unique solution $u_m^*(t)$ in the ball $U_r$, to which the sequence $\{u_m^{(n)}(t)\}$ will converge as $n\to\infty$ for any initial function $u_m^{(0)}(t)\in U_r$.

By using the same method as for $\psi_i(t)$, we are taking a linear combination of the functions $\{F_k(t)\}(k=1,2,...,m)$ and, so (3.3) and its solution $u_m^*(t)$ can be written as following:

$$H_m(u_m,t)=p(t)u_m'(t)-\sum_{k=1}^{m}a_{mk}\psi_k(t)=0 \tag{3.12}$$

and:

$$u_m^*(t)=\sum_{k=1}^{m}a_{mk}^*\int_{t_0}^{t}\frac{\varphi_k(x)}{p(x)}\mathrm{d}x \tag{3.13}$$

where $a_{mk}^*$ are determined numbers. In the same way $u_m^*(t)$ will be a solution of (3.1), for $p(\xi_i)\ne 0\ (i=1,2,...,m)$.

Beyond the above, it is possible to put $F_k(t)=\psi_k(t),(k=1,2,...,m)$. We use further the Feier interpolation process $Q_m(\xi)(m=1,2,...)$ on the interval $[-\pi,\pi]$defined for a given function $f(\xi)$ by $Q_m(\xi_k)=f(\xi_k),Q_m'(\xi_k)=0,(k=1,2,...m)$, where $\xi_k=\xi_{mk}=\pi\cos((2k-1)/2m)\pi$ are the

Chebyshev nodes, i.e., the $\xi_k$ are the zeros of the polynomial of degree $m$ which differs at least from zero in the uniform metric space on $[-\pi, \pi]$:

$$T_m(\xi) = \frac{\pi^m}{2^{m-1}} \cos\left(\pi \cos^{-1}(\xi/\pi)\right) \tag{3.14}$$

Also, the interpolation polynomial $Q_m(\xi)$ has the following form:

$$Q_m(\xi) = \sum_{k=1}^{m} \psi_k(\xi) f(\xi_k) \tag{3.15}$$

where:

$$\psi_k(\xi) = \left[\frac{2^{m-1} T_m(\xi)}{m\pi^m (\xi - \xi_k)}\right]^2 \left(\pi^2 - \xi\xi_k\right) \tag{3.16}$$

and $\lambda_m = 1$ for this.

We will further show that, as $m \to \infty$, the approximate solutions $u_m^*(t)$ converge in the form of $C_p^1([-\pi, \pi], t_0)$ to a solution of (2.1).

For this, we introduce the notation:

$$r_m = \sup\|S_1(\lambda, u, t) - S_m(\lambda, u, t)\|, \|u - u^*\| \le \omega \tag{3.17}$$

or:

$$r_m = \sup\max|r_m(u,t)|, \|u - u^*\| \le \omega, -\pi \le t \le \pi \tag{3.18}$$

where:

$$r_m(u,t) = p(t)F[\lambda, t, u(t), S(u,t)] - \sum_{i=0}^{m} \psi_i(t) p(\xi_i) F[\lambda, \xi_i, u(\xi_i), S(u, \xi_i)] \tag{3.19}$$

Besides since $r_m(u,t) = r_m(u^*, t) + r_m(u, u^*, t)$ where:

$$r_m(u,u^*,t) = H(u,u^*,t) - \sum_{i=0}^{m} \psi_i(t) H(u,u^*,\xi_i) \tag{3.20}$$

and:

$$H(u,u^*,t) = p(t)\big[F[\lambda,t,u(t),S(u,t)]\big] - F\big[\lambda,t,u^*(t),S(u^*,t)\big] \tag{3.21}$$

follows that for $u(t)$ and $u^*(t)$ in $U_r = \{u(t): \|u(t)\| \le r\}$:

$$\begin{aligned}\max_{-\pi\le t\le\pi} |r_m(u,t)| &\le \|S_1(\lambda,u,t) - S_1(\lambda,u^*,t)\| \\ +\sum_{i=0}^{m}\psi_i(t)\|S_1(\lambda,u,t) - S(\lambda,u^*t)\| + \varepsilon_m &\le 2f(\lambda,r)\|u-u^*\| + \varepsilon_m\end{aligned} \tag{3.22}$$

is valid, where $\varepsilon_m \to \infty$ and $m \to \infty$.

In addition, under the conditions that $\|u\| \le r$ and $\|u^*\| \le r$, we can take $\omega$ to be $2r$ and, therefore, follows that $r_m \le 2f(\lambda,r)\omega + \varepsilon_m$, i.e., these numbers are bounded for all $m$.

Because of the convergence of the Feier interpolation process [38] in the class of continuous functions, for any fixed function $u(t), r_m(u^*,t)$ and the remainder $r_m(u,u^*,t)$ approach zero on $[-\pi,\pi]$ as $m \to \infty$. Moreover, the remainder $r_m(u,u^*,t)$ also converge uniformly to zero with respect to function *u(t)* belonging to the $U^*_\omega = \{u(t): \|u-u^*\| \le \omega\}$.

Besides, for fixed $t$ in $[-\pi,\pi]$ and $u(t)$ in $U^*_\omega$, we split the set of numbers $1, 2, \ldots, m$ into two groups: $S^I(u)$ and $S^{II}(u)$, assigning to $S^I(u)$ those $k$ for which $|\xi_k - t| < \delta_1$ and to $S^{II}(u)$ the remaining ones. Then we have $r_m(u,u^*,t) = S_1(t) + S_2(t)$, where:

$$S_1(t) = \sum_{k\in S^I(u)} \big[H(t,u,u^*) - H(\xi_k,u,u^*)\big]\psi_k(t)$$

and:

$$S_2(t) = \sum_{k\in S^{II}(u)} \big[H(t,u,u^*) - H(\xi_k,u,u^*)\big]\psi_k(t) \tag{3.23}$$

Because of the continuity of $H(t,u,u^*)$ on $[-\pi,\pi]$:

$$|S_1(t)| \le \sum_{k \in S^I(u)} \psi_k(t)\varepsilon \le \varepsilon \sum_{k=1}^{m} \psi_k(t) = \varepsilon \tag{3.24}$$

is valid and in this $\varepsilon$ can be arbitrarily small for small values of $\delta_1$.

Also, if $k \in S^{II}(u)$, then by taking into account the explicit form of $\psi_k(t)$ and the inequalities $0 < \pi^2 - t\xi_k < 2\pi^2$ and $|T_m(t)| \le \pi^m / 2^{m-1}$, we obtain:

$$|S_2(t)| \le \frac{4M\pi^2}{m\delta_1^2} \tag{3.25}$$

where M is the largest value of $H(t,u,u^*)$ in the set $\{t \in [-\pi, \pi], u \in U_\delta\}$.

Beyond the above, the inequalities (3.24) and (3.25) are valid for all $-\pi \le t \le \pi$ and $u(t) \in U_\omega^*$. Hence, $r_m \to 0$ as $m \to \infty$.

As $u^*(t)$ is the unique solution of (2.1) in $U_r$, follows that this equation does not have any solutions in the ring $\varepsilon \le \|u - u^*\| \le \omega$, for $0 < \varepsilon < \omega$, i.e. there exists an $a(\varepsilon, \delta) > 0$ such that $\|u - S_1(\lambda, u, t)\| \ge a(\varepsilon, \delta)$ for $\varepsilon \le \|u - u^*\| \le \delta$.

In this ring therefore for any: $u(t), \|u - S_m(\lambda, u, t)\| \ge \|u - S_1(\lambda, u, t)\| - \|S_1(\lambda, u, t) - S_m(\lambda, u, t)\|$ is valid, and for sufficiently large $m$, $\|S_1(\lambda, u, t) - S_m(\lambda, u, t)\| \le r_m \le a(\varepsilon, \delta)$ and hence $\|u - S_m(\lambda, u, t)\| \ge a(\varepsilon, \delta) - r_m > 0$ for large $m$. Consequently, follows that the $u_m^*(t)$ of (3.3) cannot be in the ring $\varepsilon \le \|u - u^*\| \le \delta$ and, therefore $\|u_m^* - u^*\| < \varepsilon$, where $\varepsilon$ is positive and arbitrary, which finally proves Theorem 3.1.

## 4. Conclusions

The current research was devoted to a study of new approximation methods for the solution of the non-linear singular integro-differential equations, defined in closed-normed spaces, i.e. Banach spaces. This was an exposition of the conditions of applicability of the method of collocation to those non-linear equations and for the convergence of the method.

In addition, a system of Chebyshev functions was used in the collocation approximation method for the investigation of the existence of solutions for the system of non-linear equations applied for the numerical solution of the non-linear singular integro-differential equations. Closed-form solutions of such type of non-linear equations are not possible to be determined, because of the big complication of their term. Consequently, they are approximated only by special numerical methods.

Consequently, the collocation approximation methods can be used for the numerical solution of non-linear singular integro-differential equations defined in general problems of structural analysis, fracture mechanics, fluid mechanics, potential flows, aerodynamics, turbomachines, etc. of great importance.

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