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Non-linear Parabolic Integral Equations used in Stationary and Dynamic Viscoplasticity

E.G. Ladopoulos
Interpaper Research Organization
8, Dimaki Str.
Athens, GR - 106 72, Greece
eladopoulos@interpaper.org

Abstract
Some general problems of stationary and dynamic viscoplasticity are investigated, which reduce to the solution of non-linear parabolic integral equations. The existence and uniqueness of the solutions for this type of non-linear integral equations is further investigated, by proposing a new mathematical devise, which is an "innovative" and "groundbreaking" method. Non-linear stationary viscoplasticity reduces to the solution of a boundary value problem and the existence of solutions of this problem is proved by minimizing a special functional. Non-linear dynamic viscoplasticity is further investigated, by considering the solutions of the non-linear parabolic integral equations to be defined in a separable, reflexive Banach space. An application is finally given to the determination of the limit load for a rigidly plastic solid.

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Non-linear Parabolic Integral Equations, Stationary Viscoplasticity, Dynamic Viscoplasticity, Banach Space, Rigidly Plastic Solid, Banach Spaces

1. Introduction

Theory of stationary and dynamic viscoplasticity, widely applicable in a big level of problems of engineering science, was formulated and generated almost during the last decades. P. Perzyna [1], [2] has introduced a constitutive model for viscoplastic material in connection with a general theory of approaching the inelastic behavior of solids [3]. Further results for the theory of viscoplastic media, were adopted some years later, by P.P. Mosolov and V. P. Miasnikov [4], P.M. Ogibalov and A.K. Mirzadzhanzade [5], and A.Phillips and Wu Han-Chin [6].

Beyond the above, O.C.Zienkiewicz and I.C.Cormeau [7], [8] and O.C. Zienkiewicz, C. Humpheson and R.W. Lewis [9] introduced a general quasi-static formulation for solving viscoplastic material problems. S.R. Bodner and V. Partom [10] have further developed a general model, which describes the mechanical response of a wide class of materials. Moreover, I. Cormeau [11] introduced an "initial strain" numerical scheme for the solution of viscoplasticity problems, by using a non-linear first order system of ordinary differential equations.

Two years later M. Chaudonneret [12] used a direct boundary element formulation for the viscoplastic analysis of a notched plate. The integral equations presented were based on an "initial stress" form of the viscoplastic strains influence and the numerical method was carried out by using linear boundary elements and constant rectangular cells for integrating the nonlinear term. The implications of square-root singular crack-tip fields for viscoplastic solids have been investigated by E. W. Hart [13], who related the stress intensity factor at the tip to the corresponding far-field quantity.

A further development of the theory of viscoplasticity was presented by J.C.F.Telles and C.A.Brebbia [14] and M.Brunet [15], by introducing a viscoplastic boundary element method which is capable of handling plasticity, creep and viscoplasticity in a unified approach by using four different yield criteria. Furthermore, a review of a unified elasto-viscoplastic theory is given by S.R.Bodner [16], by summarizing all the well known previous results.

Furthermore, L.B.Freund and J.W.Hutchinson [17] developed a relation between the tip and the far-field stress intensity factors for a viscoplastic solid, for the dynamic steady-state mode I case, by using a linear relation between the stress and the inelastic strain rate. Also, J.D.Achenbach, N.Nishimura and J.C.Sung [18] used an integral representation for the particle velocity in terms of a Green's function and certain linear combinations of the inelastic strain rates, in order to introduce a numerical method to compute full field solutions and to develop unequivocal asymptotic expressions for the fields near a stationary crack tip defined in a viscoplastic solid.

Over the past years, E.G.Ladopoulos [19] – [23] has introduced the finite-part singular integral equations for solving some general problems of elasto-plasticity and fracture mechanics theory. On the other hand, this type of linear singular integral equations, has been extended by E.G.Ladopoulos *et al.* [24], [25} and E.G.Ladopoulos [26] - [31] to non-linear forms too, for solving two-dimensional fluid mechanics problems, aerodynamics and petroleum engineering problems, while in the present report some non-linear parabolic integral equations are used for the solution of important problems of stationary and dynamic viscoplasticity.

Consequently, non-linear stationary and dynamic viscoplasticity is investigated, by reducing the problem to the solution of a non-linear parabolic integral equation, under some special conditions. Some existence and uniqueness theorems are presented, for the solution of this type of non-linear integral equations. An application is finally given, to the determination of the limit load for a rigidly plastic solid.

2. Non-linear Stationary Viscoplasticity

Consider the principle of virtual power for a viscoplastic solid, which fills a domain E in R^3, defined by the following non-linear parabolic integral equation: [32]

$$\int_E \left[\rho \frac{\partial \mathbf{u}}{\partial t} \mathbf{b} + \sum_{ij} \sigma_{ij} b_{ij} \right] \mathrm{d}\, E = A(\mathbf{b}), \ \ \forall (\mathbf{b}) \tag{2.1}$$

where:

$$b_{ij} = \frac{1}{2} \left(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i} \right) \tag{2.2}$$

in which $A(\mathbf{b})$ denotes the power of the external forces in the variation $\mathbf{b}$ of the displacement field $\mathbf{u}$, σ_{ij} is the stress field, ρ the density of the viscoplastic solid and t the time.

The stress field σ_{ij} for a viscoplastic solid is further defined by the formula:

$$\sigma_{ij} = \frac{\partial f(\mathbf{x}, \varepsilon_{ij})}{\partial \varepsilon_{ij}}, \quad for \ |\boldsymbol{\varepsilon}| > 0$$

$$f^*(\mathbf{x}, \varepsilon_{ij}), \quad for \ |\boldsymbol{\varepsilon}| > 0 \tag{2.3}$$

where:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{2.4}$$

in which ε_{ij} denotes the strain tensor and $f(\mathbf{x}, \varepsilon_{ij})$ is the potential of an incompressible viscoplastic solid, which possess the following condition:

$$f > c\ |\boldsymbol{\varepsilon}| \tag{2.5}$$

with:

$$|\boldsymbol{\varepsilon}| = \left(\sum_{ij} \varepsilon_{ij}^2 \right)^{1/2} \tag{2.6}$$

where c is a positive constant.

Furthermore, for slow stationary motions of the viscoplastic solid the following approximation is valid:

$$\int_{\mathrm{E}} \sum_{ij} \sigma_{ij} b_{ij}\, \mathrm{d}\, E = A(\mathbf{b}) \tag{2.7}$$

Therefore, the boundary value problem (2.3), (2.4) and (2.7) has to be solved.

Theorem 2.1
Let $\mathbf{u}(\mathbf{x}), \sigma_{ij}(\mathbf{x})$ be a solution of the boundary value problem (2.3), (2.4) and (2.7). Then the displacement tensor $\mathbf{u}(\mathbf{x})$ minimizes the following functional:

$$B(\mathbf{u}) = \int_{\mathrm{E}} f(\mathbf{x}, \varepsilon_{ij}(\mathbf{x}))\, \mathrm{d}\, E - A(\mathbf{u}) \tag{2.8}$$

and vice versa, if $\mathbf{u}(\mathbf{x})$ minimizes (2.8), then there exists a solution $\mathbf{u}$, σ_{ij} of the problem (2.3), (2.4) and (2.7).

Proof. Let us replace condition (2.3) by the following formulas:

$$\begin{aligned} \sigma_{ij} &= \frac{\partial f(\mathbf{x}, \varepsilon_{ij})}{\partial \varepsilon_{ij}} + \rho_{ij}(\mathbf{x}, \boldsymbol{\varepsilon}), \quad \text{for } |\boldsymbol{\varepsilon}| > 0 \\ f^*(\mathbf{x}, \sigma_{ij}) &= 0, \quad \text{for } |\boldsymbol{\varepsilon}| = 0 \end{aligned} \tag{2.9}$$

Moreover, by $L_p^n(E)$, ($1 \leq p < \infty$), we denote the Lebesgue space of measurable vector-functions $\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots\ldots u_n(\mathbf{x})\}$ for which the integral of $|\mathbf{u}(\mathbf{x})|^p$ with respect to E is finite.

Therefore, because of the boundedness of *E*, the functional (2.8) is defined on $L_p^n(E)$. By using the theorem on the general form of a linear continuous functional in $L_p^n(E)$ [33], then the following inner product is valid, when the functional $B(\mathbf{u})$ has the form (2.8):

$$< L(\mathbf{u}), \mathbf{b} >= \int_E \sum_{ij} \sigma_{ij} b_{ij} \, \mathrm{d}\, E \tag{2.10}$$

and thus the existence of a solution **u**, σ_{ij} of the boundary value problem (2.3), (2.4) and (2.7) was proved.

Lemma 2.1
Let the solution of the boundary value problem (2.3), (2.4) and (2.7) be such that $|\boldsymbol{\varepsilon}(\mathbf{x})| \geq h > 0$, everywhere in *E*. Then this problem reduces to the usual Euler equation for the functional (2.8).

Proof. This Lemma appears to be a direct result of Theorem 2.1. Therefore, the non-differentiability of $f(\mathbf{x}, \varepsilon_{ij})$ for $|\boldsymbol{\varepsilon}| = 0$ is inessential and the functional (2.8) is differentiable on such an extremal. Then conditions (2.3) and (2.7) are the usual Euler equation for the functional (2.8).

Theorem 2.2
Consider the stress field $\sigma(\mathbf{x})$ satisfy the following condition:

$$\int_E [f(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{x}) + \mathbf{b}(\mathbf{x})) - f(\mathbf{x}, \boldsymbol{\varepsilon}) - \sigma(\mathbf{x})\mathbf{b}(\mathbf{x})] \, \mathrm{d}\, E \geq 0, \quad \forall \mathbf{b}(\mathbf{x}) \in L_p^n(E) \tag{2.11}$$

Then, for almost all $\mathbf{x}$ from *E* the following inequality is valid:

$$f(\mathbf{x}, \boldsymbol{\varepsilon}(x) + \mathbf{b}) - f(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{x})) \geq \sigma(\mathbf{x})\mathbf{b}, \quad \forall \mathbf{b} \in R^n \tag{2.12}$$

Proof. Consider the assertion in this Theorem to be false, then for some $\mathbf{b} \in R^n$ there exists a set *N* of positive measure in *E*, on which the opposite inequality to (2.12) is satisfied.

Therefore, by setting:

$$\mathbf{b}(\mathbf{x}) = \begin{cases} \overline{\mathbf{b}}, & \text{in N} \\ 0, & \text{outside N} \end{cases} \tag{2.13}$$

one arrives at a contradiction to (2.11). [34]

3. Non-linear Dynamic Viscoplasticity

Consider the principle of virtual power for slow dynamic motion of a viscoplastic solid, defined by the following non-linear parabolic integral equation: [32]

$$\int_E \left[\rho \frac{\partial \mathbf{u}}{\partial t} \mathbf{b} + \sum_{ij} \sigma_{ij} b_{ij} \right] \mathrm{d}\, E = A(\mathbf{b}) \tag{3.1}$$

under the conditions:

$$\begin{aligned} div\mathbf{u} &= 0 \\ \mathbf{u}\big|_{t=0} &= \mathbf{u_0}(x) \end{aligned} \tag{3.2}$$

where the stress field σ_{ij} is determined by (2.3).

Definition 3.1
Let B a separable, reflexive Banach space. By the notation $M_B^p[0,T]$, $p \geq 1$ we denote the space of measurable functions $g(t)$ for which the norm $\|g\|_B^p$ is summable in $[0,T]$.
Moreover, by $M_B^\infty[0,T]$ we denote the space of measurable functions for which the norm $\|g\|_B$ is bounded in $[0,T]$.

Definition 3.2
By $L_B^p[0,T]$ is defined the factor space of $M_B^p[0,T]$ in the space of negligible functions. Also, $L_B^p[0,T]$ is a Banach space with the norm:

$$\left\{ \int_0^T \|g\|_B^p \,\mathrm{d}\, t \right\}^{1/p}, \ 1 \leq p < \infty \tag{3.3}$$

Theorem 3.1
Consider by u an element from the Banach space B, ξ_n, $n=1,2,...$, a compact set in B and S the following set of elements:

$$S : \xi_{n,k} = \left(\frac{\xi_n + (2^k - 1)u}{2^k}\right), \ n = 1,2,..., \ \ k = 0,1,2,... \tag{3.4}$$

In addition, for any ξ from S and some $L(\xi)$ from $\partial F(\xi)$, where F is a convex, finite functional in B, if the following inequality is satisfied:

$$< y - L(\xi), u - \xi > \geq 0 \tag{3.5}$$

then, the functional y enters into $\partial F(u)$.

Proof. From the inequality (3.5) and the definition of $\partial F(\xi)$, we obtain:

$$< y, \xi - u > \leq < L(\xi), \xi - u > \leq F(2\xi - u) - F(\xi) \qquad (3.6)$$

Therefore, by substituting the elements $\xi_{n,k}$ into (3.6) and then adding the additional inequalities, one has:

$$< y, 2(\xi_n - u) > \leq F(u + 2(\xi_n - u)) - F(u) \qquad (3.7)$$

Thus, from (3.7) the assertion of this Theorem is completed.

Theorem 3.2
Consider by $D(t,u)$ a number function defined in $[0,T]$xB, $D(t,0)=0$, and $D(t,u(t)) \in M^{\infty}[0,T]$, for $u(t) \in M_B^{\infty}[0,T]$.
Therefore, if:

$$\int_0^T D(t,u(t))\,\mathrm{d}t \geq 0, \;\; \forall u(t) \in M_B^{\infty}[0,T] \qquad (3.8)$$

then $D(t,u(t)) \geq 0$ for almost all t from $[0,T]$ and all u from B.

Proof. The proof of this Theorem is analogous to the proof of Theorem 2.2.

Theorem 3.3
Consider by $\{u_n(t)\}$ a sequence of linear continuous functions, with $\|\mathrm{d}u_n/\mathrm{d}t\|_H \leq C$, C=constant. Also, by $u_n(t)$ we denote a function, which converge uniformly to $u(t)$ in $[0,T]$.

Then, there exists a $u'(t) \in M_H^{\infty}[0,T]$ and the following limit is valid:

$$\int_0^T \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}t}, \xi \right\rangle_H \mathrm{d}t \rightarrow \int_0^T \langle u', \xi \rangle_{\mathrm{H}}\, \mathrm{d}t, \;\; \forall \xi(t) \in M_H^1[0,T] \qquad (3.9)$$

Proof. This Theorem is easily proved from the fact that $L_{B^*}^{\infty}[0,T]$ is conjugate to $L_B^1[0,T]$, where B^* is a space of linear continuous functionals on B [35], and the properties of weak compactness of the space conjugate to the separable space. [33]

Theorem 3.4
Let $\mathbf{u}(t,x) \in C_H[0,T] \cap M_B^{\infty}[0,T]$, $\dfrac{\partial \mathbf{u}}{\partial t} \in M_H^{\infty}[0,T]$ and $A(\mathbf{b}) \in C_H[0,T]$.
The function $\mathbf{u}$ denotes the generalized solution of the non-linear parabolic integral equation (3.1), if for almost all t from $[0,T]$ the following condition is satisfied:

$$\left\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{b} \right\rangle_H + \langle L(\mathbf{u},t), \mathbf{b} \rangle = \langle A(\mathbf{b}), \mathbf{b} \rangle_H \qquad (3.10)$$
$$\mathbf{u}(0) = \mathbf{u_0}, \;\; \forall \mathbf{b} \in B \cap H$$

where $L(\mathbf{u},t) \in M^{\infty}_{B^*}[0,T]$.

Proof. Let us express the non-linear parabolic integral equation (3.1) as following:

$$N[\mathbf{u}(t),t] + \frac{\partial \mathbf{u}}{\partial t} = A(t) \tag{3.11}$$

Therefore, let us have:

$$\sum_i \Delta t_i^n = T, \quad i = 1,2,....,n \tag{3.12}$$

Then, we examine the elements u_i^n from $B \cap H$, such that the following condition is valid: [36]

$$\inf_u \left\{ \frac{1}{2\Delta t_i^n} \left\| u - u_{i-1}^n \right\|_{H^2} + F(u) - \left\langle A(t_i^n), u \right\rangle_H \right\} = \\ = \frac{1}{2\Delta t_i^n} \left\| u_i^n - u_{i-1}^n \right\|_{H^2} + F(u_i^n) - \left\langle A(t_i^n), u_i^n \right\rangle_H \tag{3.13}$$

Hence, from (3.13) follows that there exists a functional $L(u_i^n)$ such that:

$$\left\langle \frac{u_i^n - u_{i-1}^n}{\Delta t_i^n}, b \right\rangle_H + \left\langle L(u_i^n), b \right\rangle = \left\langle A(t_i^n), b \right\rangle_H, \ \forall b \in B \cap H \tag{3.14}$$

Also, the following formula is valid:

$$\left\| u_i^n \right\|_B + \left\| u_i^n \right\|_H \le c, \ c = \text{constant} \tag{3.15}$$

Furthermore, we assume that $A(t)$ and the initial element u_0 satisfy the conditions:

$$\left\| A(t_1) - A(t_2) \right\|_H \le r \left| t_1 - t_2 \right| \max_{i=1,2} \left\| A(t_i) \right\|_H \tag{3.16}$$

$$\left\| (u_1^n - u_0) / \Delta t_1^n \right\|_H \le \zeta, \ \forall \Delta t_1^n \tag{3.17}$$

where r,ζ are constants

So, one obtains:

$$\left\| (u_1^n - u_{i-1}^n) / \Delta t_1^n \right\|_H \le \lambda, \ \lambda = \text{constant}, \ i = 2,3,...,n \tag{3.18}$$

We introduce the functions $u^n(t)$ and $u^{*n}(t)$ such that:

$$u^n(t) = u_{i-1}^n \ , \ \text{for} \ \ t_{i-1}^n \le t \le t_i^n \tag{3.19}$$

and:

$$u^{*n}(t) = u_i^n \ , \ \text{for} \ \ t = t_i^n \tag{3.20}$$

Hence, from (3.14) follows:

$$\int_0^T \left\langle \frac{\mathrm{d}\,u^{*n}}{\mathrm{d}\,t}, b(t) \right\rangle_H \mathrm{d}\,t + \int_0^T \left\langle L(u^n(t)), b(t) \right\rangle \mathrm{d}\,t = \int_0^T \left\langle A^n(t), b(t) \right\rangle_H \mathrm{d}\,t \ , \ \forall b(t) \in M_{B\cap H}^1[0,T] \tag{3.21}$$

with:

$$\begin{aligned} L(u^n(t)) &= L(u_{i-1}^n) \\ A^n(t) &= A(t_{i-1}^n) \ , \ t_{i-1}^n \le t \le t_i^n \end{aligned} \tag{3.22}$$

From Theorem 3.3 and (3.18) follows that there exists a $u'(t)$. Consequently, because of (3.15) we obtain that the function $L(u^n(t))$ converge to $h(t)$ in $M_{B^*}^{\infty}[0,T]$. So, we shall show that $h(t)$ is from $N[u(t),t]$.

From (3.21) one obtains the following formula:

$$\int_0^T \left[\langle u', b \rangle_H + \langle h, b \rangle - \langle A, b \rangle_H \right] \mathrm{d}\,t = 0 \tag{3.23}$$

$$\forall b \in M_{B\cap H}^1[0,T]$$

Hence, let $b(t)$ be from $M_B^{\infty}[0,T]$, $L\{b(t)\}$ from $M_{B^*}^{\infty}[0,T]$ and from $N[u(t),t]$. Also, for $b(t)=u^n(t)$ we obtain from (3.21): [36]

$$\int_0^T \left\langle L(u^n(t)), u^n(t) \right\rangle \mathrm{d}\,t = \int_0^T \left[\left\langle A^n(t), u^n(t) \right\rangle_H - \left\langle \frac{\partial \overline{u}^n}{\partial t}, u^n \right\rangle_H \right] \mathrm{d}\,t \tag{3.24}$$

Moreover, (3.24) takes the following form:

$$\frac{1}{2}\|u_0\|_H^2 - \frac{1}{2}\|u_n^n\|_H^2 + \frac{1}{2}\sum_{i=1}^{n}\|u_i^n - u_{i-1}^n\|_{H^2} + \int_0^T \left\langle A^n(t), u^n(t)\right\rangle_H \mathrm{d}t \tag{3.25}$$

which tends to the limit:

$$\frac{1}{2}\|u_0\|_{H^2} - \frac{1}{2}\|u(T)\|_{H^2} + \int_0^T \left\langle A(t), u(t)\right\rangle_H \mathrm{d}t \tag{3.26}$$

Therefore, from (3.23) to (3.26) one has:

$$\int_0^T \left\langle h - L(b(t))\right\rangle \mathrm{d}t \geq 0\,,\; u - b > \mathrm{d}t\,,\; \forall b(t) \in M_B^{\infty}[0,T] \tag{3.27}$$

Finally, from (3.27) and Theorems 3.1 and 3.2 follows that h is from $N[u(t),t]$ and thus, the existence of the generalized solution (3.10) was proved.

4. Application to the Determination of the Limit Load for a Rigidly Plastic Solid.

As an application of the theory of non-linear stationary viscoplasticity, the limit load p_0 for the external forces will be determined, when the kinematically allowable velocity forms a linear space, with volume density $\mathbf{v}$ and surface density $\mathbf{h}$ of a rigidly plastic solid. In this case the potential function $f(\mathbf{x}, \varepsilon_{ij})$ is valid such that: [4]

$$f(\mathbf{x}, \mu\varepsilon_{ij}) = \mu f(\mathbf{x}, \varepsilon_{ij})\,,\; \forall \mu \geq 0 \tag{4.1}$$

Theorem 4.1

Consider a rigidly plastic solid, while its kinematically allowable velocity forms a linear space. Let p_n a nonnegative number such that some functions from $M^{\infty}(E)$ are valid, for which the following equalities are satisfied:

$$\int_E \sum_{ij} \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}E = p_n D(\mathbf{u})\,,\; \forall \mathbf{u}(x) \in U \tag{4.2}$$

$$f^*(\mathbf{x}, \sigma_{ij}(\mathbf{x})) = 0$$

with:

$$D(\mathbf{u}) = \int_E \mathbf{v}\mathbf{u}\,\mathrm{d}E + \int_{\partial E} \mathbf{h}\mathbf{u}\,\mathrm{d}S \tag{4.3}$$

where f^* denotes the conjugate of the potential function $f(\mathbf{x}, \varepsilon_{ij})$ given by (4.1).

Moreover, if $p_d = sup\, p_n$, then:

$$p_0 = p_d \tag{4.4}$$

where p_0 is the limit load for the external forces.

Proof. The limit load p_0 for the external forces is given by the following formula: [4]

$$\frac{1}{p_0} = \sup D(\mathbf{u}, (\mathbf{x}))\left[\int_E f(\mathbf{x}, \varepsilon_{ij}(\mathbf{x}))\,\mathrm{d}\,E\right]^{-1}, \quad \forall \mathbf{u}(x) \in U \tag{4.5}$$

where $p_0 \geq p_n$ is given by (4.3).

By assuming $D(\mathbf{u}) > 0$, then we obtain the upper bound for the load p_0:

$$p_0 \leq p_u = \frac{1}{D(\mathbf{u})}\int_E f(\mathbf{x}, \varepsilon_{ij}(\mathbf{x}))\,\mathrm{d}\,E \tag{4.6}$$

From (4.2) since $f^*(\mathbf{x}, \sigma_{ij}) = 0$, then one has:

$$\sum_{ij} \sigma_{ij}\varepsilon_{ij} \leq f(\mathbf{x}, \varepsilon_{ij}) \tag{4.7}$$

Therefore, we obtain:

$$\frac{1}{p_n} \geq \frac{D(\mathbf{u})}{\int_E f\,\mathrm{d}\,E}, \quad \forall \mathbf{u} \in U \tag{4.8}$$

Hence, if $D(\mathbf{u}) > 0$, then:

$$p_0 \geq p_n \tag{4.9}$$

and it is obvious that:

$$p_0 \geq p_d \tag{4.10}$$

Furthermore, we can consider that $p_0>0$, since if $p_0=0$, then we shall have $\sigma_{ij}=0$ for $p_n=0$. Consequently, let the following functional, where p is a positive number, $0<p<p_0$:

$$F_p(\mathbf{u}) = \int_E f(\mathbf{x}, \varepsilon_{ij}(\mathbf{x}))\,\mathrm{d}\,E - pD(\mathbf{u}) \tag{4.11}$$

It is obvious that $F_p(\mathbf{u}) \geq 0$ for all u from U and hence, $\mathbf{u}$=0 is a vector field minimizing $F_p(\mathbf{u})$. Therefore, from Theorem 2.1 follows that there exists a σ_{ij}, such that (4.2) is satisfied for p_n=p and thus, Theorem 4.1 was proved.

5. Conclusions

A non-linear parabolic integral equations analysis was presented, to the solution of some general problems of stationary and dynamic viscoplasticity. Hence, some existence and uniqueness theorems were proposed, for the solution of the non-linear parabolic integral equations.

Furthermore, a proof was given for the equivalence of the differential and variational formulations of problems concerning the motion of a viscoplastic medium in the presence of domains of the rigid state of the medium and flow domains.

By considering the principle of virtual power for a viscoplastic solid, then the general theory of non-linear stationary viscoplasticity was reduced to the solution of a boundary value problem. Consequently, an existence theorem was proved for the solution of the above non-linear boundary value problem.

Beyond the above, by considering the principle of virtual power for slow dynamic motion of the viscoplastic solid, then the general theory of non-linear dynamic viscoplasticity, was reduced to the solution of a non-linear parabolic integral equation. Some existence theorems were proved for the solution of the above non-linear integral equation over a reflexive Banach space.

An application was finally given to the determination of the limit load for a rigidly plastic solid. Hence, the agreement between the upper bounds of the static load coefficients and the lower bound of the kinematic limit load coefficients was shown, for the above rigidly plastic solid.

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