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Three-dimensional Singular Integral Operators Method for Composite Solids Stress Analysis

E.G. Ladopoulos
Interpaper Research Organization
8, Dimaki Str.
Athens, GR - 106 72, Greece
eladopoulos@interpaper.org

Abstract
A new method is investigated by applying the *Singular Integral Operators Method (S.I.O.M.)*.for the solution of the anisotropic elastic stress analysis, which defines the basic feature of the mechanical behavior of composite solids. So, the above 'innovative" method depends on the existence and explicit definition of the fundamental solution to the governing partial differential equations. Consequently, after the determination of the fundamental solution, a real variable boundary integral formula is generated. In addition the construction of the solution for the composite solids problem is presented as is the derivation of the expression for the surface tractions necessary to maintain the fundamental solution in a bounded region. Several parameters, like intensity factors, incorporate stress kernels, geometry and crack size, may be evaluated by the elastic stress analysis of cracked structures. Hence, by using the S.I.O.M., then the anisotropic elastic stress of composite solids will be determined.

Key Word and Phrases
Singular Integral Operators Method (S.I.O.M.), Composite Solids, Anisotropic Materials, Anisotropic Elastic Stress Analysis, Somigliana Identity.

1. Introduction

Many homogeneous solids like paper, textolite, plywood, delta wood, pine wood, metal systems, reinforced concrete and laminates are often anisotropic (or at least orthotropic from point to point) and find wide application in modern technology. Furthermore, all the composite materials have anisotropic behavior. Plates with artificially-made differences between the flexural rigidities for different directions may also be considered as anisotropic solids. Plates strengthened by stiffening ribs and corrugated plates may be regarded as anisotropic materials. If a crack or hole is presented and is not associated with a plane or elastic symmetry, then the problem must be treated as one of general anisotropy.

During the past years, special effort has gone into studying stress fields in anisotropic solids, because numerous engineering materials under normal or loading conditions show different mechanical properties along certain preferred directions. Among them we shall mention the following authors, following classical lines: S.G.Lekhnitskii [1]-[5], G.N.Savin [6]-[10], M.O.Basheleishvili [11], [12], J.R.Willis [13], H.T.Rathod [14], S.Krenk [15], G.C.Sih and H.Liebowitz [16], G.C.Sih and M.K.Kassir [17] and G.C.Sih et al. [18].

On the contrary, by using an integral transform method obtained by I.N.Sneddon [19], [20] the governing partial differential equation of anisotropic elasticity is solved, while G.E.Tupholme [21], D.D.Ang and M.L.Williams [22], O.L.Bowie and C.E.Freese [23] have studied some fracture mechanics problems of orthotropic media.

Singular integral equation methods for solving two- and three-dimensional problems of cracks and holes in anisotropic bodies have been introduced by F.J.Rizzo and D.J.Shippy [24], S.M.Vogel and F.J.Rizzo [25], M.D.Snyder and T.A.Cruse [26], [27], E.G.Ladopoulos [28], [29], K.S.Parihar and S.Sowdamini [30], T.Mura [31], C.Ouyang and Mei-Zi Lu [32], R.P.Gilbert et al. [33], R.P.Gilbert and M.Schneider [34], R.P.Gilbert and R.Magnanini [35] and U.Zastrow [36] - [38].

Structural and engineer members are often loaded in ways that produce a three-dimensional stress state. Generally, although the loading may appear outwardly simple, a complex state of stress can exist inside the medium, particularly in the neighbourhood of the cracks and the holes, where the stresses can undergo sharp variations.

In the neighborhood of cracks of irregular shapes or holes, the stress state is often triaxial in nature and the problem of predicting the surface of crack propagation is very difficult. Consequently, by the present research, all the methods which were applied to boundary value problems of homogeneous and piecewise homogeneous isotropic bodies, may be extended to anisotropic elastic bodies. This requires, a further elaboration of the theory of the various kinds of fundamental solutions for systems of elliptic equations with discontinuous coefficients and also an extension of the theory of many-dimensional singular integral equations to systems of equations having these fundamental solutions as their kernels. These problems, which present new difficulties, are of considerable interest and should be made the subject of future investigations.

Thus, some parameters, such as intensity factors, incorporate stress levels, geometry and crack size, may be evaluated from the stress analysis of cracked structures. These parameters deduced from elastic fracture mechanics analysis can correlate crack growth rate in specimens and structures which see nominally - elastic loading. Elastic fracture mechanics analysis idealizes the physical crack or inclusion problem in three different ways : Firstly the crack plane is generally taken to be flat, secondly the crack is assumed to be sufficiently large that the local material microstructure can be modeled as a continuum and, thirdly inelastic crack tip effects such as plasticity are restricted to a sufficiently small volume that they can be neglected. The validity of these assumptions must be verified on the basis of the actual behavior of the cracked structure.

In some problems such as planar structures containing through cracks and holes, the stress and strain fields ahead of the cracks and the hole, may be approximated as two dimensions. In order to form the three-dimensional anisotropic problem, the Somigliana identity has been used which provides a formal representation for the displacement of an interior point in the body in terms of integrals involving boundary data.

By this theory it is possible to provide integral relations between boundary tractions and displacements, which in a well-posed boundary-value problem are not concurrently assigned over the entire surface.

Hence, the Singular Integral Operators Method (S.I.O.M.) which was used very successfully for the solution of several engineering problems of fluid mechanics, hydraulics, aerodynamics, solid mechanics, potential flows and structural analysis, is further extended by the present investigation for the solution of composite elastic stress analysis problems.

2. General Theory of Anisotropic Elastic Stress Analysis

Let us express the stresses $(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy})$ in terms of strains $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy})$ through a set of constants C_{ij}, which are called the moduli of elasticity:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \times \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} \tag{2.1}$$

On the other hand, in order to express the strains in terms of stresses, let us use another set of 36 constants a_{ij} $(i, j = 1,2,\ldots,6)$, known as the coefficients of deformation:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \times \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \tag{2.2}$$

Consequently, by considering the case where the material is "transversely isotropic", which means, that it possesses an axis of elastic symmetry such that the material is isotropic in the planes normal to this axis, then the following formula is valid between the stresses and strains:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(a_{11} - a_{12}) \end{bmatrix} \times \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \tag{2.3}$$

where z is the direction of the elastic symmetry.

The coefficients of deformation in (2.3) are expressed as: [1]

$$a_{11} = \frac{1}{E_1}, \quad a_{12} = -\frac{v_1}{E_1},$$

$$a_{33} = \frac{1}{E_2}, \quad a_{13} = -\frac{v_2}{E_2}, \tag{2.4}$$

$$a_{44} = \frac{1}{G_2}, \quad 2(a_{11} - a_{12}) = \frac{2(1 + v_1)}{E_1} = \frac{1}{G_1}$$

where E_1, G_1 and v_1 are the Young's modulus, shear modulus, and Poisson's ratio, respectively, in the plane of isotropy and E_2, G_2 and v_2 are the same quantities in the transverse direction.

Furthermore, in order to express the stress components in terms of strains for a "transversely isotropic" material we obtain the following formula:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \times \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} \tag{2.5}$$

in which the elastic moduli C_{ij} may be expressed as following: [1],[2]

$$C_{11} = 2G_1 \left(1 - v_2^2 \frac{E_1}{E_2} \right) \Big/ \left(1 - v_1 - 2v_2^2 \frac{E_1}{E_2} \right)$$

$$C_{12} = 2G_1 \left(v_1 + v_2^2 \frac{E_1}{E_2} \right) \Big/ \left(1 - v_1 - 2v_2^2 \frac{E_1}{E_2} \right)$$

$$C_{13} = E_1 v_2 \Big/ \left(1 - v_1 - 2v_2^2 \frac{E_1}{E_2} \right)$$

$$\tag{2.6}$$

$$C_{33} = E_2 (1 - v_1) \Big/ \left(1 - v_1 - 2v_2^2 \frac{E_1}{E_2} \right)$$

$$C_{44} = G_2$$

$$\frac{1}{2}(C_{11} - C_{12}) = G_1$$

Beyond the above, in the case of isotropic material, $v_1 = v_2$, $E_1 = E_2$ and $G_1 = G_2$ and so the elastic moduli C_{ij} may be related to the Lamé coefficients λ and μ as:

$$C_{11} = C_{12} = \lambda + 2\mu$$

$$C_{12} = C_{13} = \lambda \tag{2.7}$$

$$C_{44} = \mu$$

The strain components in (2.5) are expressed by the formulas:

$$\varepsilon_x = \frac{\vartheta u_x}{\vartheta x}\,,\ \varepsilon_y = \frac{\vartheta u_y}{\vartheta y}\,,\ \varepsilon_z = \frac{\vartheta u_z}{\vartheta z}$$

$$\gamma_{yz} = \frac{\vartheta u_z}{\vartheta y} + \frac{\vartheta u_y}{\vartheta z}$$

$$\gamma_{zx} = \frac{\vartheta u_x}{\vartheta z} + \frac{\vartheta u_z}{\vartheta x} \qquad (2.8)$$

$$\gamma_{xy} = \frac{\vartheta u_x}{\vartheta y} + \frac{\vartheta u_y}{\vartheta x}$$

where u_x, u_y and u_z are the components of displacements in cartesian coordinates.

3. Fundamental Solutions of Composite Stress Analysis

Let us consider a body in three-dimensional space, which has a bounding surface L. According to Betti's reciprocal theorem and by considering absence of body forces, we obtain: [24],[29]

$$\int_L (u_i T_{ij} - t_i U_{ij})\,\mathrm{d}\,R + \int_\Gamma (U_i T_{ij} - t_i U_{ij})\,\mathrm{d}\,R = 0 \qquad (3.1)$$

where dR is an element of surface area at R, which is a point on L. Also Γ is the boundary of the finite or infinite domain of space in coordinates x_1, x_2, x_3, in which exist the anisotropic elastic body. This boundary Γ is a connected closed Lyapounov surface.

In (3.1) u_i and t_i are the displacement and traction components, $U_{ij}(x,y)$ the displacement at point x in response to a concentrated unit body force acting in the j coordinate direction at point y, and T_{ij} the suitable boundary tractions.

Furthermore, Betti's theorem (eq. (3.1)) results in Somigliana's identity [24]:

$$u_i(y) = \frac{1}{a}\int_L \left[t_j(x)U_{ij}(x,y) - u_j T_{ij}(x,y)\right]\cdot \mathrm{d}\,R \tag{3.2}$$

where the point dependence is explicitly indicated and a is the magnitude of the force components.

The following two limiting formulas have to exist:

$$\lim_{\varepsilon\to 0}\int_\Gamma u_i T_{ij}\ \mathrm{d}\,R = a u_j(y) \tag{3.3}$$

$$\lim_{\varepsilon\to 0}\int_\Gamma t_i U_{ij}\ \mathrm{d}\,R = 0 \tag{3.4}$$

where ε is the radius of a sphere centre y, with the boundary Γ, and $u_j(y)$ the displacement at the origin corresponding to u_i and t_i on L.

In order to derive the formula of the fundamental solution, we adopt the method of decomposition into plane waves used in [39]. Hence, consider the function g, which is an arbitrary distribution and vanishes outside a finite sphere.

The next formula is a solution of the differential equation:

$$\Delta_y u(y) = g(y) \tag{3.5}$$

$$u(y) = \int_A g(x)\left(-\frac{1}{4\pi|x-y|}\right)\mathrm{d}\,x \tag{3.6}$$

where Δy denotes the Laplacean with respect to y_i.

The following identity is easily seen to be:

$$\int_{|\zeta|=1} \left|(x_i - y_i)\zeta_i\right| \mathrm{d}\,R = 2\pi|x-y| \tag{3.7}$$

From (3.5) and (3.7) we obtain the result:

$$\Delta_y|x-y| = \frac{2}{|x-y|} \tag{3.7a}$$

Consequently, from (3.5), (3.6) and (3.7a) one has:

$$g(y) = \frac{1}{16\pi^2}\Delta_y^2 \int_A \int_{|\zeta|=1} g(x)\left|(x_i - y_i)\zeta_i\right| \mathrm{d}\,R\,\mathrm{d}\,x \tag{3.8}$$

Beyond the above, consider the function $h(\zeta,p)$ which is given by the formula:

$$h(\zeta, p) = \int_{(x\cdot\zeta)=p} g(x)\,\mathrm{d}\,R \tag{3.9}$$

Also, the following formula is valid:

$$\int_A \int_{|\zeta|=1} g(x)\left|(x_i - y_i)\zeta_i\right| \mathrm{d}\,R\,\mathrm{d}\,x = \int_{|\zeta|=1} \mathrm{d}\,R \int_{-\infty}^{\infty} |p|\,\mathrm{d}\,p \int_{(x-y)\cdot\zeta=p} g(x)\,\mathrm{d}\,x$$

$$= \int_{|\zeta|=1} \mathrm{d}\,R \int_{-\infty}^{\infty} |p| h(\zeta, p + y\cdot\zeta)\,\mathrm{d}\,p \tag{3.10}$$

and:

$$\Delta_y \int_{-\infty}^{\infty} |p| h(\zeta, p + y\cdot\zeta)\,\mathrm{d}\,p$$

$$= \Delta_y \left[\int_{(y\cdot\zeta)}^{\infty} (p - y\cdot\zeta) h(\zeta, p)\right] \mathrm{d}\,p - \int_{-\infty}^{(y\cdot\zeta)} (p - y\cdot\zeta) h(\zeta, p)\,\mathrm{d}\,p = 2h(\zeta, y\cdot\zeta)$$

(3.11)

From (3.8), (3.10) and (3.11) one has:

$$g(y) = -\frac{1}{8\pi^2}\Delta_y \int_{|\zeta|=1} h(\zeta, y\cdot\zeta)\,\mathrm{d}\,R \tag{3.12}$$

By considering the case where:

$$g(y) = \delta(y) \tag{3.13}$$

then one obtains:

$$h(\zeta, y\cdot\zeta) = \delta(y\cdot\zeta) \tag{3.14}$$

From (3.12), (3.13) and (3.14) we have the expression for the three-dimensional delta function:

$$\delta(x-y) = -\frac{1}{8\pi^2}\Delta_y \int_{|\zeta|=1} \delta((x-y)\cdot\zeta)\,\mathrm{d}\,R \tag{3.15}$$

So, from (3.15) we derive the fundamental solution for the displacements:

$$U_{ij}(x,y) = \frac{1}{8\pi^2}\Delta_y \int_{\zeta=1} W_{ij}(x,y,\zeta)\,\mathrm{d}\,R \tag{3.16}$$

where the function W_{ij} is given by:

$$W_{ij}(x,y,\zeta) = \begin{cases} W_{ij}(x,y,\zeta), & (x-y)\cdot\zeta > 0 \\ 0, & (x-y)\cdot\zeta \le 0 \end{cases} \tag{3.17}$$

According to the Cauchy-Kowalewski theorem one has:

$$W_{ij} = P_{ij}(\zeta)(x_k - y_k)\zeta_k \tag{3.18}$$

Hence, from (3.16), (3.17) and (3.18) we obtain:

$$U_{ij}(x,y) = \frac{1}{8\pi^2}\Delta_y \int_{\substack{|\zeta|=1 \\ (x-y)\cdot\zeta>0}} P_{ij}(\zeta)\cos\varphi\,\mathrm{d}\,R \tag{3.19}$$

By using (3.7), then (3.19) takes the simpler form:

$$U_{ij}(x,y) = \frac{1}{4\pi^2|x-y|} \int_{\substack{|\zeta|=1 \\ (x-y)\cdot\zeta>0}} P_{ij}(\zeta)\cos\varphi\,\mathrm{d}\,R \tag{3.20}$$

where φ is the angle between the vectors x-y and ζ.

From (3.20) we derive a simpler form, if the part of the integration over the unit hemispherical shell of (3.20) involving the azimuthal angle, is carried out:

$$U_{ij}(x,y) = \frac{1}{8\pi^2 |x-y|} \int_{\substack{|\zeta|=1 \\ (x-y)\cdot\zeta>0}} P_{ij}(\zeta)\,\mathrm{d}\,s \tag{3.21}$$

where $\mathrm{d}s$ is an element of arc length.

Consequently, (3.21) gives the solution for the general case of three-dimensional elasticity.

Beyond the above, $P_{ij}(\zeta)$ in (3.21) is given by:

$$P_{ij}(\zeta) = \frac{1/2\,\varepsilon_{imn}\varepsilon_{jrs}Q_{mr}(\zeta)Q_{ns}(\zeta)}{\det Q} \tag{3.22}$$

and:

$$Q_{ik}(\zeta) = C_{ijkl}\zeta_j\zeta_l$$

where the constants C_{ijkl} are the elasticities, Q_{ij} is the characteristic matrix and the quantities ε_{imn} and $\det Q$ are the alternating symbol and determinant of Q_{ij}, respectively. On the other hand, the suitable boundary tractions T_{ij} are given by the formula:

$$T_{im}(x,y) = C_{ijkl}U_{km}(x,y)_{,l}\,n_j \tag{3.22a}$$

in which n_i are the components of the unit outward at the point x on L. Moreover, let us take a new point $\mathbf{x}^1$ relative to the point x. Then for the vectors $\mathbf{x}$, $\mathbf{x}^1$ we have:

$$\mathbf{x}^1 = \mathbf{x} + \delta\mathbf{x} \tag{3.23}$$

By the same way, the new point ζ_1 relative to the point ζ is valid as:

$$\boldsymbol{\zeta}^1 = \boldsymbol{\zeta} + \delta\boldsymbol{\zeta} \tag{3.24}$$

Consequently, from (3.22a) we obtain in an analogous way, the displacement tensor:

$$U_{ij}(x^1,y) = \frac{1}{8\pi^2 |x^1-y|} \oint_{|\zeta^1|=1} P_{ij}(\zeta^1)\,\mathrm{d}\,s \tag{3.25}$$

So, from (3.23) and (3.24), eq. (3.25) takes the form:

$$U_{ij}(x^1, y) = \frac{1}{8\pi^2 |x + \delta x - y|} \oint_{|\zeta^1|=1} P_{ij}(\zeta + \delta\zeta)\,\mathrm{d}\,s \tag{3.26}$$

Beyond the above, we introduce the expressions:

$$\lambda_i = \frac{x_i - y_i}{|x - y|} \tag{3.27}$$

and:

$$\lambda_i^1 = \frac{x_i^1 - y_i}{|x^1 - y|} = \frac{x_i + \delta x_i - y_i}{|x + \delta x - y|} \tag{3.28}$$

So, it is easy to show that:

$$\delta\zeta_i = \frac{-\lambda_k^1 \zeta_k}{1 + \lambda_k \lambda_k^1} (\lambda_i^1 + \lambda_i) \tag{3.29}$$

The insertion of (3.27), (3.28) and (3.29) into (3.26) results the displacements:

$$U_{ij,k} = -\frac{(x_k - y_k)}{8\pi^2 |x - y|^3} \oint_{|\zeta|=1} P_{ij}(\zeta)\,\mathrm{d}\,s - \frac{1}{8\pi^2 |x - y|^3} \int_{|\zeta|=1} \zeta_k \tag{3.30}$$

$$\times \frac{\lfloor (x_q - y_q) R_{jiq} + (x_r - y_r) R_{jir} + (x_s - y_s) R_{jis} + (x_t - y_t) R_{jit} \rfloor \mathrm{d}\,s}{\det Q} + \frac{1}{8\pi^2 |x - y|^3} \int_{|\zeta|=1} P_{ij} \zeta_k$$

$$\times \frac{\lfloor (x_1 - y_1) W_1 + (x_m - y_m) W_m + (x_n - y_n) W_n + (x_p - y_p) W_p + (x_r - y_r) W_r + (x_s - y_s) W_s \rfloor \mathrm{d}\,s}{\det Q}$$

Thus, (3.30) gives the solution for the general case of three-dimensional elasticity, while the boundary tractions P_{ij} are given by (3.22).

4. Three-dimensional Singular Integral Operators Method

In order to evaluate numerically the three-dimensional singular integral of (3.30), let us follow the next method.

The following integral is a three-dimensional singular integral defined on a three-dimensional finite region V, containing the third-order pole (x^1,y^1,z^1), whose boundary is a closed Lyapounov surface S: [40] - [42]

$$\Phi(x^1,y^1,z^1)=\int_V \frac{g(x^1,y^1,z^1,\theta,\varphi)}{r^3}\,u(x,y,z)\,\mathrm{d}V$$

$$=\int_V \frac{g(x^1,y^1,z^1,\theta,\varphi)}{\left[(x-x^1)^2+(y-y^1)^2+(z-z^1)^2\right]^{3/2}}\,u(x,y,z)\,\mathrm{d}V \tag{4.1}$$

Thus, we introduce the following system of spherical coordinates:

$$x=x^1+r\sin\theta\cos\varphi$$

$$y=y^1+r\sin\theta\sin\varphi,\ (0\le\theta\le\pi)$$

$$z=z^1+r\cos\theta,\quad (0\le\varphi\le 2\pi) \tag{4.2}$$

$$r^2=(x-x^1)^2+(y-y^1)^2+(z-z^1)^2$$

Then, from eq. (4.2) we obtain:

$$\Phi=\lim_{\varepsilon\to 0}\int_0^{\pi}\int_0^{2\pi}\int_{\varepsilon}^{R(\theta,\varphi)}\sin\theta\times f(\theta,\varphi)\,\frac{u(r,\theta,\varphi)}{r}\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r \tag{4.3}$$

By integrating (8.4.3) with respect to θ and φ and using the trapezoidal rule with abscissas G, D, then we obtain the formula:

$$\Phi=\frac{2\pi^2}{GD}\sum_{i=1}^{G}\sum_{j=1}^{D}\sin(\theta_i)\varphi(\theta_i,\varphi_i) \tag{4.4}$$

in which:

$$\theta_i = \frac{\pi}{G}(i-1)$$

(4.5)

$$\varphi_j = \frac{2\pi(j-1)}{D}$$

and:

$$\varphi(\theta,\varphi) = g(\theta,\varphi) \int_{O}^{R(\theta,\varphi)} \frac{u(r,\theta,\varphi)}{r} \mathrm{d}r \tag{4.6}$$

For the numerical evaluation of the integral in (8.4.6) we use the following numerical integration rule:

$$\int_{0}^{R(\theta,\varphi)} \frac{u(r,\theta,\varphi)}{r} \mathrm{d}r \cong \sum_{k=1}^{L} A_k u[R(\theta,\varphi)\rho_k, \theta, \varphi] + u(0,\theta,\varphi) \ln[R(\theta,\varphi)] \tag{4.7}$$

where ρ_k are the abscissas and A_k the weights for the integration interval [0,1].

5. Conclusions

The Singular Integral Operators Method (S.I.O.M.) has been investigated for the determination of the anisotropic elastic stress components of composite solids. Consequently, by using the the anisotropic theory, then the mechanical behavior of the composite solids can be explained. Such composite materials have an increasing application in engineering, like the aerospace industry and they have also several possible fracture modes, such as fiber fracturing, crack bridging, matrix crazing and fiber-matrix debounding.

In addition as it is easily seen for problems in which the nominal stress field is development upon the anisotropy of the solid it would be expected that the anisotropic stress intensity factor may be much different from the isotropic result. Moreover, the proposed method depends on the existence and explicit definition of a fundamental solution to the governing partial differential equations.

The new formulas of three-dimensional elasticity in anisotropic bodies can be further reduced to the solution of elasticity problems in orthotropic solids. So, the current method was devoted to a basic description of a very specific numerical scheme, the vigorous foundation and comparison of some regularization algorithms and their application to the numerical solution of singular integral equations. Consequently, particular attention has been concentrated on the construction and verification of regularization algorithms effectible in elasticity.

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