

Free Surface Profile in Open-Channel Transitions by Singular Integral Operators Method

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Abstract

By using a method of singular integral equations, then the free surface profile of potential flow is calculated in open-channel transitions. In these free surface hydraulics applications the analysis of fluid motion is too complicated, as both the subcritical and supercritical flows are presented simultaneously. For the numerical evaluation of the singular integral equations are used both constant and linear elements of the Singular Integral Operators Method (S.I.O.M.). An application is finally given to the determination of the free-surface profile in a special open – channel transition and comparing the numerical results of the SIOM with corresponding results by finite differences.

Key Word and Phrases

Open Channel Transition, Open Channel Hydraulics, Singular Integral Equations, Singular Integral Operators method (S.I.O.M.), Free – surface Profile, Constant & Linear Elements, Potential Flows.

1. Introduction

The study of open-channel transitions belongs to a major field of hydraulics engineering and fluid mechanics applications. Thus, contractions and expansions of flow belong to a very important chapter of open-channel hydraulics.

Open-channel transitions are used in many hydraulic structures, such as in sluice gates, spillways, steep chutes and culverts. The fluid motion analysis in such hydraulics applications are very complicated, as both the subcritical and supercritical flows are present simultaneously.

In the past the two-dimensional St.Venant equations based on hydrostatic pressure distribution and shallow water theory have been used with success in order to describe open-channel transitions. The above equations are non-linear first-order hyperbolic partial differential equations and are evaluated only by numerical methods. Some important studies on open-channel transitions were firstly published by A.T.Ippen and J.H.Dawson [1] and A.T.Ippen and D.R.F.Harleman [2], [3].

A few years later J.A.Liggett and S.U.Vasudev [4], M.Pandolfi [5], F.Villegas [6], R.Rajar and M.Cetina [7] and O.F.Jimenez and M.H.Chaudhry [8] used several numerical methods for the computation of supercritical flows in open-channels. Some of the above computational results are in good agreement with the corresponding experimental applications.

Furthermore, R.J.Fennema and M.H.Chaudhry [9] and S.M.Bhallamudi and M.H.Chaudhry [10] used finite differences for the numerical evaluation of the two-dimensional St.Venant equations in order to simulate free-surface flows. Their computational method was used for the determination of the free-surface profile in open-channel transitions. They made efforts in order to improve the numerical solution in open-channel hydraulics and especially where the flow phenomenon occurs in different length scales, in different regions of the flow domain.

On the other hand, J.F.Thompson et al. [11] and R.G.Hindman et al. [12] in order to generate the motion of the dynamic grid system took the time derivative of the elliptic governing differential equations, of the coordinate mapping in order to solve the two-dimensional time-dependent Euler equations.

Beyond the above, H.A.Dwyer et al. [13] proposed adaptive grid methods for the solution of problems in open-channel hydraulics and heat transfer. Also, M.M.Rai and D.A.Anderson [14] studied some applications of adaptive grids to free-surface flow problems with asymptotic solutions. According to them, grid locations are directly calculated from the grid speed equation.

For their method the two-dimensional St.Venant equations describing flows in open-channel transitions are solved.

Recently, M.H.Chaudhry [15] and M.Rahman and M.H.Chaudhry [16] used MacCormack second-order accurate explicit predictor-corrector scheme in order to solve the two-dimensional depth averaged shallow water equations for the numerical simulation of the supercritical free-surface flows in open-channel transitions. For their computational method they used an adaptive grid system in order to have a resolution of the changes of the flow variables both for subcritical and supercritical flows.

Over the past years E.G.Ladopoulos [17] – [22] and E.G.Ladopoulos and V.A.Zisis [23], [24] introduced and investigated linear and non-linear singular integral equations methods for the solution of fluid mechanics and hydraulics problems. In the present research these methods will be extended to the solution of open-channel transitions flows.

Hence, the Singular Integral Operators Method (S.I.O.M.) [22], [25]-[32] is applied to the determination of the free-surface profile in open-channel transitions, by using the Laplacean equation of potential flow. For the numerical evaluation of the singular integral equations are used both constant and linear elements. An application is finally given to the determination of the free-surface profile in open-channel transitions.

2. Potential Flow Theory for Open-Channel Transitions

Let an homogeneous, incompressible and inviscid fluid, which flows through an open-channel transition. As the flow is irrotational, then for the stream function f , with $\mathbf{f} = \nabla f$, is valid: [22]

$$\nabla \times \mathbf{f} = 0 \quad (2.1)$$

Moreover, because of the conservation of mass for an incompressible fluid, then one has:

$$\nabla \cdot \mathbf{f} = 0 \quad (2.2)$$

By using (2.1) and (2.2.) one obtains the equation of Laplace which is the governing equation in the domain Ω :

$$\nabla^2 f = 0 \quad (2.3)$$

The boundary conditions corresponding to the flow for open-channel transitions are:

- a. Essential conditions of the type: $f=Q$ on the axis of symmetry of the transition and $f = Q$ on the boundary wall where Q is the flow discharge. (2.4)

- b. Natural conditions of the next type:

$$v = \frac{\partial f}{\partial n} \quad (2.5)$$

in which v denotes the velocity and n the unit normal from the free surface. (Fig. 1)

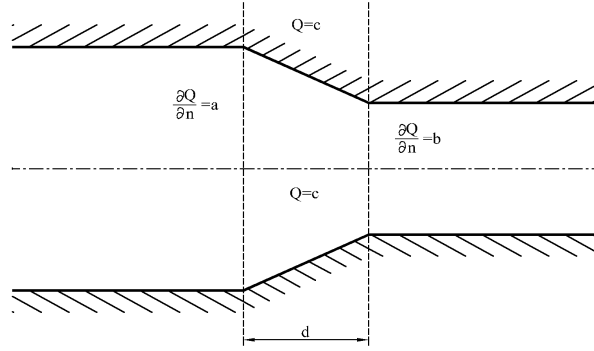


Fig. 1 Boundary Conditions for Open-Channel Transitions

Thus, with a known flow rate Q and known velocities upstream and downstream the transition under study, then the remaining velocities on the boundary of the transition and in internal points can be calculated.

Furthermore, the free surface elevations can be determined in every boundary or internal point of the transition.

3. Potential Flow Analysis by Singular Integral Equations

Consider a weighting function f^* , so that it has continuous first derivatives. Thus the function f^* produces the following weighted residual statement:

$$\int_{\Omega} (\nabla^2 f) f^* d\Omega = \int_{\Gamma_2} \left(\frac{\partial f}{\partial n} - \bar{v} \right) f^* d\Gamma - \int_{\Gamma_1} (f - \bar{f}) \frac{\partial f^*}{\partial n} d\Gamma \quad (3.1)$$

in which by $(-)$ are meant average values and Γ_1 , Γ_2 are the boundaries where the essential and the natural conditions are affected, respectively.

Furthermore, integrating by parts the left hand side of (3.1) one has:

$$-\int_{\Omega} \left(\frac{\partial f}{\partial x_k} \frac{\partial f^*}{\partial x_k} \right) d\Omega = \int_{\Gamma_2} \frac{\partial f}{\partial n} f^* d\Gamma - \int_{\Gamma_2} \bar{v} f^* d\Gamma - \int_{\Gamma_1} f \frac{\partial f^*}{\partial n} d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\partial f^*}{\partial n} d\Gamma \quad (3.2)$$

By integrating again the left hand side of (3.2) we obtain:

$$\int_{\Omega} f (\nabla^2 f^*) d\Omega = \int_{\Gamma_2} f \frac{\partial f^*}{\partial n} d\Gamma - \int_{\Gamma_2} \bar{v} f^* d\Gamma - \int_{\Gamma_1} \frac{\partial f}{\partial n} f^* d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\partial f^*}{\partial n} d\Gamma \quad (3.3)$$

In order to find a solution satisfying the Laplace equation, then the governing equation is:

$$\nabla^2 f^* + \Delta_i = 0 \quad (3.4)$$

where Δ_i is the Dirac delta function.

The solution of (3.4) is called the fundamental solution and has the property such that:

$$\int_{\Omega} f(\nabla^2 f^* + \Delta_i) d\Omega = \int_{\Omega} f \nabla^2 f^* d\Omega + f_i \quad (3.5)$$

where f_i denotes the value of the unknown function at the point "i" where a concentrated load is acting.

Hence, if (3.4) is satisfied by the fundamental solution then follows:

$$\int_{\Omega} f(\nabla^2 f^*) d\Omega = -f_i \quad (3.6)$$

By using (3.6), then eqn (3.3.) takes the form:

$$f_i + \int_{\Gamma_2} f \frac{\mathcal{G}f^*}{\mathcal{G}n} d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\mathcal{G}f^*}{\mathcal{G}n} d\Gamma = \int_{\Gamma_2} \bar{v} f^* d\Gamma + \int_{\Gamma_1} \frac{\mathcal{G}f}{\mathcal{G}n} f^* d\Gamma \quad (3.7)$$

Beyond the above, by taking the point "i" on the boundary, then the term f_i in (3.7) must be multiplied by 1/2 for a smooth boundary. On the other hand, if the boundary is not smooth at the point "i" then the number 1/2 must be replaced by a constant which can be determined from constant potential considerations.

Then (3.7) takes the form:

$$c_i f_i + \int_{\Gamma} f \frac{\mathcal{G}f^*}{\mathcal{G}n} d\Gamma = \int_{\Gamma} \frac{\mathcal{G}f}{\mathcal{G}n} f^* d\Gamma \quad (3.8)$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and has been assumed that $f = \bar{f}$ on Γ_1 and $\frac{\mathcal{G}f}{\mathcal{G}n} = v = \bar{v}$ on Γ_2 .

In addition, the constant c_i can be determined by the relation:

$$c_i = \frac{\Theta}{2\pi} \quad (3.9)$$

in which Θ denotes the internal angle of the corner in rad.

(a) Constant Elements

In order (3.8) to be numerically evaluated by using constant elements, then the above equation may be written as:

$$c_i f_i + \sum_{j=1}^n f_j \int_{\Gamma_j} \frac{\mathcal{G}f^*}{\mathcal{G}n} d\Gamma = \sum_{j=1}^n \frac{\mathcal{G}f_j}{\mathcal{G}n} \int_{\Gamma_j} f^* d\Gamma \quad (3.10)$$

Also, (3.10) may be further written as:

E.G. Ladopoulos

$$c_i f_i + \sum_{j=1}^n f_j A_{ij}^* = \sum_{j=1}^n \frac{\mathcal{G}_j}{\mathcal{G}_n} B_{ij} \quad (3.11)$$

where: $A_{ij} = A_{ij}^*$, when $i \neq j$
 $A_{ij} = A_{ij}^* + c_i$, when $i=j$ (3.12)

Thus, (3.11) takes the form:

$$\sum_{j=1}^n A_{ij} f_j = \sum_{j=1}^n B_{ij} \frac{\mathcal{G}_j}{\mathcal{G}_n} \quad (3.13)$$

or in matrix form can be written as:

$$\mathbf{A} \mathbf{f} = \mathbf{B} \mathbf{v} \quad (3.14)$$

On the other hand, by reordering the above equation so that all the unknowns are on the left hand side, then one has:

$$\mathbf{C} \mathbf{X} = \mathbf{D} \quad (3.15)$$

in which \mathbf{X} denotes the vector of unknowns f and v .

Thus, once the values of f and v on the whole boundary are known, then f can be calculated at any interior point:

$$f_i = \sum_{j=1}^n \frac{\mathcal{G}_j}{\mathcal{G}_n} B_{ij} - \sum_{j=1}^n f_j A_{ij}^* \quad (3.16)$$

(b) Linear Elements

In order (3.8) to be numerically evaluated by using linear elements, then the above equation may be written as:

$$c_i f_i + \sum_{j=1}^n \int_{\Gamma_j} f \frac{\mathcal{G}_j^*}{\mathcal{G}_n} d\Gamma = \sum_{j=1}^n \int_{\Gamma_j} \frac{\mathcal{G}_j}{\mathcal{G}_n} f^* d\Gamma \quad (3.17)$$

In this case, in contrary to eqn (3.10), the variables f_j and $\frac{\mathcal{G}_j}{\mathcal{G}_n}$ cannot be taken out of the integral as they vary linearly within the element.

Hence, by using linear elements then (3.17) can be further written as:

$$c_i f_i + \sum_{j=1}^n f_j A_{ij}^* = \sum_{j=1}^n \frac{\mathcal{G}_j}{\mathcal{G}_n} B_{ij} \quad (3.18)$$

By the same way, as for (3.13), the above equation takes the form:

$$\sum_{j=1}^n A_{ij} f_j = \sum_{j=1}^n B_{ij} \frac{\partial f_j}{\partial n} \quad (3.19)$$

and in matrix form:

$$\mathbf{A} \mathbf{f} = \mathbf{B} \mathbf{v} \quad (3.20)$$

Finally, by using either the constant elements or the linear elements, then the velocities $v = \partial f / \partial n$ are computed through the open-channel transition.

Also, the free surface elevations y , are further computed by the formula:

$$y = \frac{Q}{d \cdot v} \quad (3.21)$$

with d the width of the transition, and thus the free-surface profile is fully determined.

4. Application to the determination of the free-surface profile of an expansion

As an application of the previous mentioned theory, the free-surface profile will be determined in a channel expansion, with inlet conditions of velocity $u_0 = 1.167 \text{ m/sec}$, water depth $h_0 = 0.06 \text{ m}$, which corresponds to a Froude number $F_0 = 1.521$.

Furthermore, the outlet conditions of the channel expansion are: velocity $u = 0.222 \text{ m/sec}$, water depth $h = 0.07 \text{ m}$, corresponding to a Froude number $F_0 = 0.268$. The width of the inlet channel is 0.10 m , the width of the outlet channel 0.45 m , and the length of the expansion $L = 1.83 \text{ m}$ (see: Figure 2). Also, a steady flow of constant flow discharge $Q = 0.007 \text{ m}^3/\text{sec}$ is assumed.

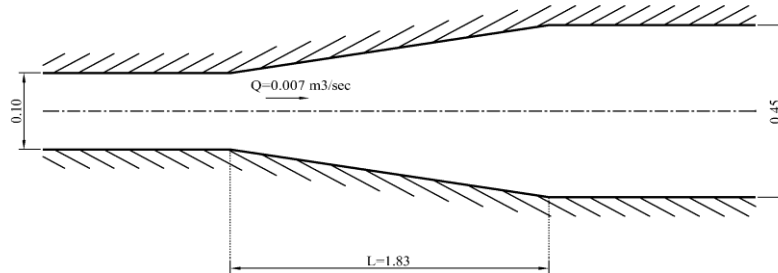


Fig. 2 Channel Expansion.

The same problem has been previously solved by S. M. Bhallamudi and M. H. Chaudhry [10] by using a uniformly distributed grid of steady flow by applying a numerical method of finite differences. Furthermore, same problem was solved by M.Rahman and M.H.Chaudhry [16] by using an adaptive grid system. Thus a comparison will be made between the results of the Singular Integral Operators Method (S.I.O.M.) and the two different methods of finite differences, the uniformly distributed grid and the adaptive grid.

This problem has been solved by using both constant and linear elements. Thus, Figure 3 shows the distribution of water depth along the channel centerline for the channel expansion of Fig. 2. Also, Figure 3a shows the same distribution in 3-dimensional form.

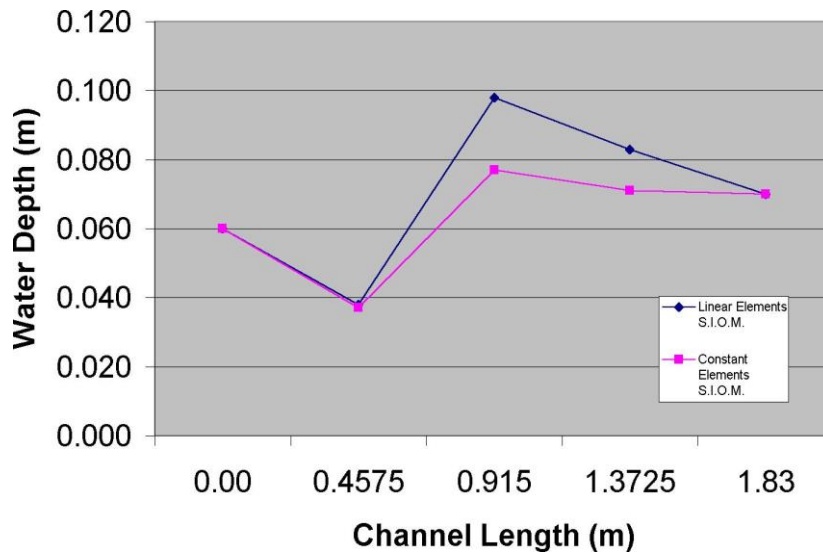


Fig. 3 Distribution of Water Depth along the Channel Centerline

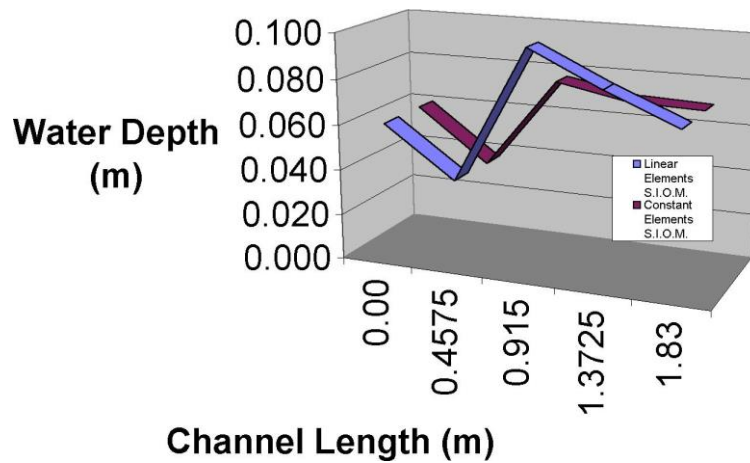


Fig. 3a 3-D Distribution of Water Depth along the Channel Centerline

As it can be seen from Figures 3 and 3a there is a small disagreement between the results of the constant and linear elements of the S.I.O.M. Furthermore, Figure 4 shows the distribution of water depth along the channel boundary for the channel expansion of Fig. 2. Also, Figure 4a shows the same distribution in 3-dimensional form.

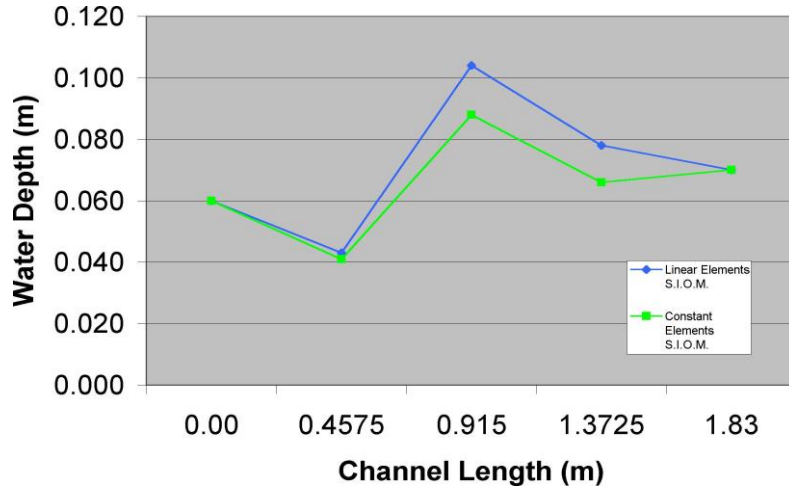


Fig. 4 Distribution of Water Depth along the Channel Boundary.

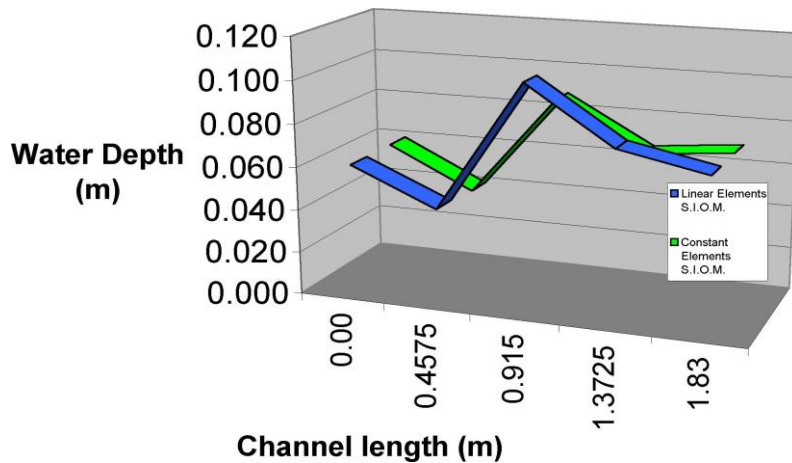


Fig. 4a 3-D Distribution of Water Depth along the Channel Boundary

Finally, as follows from Figures 3, 3a, 4 and 4a the results by using the S.I.O.M. (especially the linear elements) are in fair agreement with the corresponding results by using the two different methods of finite differences, the uniformly distributed grid [10] and the adaptive grid system [16].

5. Conclusions

The Singular Integral Operators Method (S.I.O.M.) was applied to the determination of the free-surface profile of potential flow in open-channel transitions. The study of transitions is very important in free-surface hydraulics, as these are used in many hydraulic structures, like sluice gates, spillways, steep chutes and culverts. As the flow in open-channel transitions contains both the subcritical and the supercritical flows, the analysis becomes too complicated. Thus in the past several numerical methods have been used in order to calculate the free-surface profile in open-channel transitions. The potential flow model which was presented in this research was found to be very effective to produce good solutions both for subcritical and supercritical flows.

The governing equation for solving potential flow problems is the equation of Laplace. By using therefore the Laplacean and choosing the proper boundary conditions, then the unsteady flow in open-channel transitions is calculated by using a numerical method based on the singular integral equations. For the numerical evaluation of the singular integral equations were used both constant and linear elements. An application was given to the determination of the free-surface profile in a special open-channel transition and comparing the numerical results with corresponding results by finite differences.

Thus the proposed method by using the Laplacean for solving potential flow problems can be applied in many other hydraulic fields of open channel flows. In future special attention should be given to the research and application of singular integral equations methods to the solution of several important hydraulic problems of open channel flows.

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E.G. Ladopoulos

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