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Contractions and Expansions in Open Channel Hydraulics by S.I.O.M.

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Abstract
By using the very modern method of singular integral equations, then the free surface profile of potential flow is calculated in open-channel transitions. Consequently, in such free surface hydraulics applications the analysis of fluid motion is too complicated, as both the subcritical and supercritical flows are presented simultaneously. For the numerical evaluation of the singular integral equations are used both constant and linear elements of the Singular Integral Operators Method (S.I.O.M.). Finally, an application is given to the determination of the free-surface profile in a special open – channel transition and comparing the numerical results of the SIOM with corresponding results by finite differences.

Key Word and Phrases
Contractions & Expansions, Open Channel Transition, Open Channel Hydraulics, Singular Integral Equations, Singular Integral Operators method (S.I.O.M.), Free – surface Profile, Constant & Linear Elements, Potential Flows.

1. Open Channel Hydraulics

The study of open-channel transitions, which are contractions and expansions, belongs to a major field of hydraulics engineering and fluid mechanics applications. So, contractions and expansions of flow belong to a very important chapter of open-channel hydraulics. Open-channel transitions are used in many hydraulic structures, such as in sluice gates, spillways, steep chutes and culverts. The fluid motion analysis in such hydraulics applications are very complicated, as both the subcritical and supercritical flows are present simultaneously.

Over the past years the two-dimensional St.Venant equations based on hydrostatic pressure distribution and shallow water theory have been used with success in order to describe open-channel transitions. The above equations are non-linear first-order hyperbolic partial differential equations and are solved only through computational methods. Some important studies on open-channel transitions were firstly published by A.T.Ippen and J.H.Dawson [1] and A.T.Ippen and D.R.F.Harleman [2], [3]. Some years later J.A.Liggett and S.U.Vasudev [4], M.Pandolfi [5], F.Villegas [6], R.Rajar and M.Cetina [7] and O.F.Jimenez and M.H.Chaudhry [8] used several numerical methods for the computation of supercritical flows in open-channels. Some of the above computational results are in good agreement with the corresponding experimental applications.

In addition, R.J.Fennema and M.H.Chaudhry [9] and S.M.Bhallamudi and M.H.Chaudhry [10] used finite differences for the numerical solution of the two-dimensional St.Venant equations in order to simulate free-surface flows. Their computational method was used for the determination of the free-surface profile in open-channel transitions. They made efforts in order to improve the numerical solution in open-channel hydraulics and especially where the flow phenomenon occurs in different length scales, in different regions of the flow domain. On the contrary, J.F.Thompson et al. [11] and R.G.Hindman et al. [12] in order to generate the motion of the dynamic grid system took the time derivative of the elliptic governing differential equations, of the coordinate mapping in order to solve the two-dimensional time-dependent Euler equations. Furthermore, H.A.Dwyer et al. [13] proposed adaptive grid methods for the solution of problems in open-channel hydraulics and heat transfer. Besides, M.M.Rai and D.A.Anderson [14] studied some applications of adaptive grids to free-surface flow problems with asymptotic solutions. According to them, grid locations are directly calculated from the grid speed equation. For their method the two-dimensional St.Venant equations describing flows in open-channel transitions are solved.

Recently, M.H.Chaudhry [15] and M.Rahman and M.H.Chaudhry [16] used MacCormack second-order accurate explicit predictor-corrector scheme in order to solve the two-dimensional depth averaged shallow water equations for the numerical simulation of the supercritical free-surface flows in open-channel transitions. For their computational method they used an adaptive grid system in order to have a resolution of the changes of the flow variables both for subcritical and supercritical flows.

During the last years E.G.Ladopoulos [17] – [22] and E.G.Ladopoulos and V.A.Zisis [23], [24] introduced and investigated linear and non-linear singular integral equations methods for the solution of fluid mechanics and hydraulics problems. By the current investigation these methods will be extended to the solution of open-channel transitions flows.

Consequently, the Singular Integral Operators Method (S.I.O.M.) [22], [25]-[32] is applied to the determination of the free-surface profile in open-channel transitions, by using the Laplacean equation of potential flow. For the numerical solution of the singular integral equations are used both constant and linear elements. Finally, an application is given to the determination of the free-surface profile in open-channel transitions.

2. Open-Channel Transitions by Potential Flows

Consider an homogeneous, incompressible and inviscid fluid, which flows through an open-channel transition. As the flow is irrotational, then for the stream function f, with $\mathbf{f} = \nabla f$, is valid: [22]

$$\nabla \times \mathbf{f} = 0 \tag{2.1}$$

Besides, because of the conservation of mass for an incompressible fluid, then we have:

$$\nabla \bullet \mathbf{f} = 0 \tag{2.2}$$

By using (2.1) and (2.2.) we obtain the equation of Laplace which is the governing equation in the domain Ω:

$$\nabla^2 f = 0 \tag{2.3}$$

The boundary conditions corresponding to the flow for open-channel transitions are:

a. Essential conditions of the type: $f=Q$ on the axis of symmetry of the transition and $f = Q$ on the boundary wall where Q is the flow discharge. (2.4)

b. Natural conditions of the next type:

$$v = \frac{\vartheta f}{\vartheta n} \tag{2.5}$$

where v denotes the velocity and n the unit normal from the free surface. (Fig. 1)

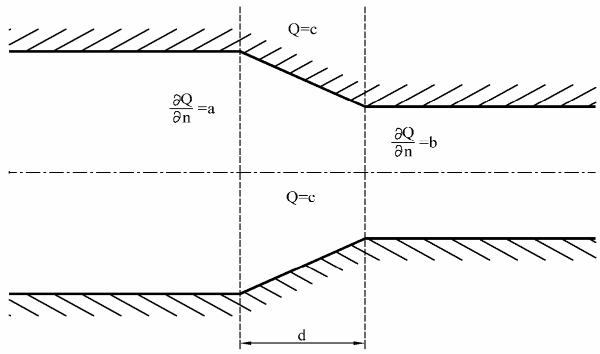

Fig. 1 Boundary Conditions for Open-Channel Transitions

Hence, with a known flow rate Q and known velocities upstream and downstream the transition under study, then the remaining velocities on the boundary of the transition and in internal points can be calculated.

In addition, the free surface elevations can be determined in every boundary or internal point of the transition.

3. Singular Integral Equations for Potential Flow Analysis

Let us consider a weighting function f^*, so that it has continuous first derivatives. Hence, the function f^* produces the following weighted residual statement:

$$\int_{\Omega} (\nabla^2 f) f^* d\Omega = \int_{\Gamma_2} \left(\frac{\vartheta f}{\vartheta n} - \bar{v} \right) f^* d\Gamma - \int_{\Gamma_1} (f - \bar{f}) \frac{\vartheta f^*}{\vartheta n} d\Gamma \tag{3.1}$$

where by (-) are meant average values and Γ_1, Γ_2 are the boundaries where the essential and the natural conditions are affected, respectively.

Besides, integrating by parts the left hand side of (3.1) we have:

$$-\int_{\Omega} \left(\frac{\vartheta f}{\vartheta x_k} \frac{\vartheta f^*}{\vartheta x_k} \right) d\Omega = \int_{\Gamma_2} \frac{\vartheta f}{\vartheta n} f^* d\Gamma - \int_{\Gamma_2} \bar{v} f^* d\Gamma - \int_{\Gamma_1} f \frac{\vartheta f^*}{\vartheta n} d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\vartheta f^*}{\vartheta n} d\Gamma \tag{3.2}$$

By integrating again the left hand side of (3.2) one obtains:

$$\int_{\Omega} f \left(\nabla^2 f^* \right) d\Omega = \int_{\Gamma_2} f \frac{\vartheta f^*}{\vartheta n} d\Gamma - \int_{\Gamma_2} \bar{v} f^* d\Gamma - \int_{\Gamma_1} \frac{\vartheta f}{\vartheta n} f^* d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\vartheta f^*}{\vartheta n} d\Gamma \tag{3.3}$$

In order to find a solution satisfying the Laplace equation, then the governing equation is:

$$\nabla^2 f^* + \Delta_i = 0 \tag{3.4}$$

in which Δ_i is the Dirac delta function.

The solution of (3.4) is called the fundamental solution and has the property such that:

$$\int_{\Omega} f(\nabla^2 f^* + \Delta_i)d\Omega = \int_{\Omega} f\nabla^2 f^* d\Omega + f_i \tag{3.5}$$

where f_i denotes the value of the unknown function at the point "i" where a concentrated load is acting.

Consequently, if (3.4) is satisfied by the fundamental solution then follows:

$$\int_{\Omega} f(\nabla^2 f^*)d\Omega = -f_i \tag{3.6}$$

By using (3.6), then eqn (3.3.) takes the form:

$$f_i + \int_{\Gamma_2} f \frac{\vartheta f^*}{\vartheta n} d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\vartheta f^*}{\vartheta n} d\Gamma = \int_{\Gamma_2} \bar{v} f^* d\Gamma + \int_{\Gamma_1} \frac{\vartheta f}{\vartheta n} f^* d\Gamma \tag{3.7}$$

In addition, by taking the point "i" on the boundary, then the term f_i in (3.7) must be multiplied by 1/2 for a smooth boundary. On the contrary, if the boundary is not smooth at the point "i" then the number 1/2 must be replaced by a constant which can be determined from constant potential considerations.

Then (3.7) takes the form:

$$c_i f_i + \int_{\Gamma} f \frac{\vartheta f^*}{\vartheta n} d\Gamma = \int_{\Gamma} \frac{\vartheta f}{\vartheta n} f^* d\Gamma \tag{3.8}$$

in which $\Gamma = \Gamma_1 + \Gamma_2$ and has been assumed that $f = \bar{f}$ on Γ_1 and $\frac{\vartheta f}{\vartheta n} = v = \bar{v}$ on Γ_2.

Besides, the constant c_i can be determined by the relation:

$$c_i = \frac{\Theta}{2\pi} \tag{3.9}$$

where Θ denotes the internal angle of the corner in rad.

<u>(a) Constant Elements</u>

In order (3.8) to be numerically evaluated by using constant elements, then the above equation may be written as:

$$c_i f_i + \sum_{j=1}^{n} f_j \int_{\Gamma_j} \frac{\vartheta f^*}{\vartheta n} d\Gamma = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} \int_{\Gamma_j} f^* d\Gamma \tag{3.10}$$

Furthermore, (3.10) may be further written as:

$$c_i f_i + \sum_{j=1}^{n} f_j A_{ij}^{*} = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} B_{ij} \tag{3.11}$$

in which: $A_{ij} = A_{ij}^{*}$, when $i \neq j$
$A_{ij} = A_{ij}^{*} + c_i$, when $i=j$ (3.12)

Hence, (3.11) takes the form:

$$\sum_{j=1}^{n} A_{ij} f_j = \sum_{j=1}^{n} B_{ij} \frac{\vartheta f_j}{\vartheta n} \tag{3.13}$$

or in matrix form can be written as:

$$\mathbf{A\, f = B\, v} \tag{3.14}$$

On the contrary, by reordering the above equation so that all the unknowns are on the left hand side, then we have:

$$\mathbf{C\, X = D} \tag{3.15}$$

where **X** denotes the vector of unknowns f and v.

So, once the values of f and v on the whole boundary are known, then f can be calculated at any interior point:

$$f_i = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} B_{ij} - \sum_{j=1}^{n} f_j A^{*}{}_{ij} \tag{3.16}$$

<u>**(b) Linear Elements**</u>

In order (3.8) to be numerically evaluated by using linear elements, then the above equation may be written as:

$$c_i f_i + \sum_{j=1}^{n} \int_{\Gamma_j} f \frac{\vartheta f^{*}}{\vartheta n} d\Gamma = \sum_{j=1}^{n} \int_{\Gamma_j} \frac{\vartheta f}{\vartheta n} f^{*} d\Gamma \tag{3.17}$$

In this case, in contrary to eqn (3.10), the variables f_j and $\frac{\vartheta f_j}{\vartheta n}$ cannot be taken out of the integral as they vary linearly within the element.

Consequently, by using linear elements then (3.17) can be further written as:

$$c_i f_i + \sum_{j=1}^{n} f_j A_{ij}{}^{*} = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} B_{ij} \tag{3.18}$$

By the same way, as for (3.13), the above equation takes the form:

$$\sum_{j=1}^{n} A_{ij} f_j = \sum_{j=1}^{n} B_{ij} \frac{\vartheta f_j}{\vartheta n} \tag{3.19}$$

and in matrix form:

$$\mathbf{A\ f} = \mathbf{B\ v} \tag{3.20}$$

Finally, by using either the constant elements or the linear elements, then the velocities $v = \partial f / \partial n$ are computed through the open-channel transition.

Moreover, the free surface elevations y, are further computed by the formula:

$$y = \frac{Q}{d \cdot v} \tag{3.21}$$

with d the width of the transition, and thus the free-surface profile is fully determined.

4. Determination of the Free-Surface Profile of an Expansion

As an application of the previous mentioned theory, the free-surface profile will be determined in a channel expansion, with inlet conditions of velocity $u_0 = 1.167\ m/\sec$, water depth $h_0 = 0.06\,m$, which corresponds to a Froude number $F_0 = 1.521$.

In addition, the outlet conditions of the channel expansion are: velocity $u = 0.222\ m/\sec$, water depth $h = 0.07\ m$, corresponding to a Froude number $F_0 = 0.268$. The width of the inlet channel is 0.10 *m*, the width of the outlet channel 0.45 *m*, and the length of the expansion $L = 1.83\ m$ (see: Figure 2). Furthermore, a steady flow of constant flow discharge $Q = 0.007\ m^3/\sec$ is assumed.

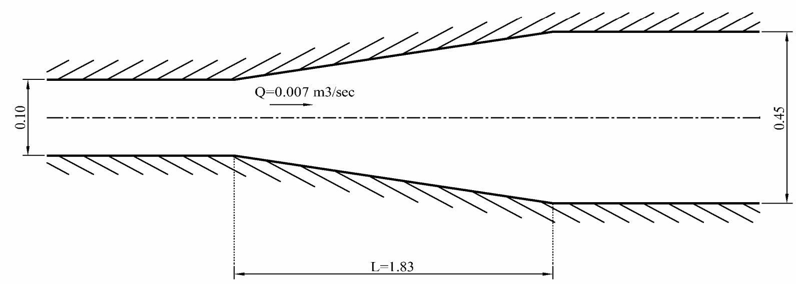

Fig. 2 Channel Expansion.

The same problem has been previously solved by S. M. Bhallamudi and M. H. Chaudhry [10] by using a uniformly distributed grid of steady flow by applying a numerical method of finite differences. Beyond the above, same problem was solved by M.Rahman and M.H.Chaudhry [16] by using an adaptive grid system. So, a comparison will be made between the results by the Singular Integral Operators Method (S.I.O.M.) and by the two different methods of finite differences, the uniformly distributed grid and the adaptive grid.

This problem has been solved by using both constant and linear elements. Hence, Figure 3 shows the distribution of water depth along the channel centerline for the channel expansion of Fig. 2. Also, Figure 3a shows the same distribution in 3-dimensional form.

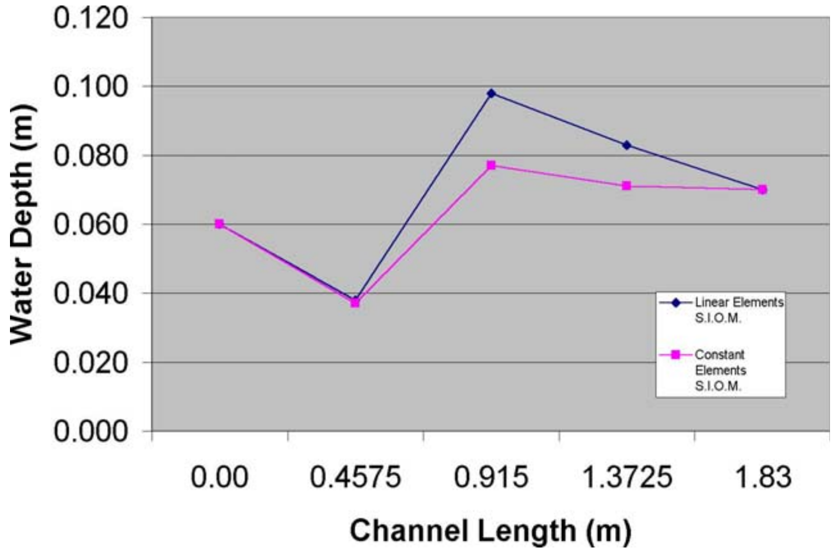

Fig. 3 Distribution of Water Depth along the Channel Centerline

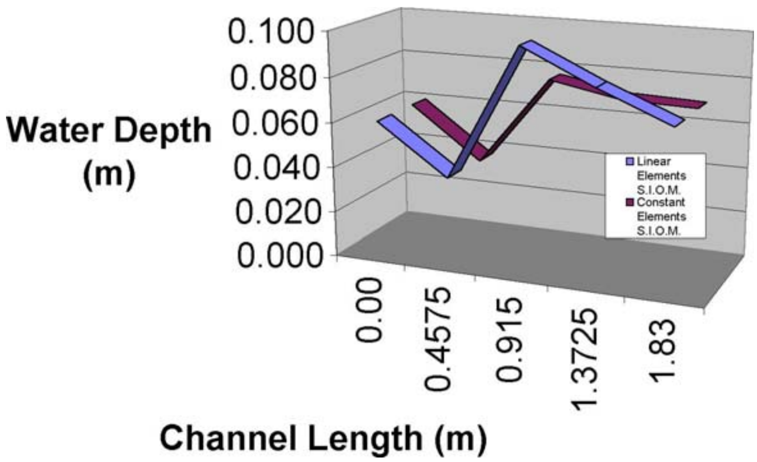

Fig. 3a 3-D Distribution of Water Depth along the Channel Centerline

As it can be seen from Figures 3 and 3a there is a small disagreement between the results of the constant and linear elements of the S.I.O.M. In addition, Figure 4 shows the distribution of water depth along the channel boundary for the channel expansion of Fig. 2. Furthermore, Figure 4a shows the same distribution in 3-dimensional form.

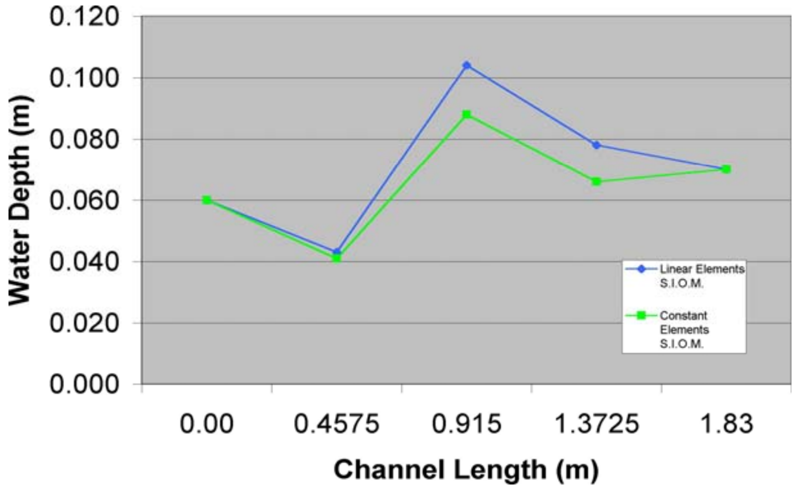

Fig. 4 Distribution of Water Depth along the Channel Boundary.

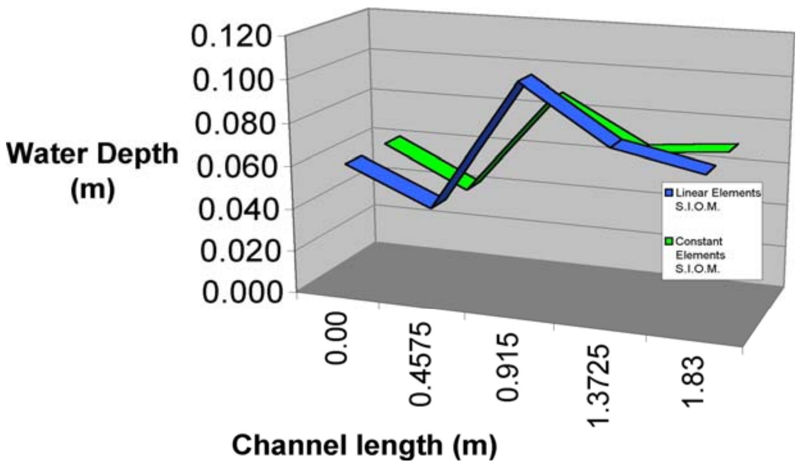

Fig. 4a 3-D Distribution of Water Depth along the Channel Boundary

Finally, as follows from Figures 3, 3a, 4 and 4a the results by using the S.I.O.M. (especially the linear elements) are in fair agreement with the corresponding results by using the two different methods of finite differences, the uniformly distributed grid [10] and the adaptive grid system [16].

5. Conclusions

The Singular Integral Operators Method (S.I.O.M.) was applied to the determination of the free-surface profile of potential flows in open-channel transitions, which are contractions and expansions. So, the study of transitions is very important in free-surface hydraulics, as these are used in many hydraulic structures, like sluice gates, spillways, steep chutes and culverts. As the flow in open-channel transitions contains both the subcritical and the supercritical flows, the analysis becomes too complicated. Consequently, in the past several numerical methods have been used in order to calculate the free-surface profile in open-channel transitions. The potential flow model which was presented in this research was found to be very effective to produce good solutions both for subcritical and supercritical flows.

The governing equation for solving potential flow problems is the equation of Laplace. By using therefore the Laplacean and choosing the proper boundary conditions, then the unsteady flow in open-channel transitions is calculated by using a numerical method based on the singular integral equations. For the numerical solution of the singular integral equations were used both constant and linear elements. An application was given to the determination of the free-surface profile in a special open-channel transition and comparing the numerical results with corresponding results by finite differences.

So, the proposed method by using the Laplacean for solving potential flow problems can be applied in many other hydraulic fields of open channel flows. In future special attention should be given to the research and application of singular integral equations methods to the solution of several important hydraulic problems of open channel flows.

References

1. *Ippen A.T. and Dawson J.H.,* 'Design of channel contraction', Symposium on high velocity flow in open channels, *Trans. ASCE,* ***116*** *(1951), 326-346.*
2. *Ippen A.T. and Harleman D.R.F., 'Studies on the validity of hydraulic analogy to supersonic flow; Parts I and II'* , USAF Technical Report No. 5985, *1950.*
3. *Ippen A.T. and Harleman D.R.F.,* 'Verification of theory for oblique standing waves', *Trans. ASCE,* ***121*** *(1956), 678-694.*
4. *Liggett J.A. and Vasudev S.U.,* 'Slope and friction effects in two-dimensional, high speed flow', *Proc. 11th Intern. Congress, IAHR, Leningrad,* ***1*** *(1965), 1-12.*
5. *Pandolfi M.,* 'Numerical experiments of free surface water motion with bores', *Proc. 4th Int. conf. Numerical Methods in Fluid Dynamics, Lecture Notes in Physics,* **No. 35**, *Springer Verlag, pp. 304-312.*
6. *Villegas F.,* 'Design of Punchina spillway', *Water Power Dam Const.,* ***28*** *(1976), 32-34.*
7. *Rajar R. and Cetina M.,* 'Two-dimensional dam-break flow in steep curved channels', *Proc XX Cong. IAHR, Inter. Assoc. Hydr. Res.,* ***11*** *(1983), 571-579.*
8. *Jimenez O.F. and Chaudhry M.H.,* 'Computation of supercritical free-surface flows', *J .Hydr. Engng ASCE,* ***114*** *(1988), 377-395.*
9. *Fennema R.J. and Chaudhry M.H.,* 'Explicit methods for 2-D transient free-surface flows', *J. Hydr. Engng ASCE,* ***116*** *(1990), 1013-1035.*
10. *Bhallamudi S.M. and Chaudhry M.H.,* 'Computation of flows in open-channel transitions', *J. Hydr. Res., IAHR,* ***30*** *(1992), 77-93.*
11. *Thompson J.F., Thames F.C. and Mastin C.M.,* 'Automatic numerical generation of body-fitted curvilinear coordinate systems for fields containing any number of arbitrary two-dimensional bodies', *J. Comput. Phys.,* ***15*** *(1974), 299-319.*
12. *Hindman R.G., Kutler P. and Anderson D.A.,* 'A two-dimensional unsteady Euler-equation solver for flow regions with arbitrary boundaries', *AIAA J.,* ***79*** *(1979).*
13. *Dwyer H.A., Kel R.J. and Sanders B.R.,* 'Adaptive grid methods for problems in fluid mechanics and heat transfer', *AIAA J.,* ***18*** *(1980).*
14. *Rai M.M. and Anderson D.A.,* 'Application of adaptive grids to fluid-flow problems with asymptotic solutions', *AIAA J.,* ***20*** *(1982), 496-502.*
15. *Chaudhry M.H., 'Open-Channel Flow',* Prentice Hall, New Jersey, *1993.*
16. *Rahman M. and Chaudhry M.H.,* 'Computation of flow in open-channel transitions', *J. Hydr. Res, IAHR,* ***35*** *(1997), 243-256.*
17. *Ladopoulos E.G.,* 'Finite – part singular integro – differential equations arising in two – dimensional aerodynamics', *Arch. Mech.,* ***41*** *(1989), 925 – 936.*
18. *Ladopoulos E.G.,* 'Non-linear singular integral representation for unsteady inviscid flowfields of 2-D airfoils', *Mech. Res. Commun.,* ***22*** *(1995), 25 – 34.*
19. *Ladopoulos E.G.,* 'Non-linear singular integral computational analysis for unsteady flow problems', *Renew. Energy,* ***6*** *(1995), 901 – 906.*
20. *Ladopoulos E.G.,* 'Non-linear singular integral representation analysis for inviscid flowfields of unsteady airfoils', *Int. J. Non-Lin. Mech.,* ***32*** *(1997), 377 – 384.*
21. *Ladopoulos E.G.,* 'Non-linear multidimensional singular integral equations in 2-dimensional fluid mechanics analysis', *Int. J. Non-Lin. Mech.,* ***35*** *(2000) , 701-708.*
22. *Ladopoulos E.G., 'Singular Integral Equations, Linear and Non-Linear Theory and its Applications in Science and Engineering',* Springer Verlag, New York, Berlin, *2000.*

23. *Ladopoulos E.G. and Zisis V.A.,* 'Existence and uniqueness for non-linear singular integral equations used in fluid mechanics', *Appl. Math.,* ***42*** *(1997), 345 – 367.*
24. *Ladopoulos E.G. and Zisis V.A.,* 'Non-linear finite-part singular integral equations arising in two-dimensional fluid mechanics', *Nonlin. Anal., Th. Meth. Appl.,* ***42*** *(2000), 277-290.*
25. *Ladopoulos E.G.,* 'On the numerical evaluation of the singular integral equations used in two and three – dimensional plasticity problems', *Mech. Res. Commun.,* ***14*** *(1987), 263 – 274.*
26. *Ladopoulos E.G.,* 'Singular integral representation of three – dimensional plasticity fracture problem', *Theor. Appl. Fract. Mech.,* ***8*** *(1987), 205 – 211.*
27. *Ladopoulos E.G.,* 'On the numerical solution of the multidimensional singular integrals and integral equations used in the theory of linear viscoelasticity', *Int. J. Math. Math. Scien.,* ***11*** *(1988), 561 – 574.*
28. *Ladopoulos E.G.,* 'Singular integral operators method for two – dimensional plasticity problems', *Comp. Struct.,* ***33*** *(1989), 859 – 865.*
29. *Ladopoulos E.G.,* 'Cubature formulas for singular integral approximations used in three – dimensional elasticity', *Rev. Roum. Sci. Tech., Méc Appl.,* ***34*** *(1989), 377 – 389.*
30. *Ladopoulos E.G.,* 'Singular integral operators method for three – dimensional elasto – plastic stress analysis', *Comp. Struct.,* ***38*** *(1991), 1 – 8.*
31. *Ladopoulos E.G.,* 'Singular integral operators method for two – dimensional elasto – plastic stress analysis', *Forsch. Ingen.,* ***57*** *(1991), 152 – 158.*
32. *Ladopoulos E.G.,* 'Singular integral operators method for anisotropic elastic stress analysis', *Comput. Struct.,* ***48*** *(1993), 965 – 973.*