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Universal Fracture Mechanics by the Generalized Sokhotski - Plemelj Formulas for Finite-Part Singular Integrals

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Abstract
The generalization of the Sokhotski – Plemelj formulas is investigated, in order to show the behavior of the limiting values of the finite – part singular integrals, defined over smooth close or open contours. Furthermore, when some corner points are included in the contours, then the Generalized Sokhotski – Plemelj formulas are even more complicated. On the other hand, when the contour is infinite, then the limiting values of the finite – part singular integrals are calculated by using a corresponding method. The proposed theory is defined as the *"Universal Fracture Mechanics"*.

Key Word and Phrases
Generalized Sokhotski – Plemelj Formulas, Universal Fracture Mechanics, Finite – Part Singular Integrals, Closed Contours, Open Contours, Infinite Contours, Hölder Continuous Function, Cauchy Singular Integrals.

1. Introduction

Finite-part singular integral equations have been widely used over the past years for the solution of several important problems of engineering mechanics, like elasticity, plasticity, thermo-elasticity, thermo-elastoplasticity, aerodynamics and fracture mechanics theory. The general property of this type of singular integral equations, consists to the generalization of the Cauchy singular integral equations, which have been systematically studied over the last decades.

In 1873, in his doctor's thesis, Yu. V. Sokhotski [1], investigated the behavior of the Cauchy singular integrals on the contours and derived some basic formulas under the Hölder condition. Some years later, Y. Plemelj [2], gave a broader derivation of the Sokhotski formulas under the same assumptions of the Hölder condition. The most important result of Plemelj is that for the first time the Cauchy singular integral was employed as a mathematical device for solving a definite boundary value problem of the theory of analytic functions. So, the formulas which show the behavior of the limiting values of the Cauchy singular integrals are known as the Sokhotski-Plemelj formulas [3]-[5].

The concept of finite-part integrals was firstly introduced by J.Hadamard [6], [7] and some basic properties were analyzed by L.Schwartz [8]. Many years later, H.R.Kutt [9] proposed some numerical formulas for the evaluation of finite-part singular integrals. He further explained the difference between a finite-part integral and a "generalized principal value integral".

A few years later M.A.Golberg [10] studied the convergence of several algorithms for solving finite-part singular integrals. His method was an extension beyond the two well-known Galerkin and collocation methods [11]. Also, A.C.Kaya and F.Erdogan [12], [13] investigated complicated problems of elasticity and fracture mechanics theory, which are reduced to the solution of finite-part singular integral equations.

Beyond the above, E.G.Ladopoulos [14]-[18] introduced several numerical methods for the evaluation of the finite-part singular integral equations of the first and the second kind and applied them to several generalized problems of elasticity, fracture mechanics and aerodynamics. Also, by the same author [19] a generalization of the Sokhotski - Plemelj formulas was given, in order to show the behavior of the limiting values of the finite-part singular integrals.

On the other hand, E.G.Ladopoulos, V.A.Zisis and D.Kravvaritis [20], [21] have used functional analysis as a tool of investigation. They have studied finite-part singular integral equations defined in general Hilbert and Lp spaces and applied them to some important crack problems.

By the current research a generalization of the Sokhotski – Plemelj formulas is introduced and investigated. These consist to the determination of the limiting values of the finite-part singular integrals defined over a smooth open or closed contour. Also, some corresponding formulas are proposed for the limiting values of the above integrals, when some corner points are included in the contour.

On the other hand, when the contour is infinite, then some other formulas are investigated for the determination of the corresponding limiting values of the finite-part singular integrals. These consist, first, when the contour is smooth and second, when some corner points are present.

By the proposed theory a widely number of crack problems can be solved. So, the proposed theory is defined as the *"Universal Fracture Mechanics"*.

2. Generalized Sokhotski – Plemelj Formulas

Definition 2.1

The following integral is a finite-part singular integral:

$$\Phi(z) = \frac{1}{2\pi i} \oint_L \frac{f(t)}{(t-z)^{\mu}} \mathrm{d}\, t, \mu \in N \tag{2.1}$$

where L is a smooth contour in the plane of the complex variable z, t is the complex coordinate of its points and $f(t)$ a Hölder continuous function of t in any plane domain D containing the interval L.

Theorem 2.1

Consider a smooth closed or open contour L and a Hölder continuous function $f(t)$ of position on the contour. Then, the finite-part singular integral (2.1) has the limiting values $\Phi^{(\mu-1)^+}(x)$ and $\Phi^{(\mu-1)^-}(x)$ at all points of the contour L not coinciding with its ends, on approaching the contour from the left or from the right along an arbitrary path, and these limiting values are given by the relations:

$$\left\{ \begin{array}{c} \Phi^{(\mu-1)^+}(x) - \Phi^{(\mu-1)^-}(x) = f^{(\mu-1)}(x) \\ \Phi^{(\mu-1)^+}(x) + \Phi^{(\mu-1)^-}(x) = \dfrac{\Gamma(\mu)}{\pi i} \oint_L \dfrac{f(t)}{(t-x)^{\mu}} \mathrm{d}\, t \end{array} \right\} \tag{2.2}$$

where $\mu \in N$ and $\Gamma(\mu)$ is the Gamma function.

Proof. Let a smooth closed contour L lies in the plane of the complex variable z. The domain within the contour L is called the interior domain and denoted as D^+, while the complementary domain to $D^+ + L$ is called the exterior domain and denoted by D^- (Fig.1). Beyond the above, in the case of an open contour we may supplement it by an arbitrary curve so that it becomes a closed one, setting on the additional curve $f(t) = 0$.

We denote, by $\Phi^{+}(x)$ the limiting value of the analytic function $\Phi(z)$ when the point z tends to the point x of the contour L from the inside domain D^{+}, and by $\Phi^{-}(x)$ from the outside domain D^{-}. For an open contour this corresponds to the limiting values from the left and from the right. Hence, for the case of a Cauchy singular integral, i.e. the integral (2.1) for $\mu = 1$, then these limiting values are given by the Sokhotski-Plemelj formulas [2], [3], [5]:

$$\left\{ \begin{aligned} \Phi^{+}(x) &= \frac{1}{2} f(x) + \frac{1}{2\pi i} \int_L \frac{f(t)}{t-x} \, \mathrm{d}\, t \\ \Phi^{-}(x) &= -\frac{1}{2} f(x) + \frac{1}{2\pi i} \int_L \frac{f(t)}{t-x} \, \mathrm{d}\, t \end{aligned} \right\} \tag{2.3}$$

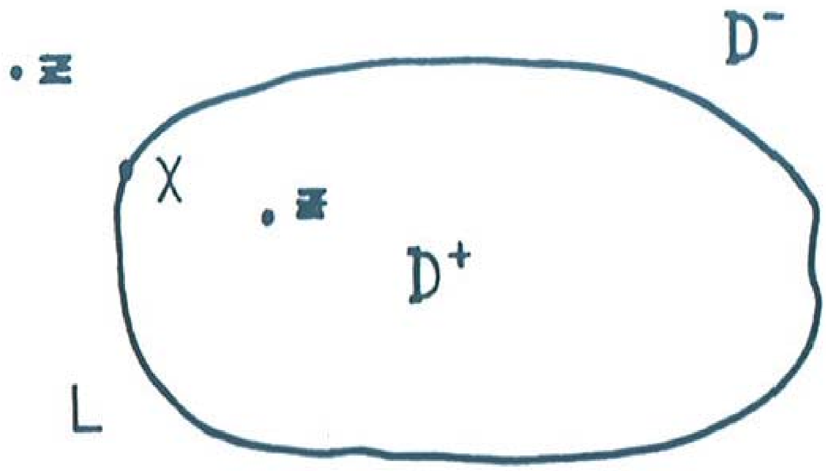

Fig. 1 A smooth closed contour L in the plane of the complex variable z.

So, in order to find the corresponding formulas for the solution of the finite-part singular integrals and consequently to generalize Cauchy's theory of functions of a complex variable and Sokhotski's-Plemelj's formulas we shall follow a new method.

The first derivative of the Cauchy singular integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{f(t)}{t-z} \, \mathrm{d}\, t \tag{2.4}$$

is valid as [5] :

$$\Phi'(z) = \frac{1}{2\pi i} \mathop{\int\!\!\!\!\!=}_L \frac{f(t)}{(t-z)^2} \, \mathrm{d}\, t \tag{2.5}$$

Furthermore, the second derivative is:

$$\Phi''(z)=\frac{1}{2\pi i}\,=\!\!\!\!\!\!\!\int_L \frac{2f(t)}{(t-z)^3}\,\mathrm{d}\,t \tag{2.6}$$

and thus, the (μ-1) derivative, $\mu \in N$, is ([5], pp. 29-31):

$$\Phi^{(\mu-1)}(z)=\frac{(\mu-1)!}{2\pi i}\,=\!\!\!\!\!\!\!\int_L \frac{f(t)}{(t-z)^{\mu}}\,\mathrm{d}\,t \tag{2.7}$$

By integrating the right-hand side of (2.5) once by parts, and assuming that the contour L is closed, one obtains:

$$\Phi'(z)=\frac{1}{2\pi i}\int_L \frac{f'(t)}{t-z}\,\mathrm{d}\,t \tag{2.8}$$

and finally, we have:

$$\Phi^{(\mu-1)}(z)=\frac{1}{2\pi i}\int_L \frac{f^{(\mu-1)}(t)}{t-z}\,\mathrm{d}\,t \tag{2.9}$$

Hence, by using (2.9), then the Sokhotski-Plemelj formulas reduce to:

$$\left\{\begin{array}{l} \Phi^{(\mu-1)^{+}}(x)=\dfrac{1}{2}f^{(\mu-1)}(x)+\dfrac{1}{2\pi i}\displaystyle\int_L \frac{f^{(\mu-1)}(t)}{t-x}\,\mathrm{d}\,t \\ \Phi^{(\mu-1)^{-}}(x)=-\dfrac{1}{2}f^{(\mu-1)}(x)+\dfrac{1}{2\pi i}\displaystyle\int_L \frac{f^{(\mu-1)}(t)}{t-x}\,\mathrm{d}\,t \end{array}\right\} \tag{2.10}$$

Beyond the above, by using (2.7) the same formulas may be written as:

$$\left\{\begin{array}{l} \Phi^{(\mu-1)^{+}}(x)=\dfrac{1}{2}f^{(\mu-1)}(x)+\dfrac{(\mu-1)!}{2\pi i}\,=\!\!\!\!\!\!\!\displaystyle\int_L \frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t \\ \Phi^{(\mu-1)^{-}}(x)=-\dfrac{1}{2}f^{(\mu-1)}(x)+\dfrac{(\mu-1)!}{2\pi i}\,=\!\!\!\!\!\!\!\displaystyle\int_L \frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t \end{array}\right\} \tag{2.11}$$

while by these formulas by subtraction and addition follow the required new formulas (2.2).

3. Generalized Sokhotski – Plemelj Formulas for Corner Points of a Contour

Equations (2.2) are used for the solution of a finite-part singular integral equation defined over a smooth closed or open contour. Beyond the above, some alterations should be made for the corner points of the contour. Hence, the following Theorem is proposed, in order to include the case where some corner points are included in the contour.

Theorem 3.1

Consider the finite-part singular integral (2.1) taken over a contour L having a finite number of corner points. Then the limiting values $\Phi^{(\mu-1)^+}(x)$ and $\Phi^{(\mu-1)^-}(x)$ of this integral exist, and formulas (2.2) are valid for the non-corner points, while for the corner points the following relations take their place :

$$\left\{\begin{array}{c} \Phi^{(\mu-1)^+}(x) - \Phi^{(\mu-1)^-}(x) = f^{(\mu-1)}(x) \\ \Phi^{(\mu-1)^+}(x) + \Phi^{(\mu-1)^-}(x) = \left(1 - \dfrac{\varphi}{2\pi}\right) f^{(\mu-1)}(x) + \dfrac{\Gamma(\mu)}{\pi i} \displaystyle\int\!\!\!\!\!\!= _L \frac{f(t)}{(t-x)^\mu}\,\mathrm{d}t \end{array}\right\} \tag{3.1}$$

where $\mu \in N$, $\Gamma(\mu)$ is the Gamma function and φ the angle between the two tangents to the contour L at the point x, measured on the left of the contour.

Proof. Let a contour L having a finite number of corner points x, while φ denotes the angle between the two tangents to the contour L at the point x, measured on the left of the contour (Fig..2).

For the case of the Cauchy singular integral (2.4), its limiting values $\Phi^+(x)$ and $\Phi^-(x)$ are given by the Sokhotski-Plemelj formulas: [5]

$$\left\{\begin{array}{l} \Phi^+(x) = \left(1 - \dfrac{\varphi}{2\pi}\right) f(x) + \dfrac{1}{2\pi i} \displaystyle\int_L \frac{f(t)}{t-x}\,\mathrm{d}t \\ \Phi^-(x) = -\dfrac{\varphi}{2\pi} f(x) + \dfrac{1}{2\pi i} \displaystyle\int_L \frac{f(t)}{t-x}\,\mathrm{d}t \end{array}\right\} \tag{3.2}$$

which can be further written as:

$$\left\{\begin{array}{c} \Phi^+(x) - \Phi^-(x) = f(x) \\ \Phi^+(x) + \Phi^-(x) = \left(1 - \dfrac{\varphi}{2\pi}\right) f(x) + \dfrac{1}{\pi i} \displaystyle\int_L \frac{f(t)}{t-x}\,\mathrm{d}t \end{array}\right\} \tag{3.3}$$

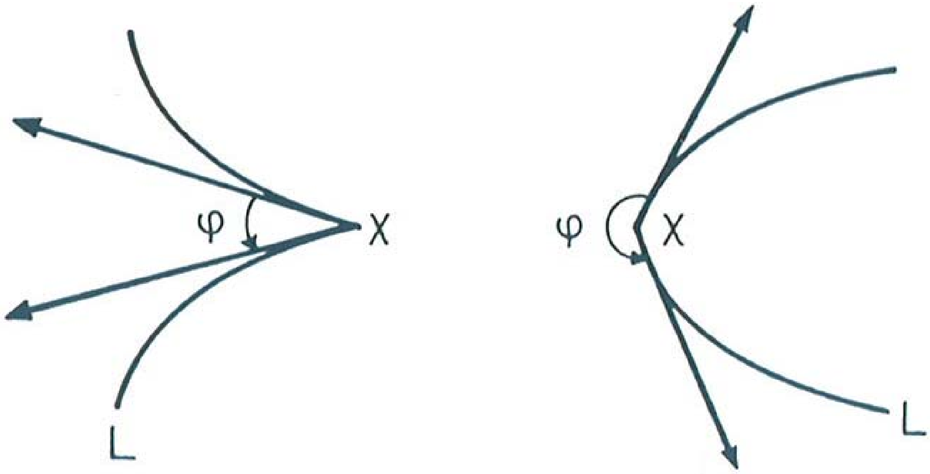

Fig. 2.2 A contour L having a finite number of corner points.

By using the (μ-1) derivative of the Cauchy singular integral (2.4) given by (2.9) and assuming that the contour L is closed, the new formulas may be written as:

$$\left\{\begin{aligned} \Phi^{(\mu-1)^+}(x) &= \left(1-\frac{\varphi}{2\pi}\right) f^{(\mu-1)}(x) + \frac{1}{2\pi i}\int_L \frac{f^{(\mu-1)}(t)}{t-x}\,\mathrm{d}t \\ \Phi^{(\mu-1)^-}(x) &= -\frac{\varphi}{2\pi} f^{(\mu-1)}(x) + \frac{1}{2\pi i}\int_L \frac{f^{(\mu-1)}(t)}{t-x}\,\mathrm{d}t \end{aligned}\right\} \tag{3.4}$$

Furthermore, by using (2.7), then the new formulas giving the limiting values of the finite-part singular integral (2.1) are written as following:

$$\left\{\begin{aligned} \Phi^{(\mu-1)^+}(x) &= \left(1-\frac{\varphi}{2\pi}\right) f^{(\mu-1)}(x) + \frac{(\mu-1)!}{2\pi i}=\!\!\!\!\!\!\int_L \frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}t \\ \Phi^{(\mu-1)^-}(x) &= -\frac{\varphi}{2\pi} f^{(\mu-1)}(x) + \frac{(\mu-1)!}{2\pi i}=\!\!\!\!\!\!\int_L \frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}t \end{aligned}\right\} \tag{3.5}$$

while from these formulas by subtraction and addition follow the required new formulas (3.1)

4. Generalized Sokhotski – Plemelj Formulas for Infinite Contours

Formulas (2.2) and (3.1) are the limiting values of the finite-part singular integral (2.1) defined over a finite contour. Furthermore, when the contour is infinite, some changes should be made to the above mentioned formulas, and therefore the following theorems will be proved.

Definition 4.1

Consider the finite-part singular integral :

$$\Phi(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{(t-z)^{\mu}}\,\mathrm{d}\,t,\mu\in N \tag{4.1}$$

where $f(t)$ is a complex function of the variable t, which obeys the Hölder condition for all finite t and tends to a definite limit $f(\infty)$ as $t\to\pm\infty$.

Theorem .4.2

Consider the finite-part singular integral (4.1) taken over an infinite contour. Then, the limiting values $\Phi^{(\mu-1)^{+}}(x)$ and $\Phi^{(\mu-1)^{-}}(x)$ of this integral exist and are equal to:

$$\left\{\begin{aligned}\Phi^{(\mu-1)^{+}}(x)-\Phi^{(\mu-1)^{-}}(x)&=f^{(\mu-1)}(x)\\ \Phi^{(\mu-1)^{+}}(x)+\Phi^{(\mu-1)^{-}}(x)&=\frac{\Gamma(\mu)}{\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t\end{aligned}\right\} \tag{4.2}$$

where μ=1,3,5,… and $\Gamma(\mu)$ is the Gamma function.

Proof. Let us suppose that Imz = 0, then the point $z = x$ is situated on the integration line. By the finite-part singular integral (4.1) we understand the principal value defined by the relation:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{(t-z)^{\mu}}\,\mathrm{d}\,t=\frac{1}{2\pi i}\lim_{\substack{\varepsilon\to 0\\ M\to\infty}}\left[\int_{-M}^{x-\varepsilon}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t+\int_{x+\varepsilon}^{M}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t\right] \tag{4.3}$$

In order to prove the existence of the limit (4.3) as $\varepsilon\to 0$, we shall return to the case of a finite contour by carrying out the transformation :

$$\begin{aligned}&\frac{1}{2\pi i}\lim_{\substack{\varepsilon\to 0\\ M\to\infty}}\left[\int_{-M}^{x-\varepsilon}\frac{f(t)}{(t-x)^{\mu}}\mathrm{d}t+\int_{x+\varepsilon}^{M}\frac{f(t)}{(t-x)^{\mu}}\mathrm{d}t\right]=\\ &\frac{1}{2\pi i}\lim_{\substack{M\to\infty\\ K\to 0}}\left[\int_{-M}^{x-K}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t+\int_{x+K}^{M}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t\right]\\ &+\frac{1}{2\pi i}\lim_{\substack{\varepsilon\to 0\\ K\to 0}}\left[\int_{x-K}^{x-\varepsilon}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t+\int_{x+\varepsilon}^{x+K}\frac{f(t)}{(t-x)^{\mu}}\,\mathrm{d}\,t\right]\end{aligned} \tag{4.4}$$

From (4.4) it can be seen that the first two integrals are independent of ε and the limit of the sum of the last two integrals exists. Hence, the principal value of the finite-part singular integral (4.1) exists, for μ = 1,3,5,… [4].

For the present case of an infinite contour, Sokhotski-Plemelj formulas can be written as following: [5]

$$\left\{ \begin{array}{l} \Phi^{+}(x) = \dfrac{1}{2} f(x) + \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{t-x} \,\mathrm{d}\, t \\ \Phi^{-}(x) = -\dfrac{1}{2} f(x) + \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{t-x} \,\mathrm{d}\, t \end{array} \right\} \tag{4.5}$$

where $\Phi^{+}(x)$ and $\Phi^{-}(x)$ are the limits of $\Phi(z)$ as z tends to x from the upper and lower semi-plane, respectively.

The $(\mu\text{-}1)$ derivative of the Cauchy singular integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} \,\mathrm{d}\, t \tag{4.6}$$

is equal to:

$$\Phi^{(\mu-1)}(z) = \frac{(\mu-1)!}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{(t-z)^{\mu}} \,\mathrm{d}\, t, \mu = 1,3,5,... \tag{4.7}$$

Also, by integrating the right-hand side of (4.7), $(\mu\text{-}1)$ times by parts, with $f(\infty)=0$, then we have:

$$\Phi^{(\mu-1)}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f^{(\mu-1)}(t)}{t-z} \,\mathrm{d}\, t \tag{4.8}$$

Thus, by using (4.8) one has:

$$\left\{ \begin{array}{l} \Phi^{(\mu-1)^{+}}(x) = \dfrac{1}{2} f^{(\mu-1)}(x) + \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f^{(\mu-1)}(t)}{t-x} \,\mathrm{d}\, t \\ \Phi^{(\mu-1)^{-}}(x) = -\dfrac{1}{2} f^{(\mu-1)}(x) + \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f^{(\mu-1)}(t)}{t-x} \,\mathrm{d}\, t \end{array} \right\} \tag{4.9}$$

On the other hand, by using (4.7) we obtain:

$$\begin{cases} \Phi^{(\mu-1)^+}(x) = \dfrac{1}{2} f^{(\mu-1)}(x) + \dfrac{(\mu-1)!}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{(t-x)^{\mu}} \mathrm{d}t \\ \Phi^{(\mu-1)^-}(x) = -\dfrac{1}{2} f^{(\mu-1)}(x) + \dfrac{(\mu-1)!}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{(t-x)^{\mu}} \mathrm{d}t \end{cases} \tag{4.10}$$

Finally, from (4.10) by subtraction and addition follow the required new formulas (4.2). The proof of the next theorem is obvious by using Theorems 3.1 and 4.1.

Theorem .4.2

Consider the finite-part singular integral (4.1) taken over an infinite contour. Then the limiting values $\Phi^{(\mu-1)^+}(x)$ and $\Phi^{(\mu-1)^-}(x)$ of this integral exist, and for the non-corner points formulas (4.2) are valid, while for the corner points the following formulas take their place :

$$\begin{cases} \Phi^{(\mu-1)^+}(x) - \Phi^{(\mu-1)^-}(x) = f^{(\mu-1)}(x) \\ \Phi^{(\mu-1)^+}(x) + \Phi^{(\mu-1)^-}(x) = \left(1 - \dfrac{\varphi}{2\pi}\right) f^{(\mu-1)}(x) + \dfrac{\Gamma(\mu)}{\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{(t-x)^{\mu}} \mathrm{d}t \end{cases} \tag{4.11}$$

where $\mu = 1,3,5,\ldots$, $\Gamma(\mu)$ is the Gamma function and φ the angle between the two tangents to the contour L at the point x.

4. Conclusions

As a conclusion to the previous outlined analysis, our proposal for investigation in the objective of fracture mechanics theory is the following: The generalized Sokhotski – Plemelj formulas were proposed in order to calculate the limiting values of the finite-part singular integrals defined over a smooth closed or open contour. Beyond the above, some other generalized formulas were proposed, when some corner points are included in the contour. Consequently, in this case the angle between the two tangents to the contour at some point is also included in the new formulas which are giving the limiting values of the finite-part singular integrals.

Furthermore, when the contour is infinite either with corner points or not, then by following a same method the limiting of the finite-part singular integrals have been calculated. So, Sokhotski-Plemelj formulas were generalized for all cases of a contour with corner points or not and for infinite contours as well, in order to determine the limiting values of the finite-part singular integrals.

By the new theory described in the current research a widely number of crack problems can be solved. So, the proposed theory is defined as the *"Universal Fracture Mechanics"*.

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