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Multidimensional Singular Integral Equations for Non-linear Plasticity Problems

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Abstract
A large class of plasticity problems can be reduced to the solution of a system of multidimensional singular integral equations. Consequently, it is of interest to evaluate numerically these systems of integral equations of the respective boundary value problem, instead of the problem itself. These numerical techniques discretize the domain of the problem under consideration into a number of elements or cells. The governing equations of the problem are then approximated over the region by functions which fully or partially satisfy the boundary conditions. By the current research the following non-linear plasticity problems are investigated: Two-dimensional plasticity problems, three-dimensional plasticity problems and three-dimensional thermoelastoplastic stress analysis.

Key Word and Phrases
Two-dimensional Plasticity, Three-dimensional Plasticity, Three-dimensional Thermoelastoplastic Stress Analysis, Multidimensional Singular Integral Equations, Non-linear Plasticity.

1. Introduction

The theory of multidimensional singular integral equations, with a wide field of applications in mathematical physics and engineering mechanics, such as thermoelastoplasticity, viscoelasticity and fracture mechanics theory, has developed comparatively slowly over a rather long period. Recently however, interest has increased sharply.

In addition a large class of plasticity problems can be reduced to the solution of multidimensional singular integral equations. So, should be evaluated numerically these singular integral equations of the respective boundary value problem, instead of the problem itself. Such numerical evaluation methods discretize the domain of the problem under consideration into a number of elements or cells. Then, the governing equations of the problem are approximated over the region by functions which fully or partially satisfy the boundary conditions.

Over the past years, some papers have been published on the application of the method of formulation of plasticity problems, by using systems of singular integral equations. J.L.Swedlow and T.A.Cruse [1], S.Mukherjee [2], A.Mendelson [3], H.D.Bui [4], and J.C.F.Telles and C.A.Brebbia [5] have studied some plasticity problems by using the Boundary Integral Equation Method (B.I.E.M.), while E.G.Ladopoulos [6] - [10] has introduced and investigated the Singular Integral Operators Method (S.I.O.M.) for solving some basic plasticity problems.

On the other hand, many scientists have studied plasticity problems following theoretical or numerical classical lines, like finite-elements, etc. Among them we shall mention the following authors : O.C.Zienkiewicz et al. [11], G.C.Nayak and O.C.Zienkiewicz [12], W.F.Chen [13], Y.Yamada et al. [14], P.V.Marcal and I.P.King [15], R.Hill [16], O.Hoffman and G.Sachs [17], P.G.Hodge and G.N.White [18], W.T.Koiter [19], D.C.Drucker and W.Prager [20], W.Prager [21], D.N.Allen and R.V.Southwell [22], W.Johnson and P.B.Mellor [23], W.D. Liam Finn [24], Z.Mroz [25], J.N.Goodier and P.G.Hodge [26], J.H.Argyris [27], G.G.Pope [28], R.von Mises [29], J.Casey and P.M.Naghdi [30] - [32], P.M.Naghdi and J.A.Trapp [33], P.V.Lade [34], P.J.Yoder and W.D.Iwan [35], J.H.Prevost [36], and P.K.Banerjee and A.S.Stipho [37].

The aim of the current research is to include an extensive generalization of the theory of plastic flow, to study its mathematical character, to articulate a reciprocal theorem for quasi-linear behaviour and to assemble the foregoing into an extended form of Somigliana's identity, giving the displacement rate in terms of the traction, where the displacement rate on the boundary involves plastic strain rate.

Complex stress plasticity problems are better formulated by using singular integral equation methods, rather than domain techniques. A simple recursive relation is presented relating the stresses to the initial elastic solution and the plastic strains.

By foundation is meant the proof of the convergence of approximate solutions to the exact, the examination of the stability of solutions, an examination of their rate of convergence and the determination of using a practical method of error estimation.

By the present study are investigated the following plasticity problems: Two-dimensional plastic stress analysis of isotropic solids, three-dimensional plasticity of isotropic solids and three-dimensional thermoelastoplastic stress analysis of isotropic solids.

2 Two-dimensional Plastic Stress Analysis of Isotropic Solids

Consider the expression between the total strain rate $\dot{\varepsilon}_{ij}$ and the displacement rate $\dot{u}_i$:

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) \tag{2.1}$$

Beyond the above, the total strain rate is represented by the formula:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p} \tag{2.2}$$

where $\dot{\varepsilon}_{ij}^{e}$ and $\dot{\varepsilon}_{ij}^{p}$ are the elastic and plastic components of the strain rate tensor.

By applying Hooke's law to the elastic part of the strain rates, then the stress rates $\dot{\sigma}_{ij}$ reduce to the following form [1], [5], [8]:

$$\dot{\sigma}_{ij} = 2G\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) + \frac{2Gv}{1-2v}\dot{u}_{k,k}\delta_{ij} - 2G\dot{\varepsilon}_{ij}^{p} \tag{2.3}$$

where v is Poisson's ratio and G the shear modulus.

From (2.1), (2.2) and (2.3) the stress rate tensor is assumed to be represented by:

$$\dot{\sigma}_{ij} = 2G\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^{p}\right) + \frac{2Gv}{1-2v}\left(\dot{\varepsilon}_{ll} - \dot{\varepsilon}_{kk}^{p}\right)\delta_{ij} \tag{2.4}$$

By inserting the plastic stress components $\dot{\sigma}_{ij}^{p}$ in (2.4), then one has:

$$\dot{\sigma}_{ij} = 2G\dot{\varepsilon}_{i,j} + \frac{2Gv}{1-2v}\dot{\varepsilon}_{ll}\delta_{ij} - \dot{\sigma}_{ij}^{p} \tag{2.5}$$

in which:

$$\dot{\sigma}_{ij}^{p} = 2G\dot{\varepsilon}_{ij}^{p} + \frac{2Gv}{1-2v}\dot{\varepsilon}_{kk}^{p}\delta_{ij} \tag{2.6}$$

as is easily seen from (2.4) and (2.5).

Consider further the equilibrium conditions in the interior of the body:

$$\dot{\sigma}_{ij,i} + \dot{g}_j = 0 \tag{2.7}$$

in which $\dot{g}_j$ are the body force rates per unit volume.

On the boundary of the body, the equilibrium conditions are:

$$\dot{p}_i - \dot{\sigma}_{ij}\dot{n}_j = 0 \tag{2.8}$$

where $\dot{p}_i$ is the traction rate tensor per unit volume and n the outward normal to the boundary of the body.

Navier's equation for the two-dimensional problem is given by the following formula, after combining eqs. (2.1), (2.2), (2.4), (2.7) and (2.8):

$$\dot{u}_{j,ll} + \frac{1}{1-2v}\dot{u}_{l,lj} = -\frac{\dot{g}_j}{G} + 2\left(\dot{\varepsilon}_{ij,i}^{p} + \frac{v}{1-2v}\dot{\varepsilon}_{kk,j}^{p}\right), \quad (i,j,k,l = 1,2) \tag{2.9}$$

A solution of Navier's equation (2.9) for the two-dimensional problem has the following form:

$$\dot{u}_i(y) = \int_{\Gamma}\left[U_{ij}(y,X)\dot{\tau}_j(X) - T_{ij}(y,X)\dot{u}_j(X)\right]\mathrm{d}\,\Gamma_x$$

$$+\int_{B} U_{ij}(y,x)\dot{g}_j(x)\,\mathrm{d}\,B_x + \int_{B} 2GU_{ij,k}\dot{\varepsilon}_{jk}^{p}(x)\,\mathrm{d}\,B_x \tag{2.10}$$

$$(i,j,k = 1,2)$$

where B is the area of cross-section of the body, Γ the boundary of B, τ_i the traction vector, X and Y are surface points and x and y are interior points.

In addition consider the formula of symmetry of the plastic strain rate tensor:

$$2GU_{ij,k}\dot{\varepsilon}^{p}_{jk} = 2G\left[\frac{1}{2}\left(U_{ij,k} + U_{ik,j}\right)\right]\dot{\varepsilon}^{p}_{jk} \tag{2.11}$$

Recalling Kelvin's solution, then (2.11) takes the form:

$$\Sigma_{jki} = 2GU_{ij,k} = -\frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)(\delta_{ij}r_{,k} + \delta_{ki}r_{,j}) - \delta_{jk}r_{,i} + 2r_{,i}r_{,j}r_{,k}\right] \tag{2.12}$$

where:

$$r = \left|x - y\right| \tag{2.13}$$

By using (2.12), then (2.10) becomes:

$$\dot{u}_i(y) = \int_{\Gamma}\left[U_{ij}(y,X)\dot{\tau}_j(X) - T_{ij}(y,X)\dot{u}_j(X)\right]\mathrm{d}\,\Gamma_x$$

$$+\int_{B}U_{ij}(y,x)\dot{g}_j(x)\,\mathrm{d}\,B_x + \int_{B}\Sigma_{jki}(y,x)\dot{\varepsilon}^{p}_{jk}(x)\,\mathrm{d}\,B_x \tag{2.14}$$

$$(i, j, k = 1, 2)$$

The singular fundamental solutions corresponding to unit loads may be written as: [7]

$$U_{ij} = -\frac{1}{8\pi(1-\nu)G}\left[(3-4\nu)\ln(r)\delta_{ij} - r_{,i}r_{,j}\right] \tag{2.15}$$

$$T_{ij} = -\frac{1}{4\pi(1-\nu)r}\left[\left[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}\right]\frac{\vartheta r}{\vartheta n} + (1-2\nu)(r_{,j}n_i - r_{,i}n_j)\right] \tag{2.16}$$

where **n** is the outward normal to the boundary of the body.

By differentiating (2.14), then one obtains the stress rate tensor:

$$\dot{\sigma}_{ij}(y) = \int_{\Gamma}\left[-\Sigma^{1}_{ijk}(y,X)\dot{\tau}_k(X) - T_{ijk}(y,X)\dot{u}_k(X)\right]\mathrm{d}\,\Gamma_x$$

$$+\int_B \left(-\Sigma^1_{ijk}(y,x)\right)\dot{g}_k(x)\,\mathrm{d}\,B_x - 2G\dot{\varepsilon}^p_{ij}(y) + \int_B \Sigma^1_{ijkl}(y,x)\dot{\varepsilon}^p_{kl}(x)\,\mathrm{d}\,B \qquad (2.17)$$

where:

$$\Sigma^1_{ijk} = -\frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)(\delta_{ij}r_{,k} + \delta_{ki}r_{,j} + \delta_{jk}r_{,i}) + 2r_{,i}r_{,j}r_{,k}\right] \qquad (2.18)$$

which is Kelvin's solution:

$$\Sigma^1_{ijkl} = \Sigma_{ijkl} + \frac{G}{2\pi(1-2\nu)r^2}\left[4\nu r_{,i}r_{,j}\delta_{kl} - 2\nu\delta_{ij}\delta_{kl}\right] \qquad (2.19)$$

where:

$$\Sigma_{ijkl} = \frac{G}{2\pi(1-\nu)r^2}\left[2(1-2\nu)(\delta_{ij}r_{,k}r_{,l} + \delta_{kl}r_{,i}r_{,j}) + 2\nu(\delta_{li}r_{,j}r_{,k} + \delta_{jk}r_{,l}r_{,i} + \right.$$

$$(2.20)$$

$$\left. + \delta_{ik}r_{,l}r_{,j} + \delta_{jl}r_{,j}r_{,k}) - 8r_{,i}r_{,j}r_{,k}r_{,l} + (1-2\nu)(\delta_{ik}\delta_{lj} + \delta_{jk}\delta_{li}) - (1-4\nu)\delta_{ij}\delta_{kl}\right]$$

3. Three-dimensional Plastic Stress Analysis of Isotropic Solids

The total strain rate:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} \qquad (3.1)$$

consists of the sum of $\dot{\varepsilon}^e_{ij}$ and $\dot{\varepsilon}^p_{ij}$ that represent, respectively, the elastic and plastic components of the strain rate. The total strain rate and the displacement rate $\dot{u}_i$ are related as:

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) \qquad (3.2)$$

By application of the Hooke’s law for the elastic part of the strain rate tensor, the following expression for the stress rates is obtained:

$$\dot{\sigma}_{ij} = G\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) + \frac{2G\nu}{1-2\nu}\dot{u}_{k,k}\delta_{ij} - 2G\dot{\varepsilon}^p_{ij} \qquad (3.3)$$

where G is the shear modulus and v the Poisson's ratio.

Making use of eqs (3.1) and (3.2), then (3.3) takes the form:

$$\dot{\sigma}_{ij} = 2G\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{p}_{ij}\right) + \frac{2Gv}{1-2v}\left(\dot{\varepsilon}_{ll} - \dot{\varepsilon}^{p}_{kk}\right)\delta_{ij} \tag{3.4}$$

In terms of the plastic stress components $\dot{\sigma}^{p}_{ij}$, the above result becomes:

$$\dot{\sigma}_{ij} = 2G\dot{\varepsilon}_{i,j} + \frac{2Gv}{1-2v}\dot{\varepsilon}_{ll}\delta_{ij} - \dot{\sigma}^{p}_{ij} \tag{3.5}$$

where the $\dot{\sigma}^{p}_{ij}$ is given by:

$$\dot{\sigma}^{p}_{ij} = 2G\dot{\varepsilon}^{p}_{ij} + \frac{2Gv}{1-2v}\dot{\varepsilon}^{p}_{kk}\delta_{ij} \tag{3.6}$$

The conditions of equilibrium are: [1], [5]

$$\dot{\sigma}_{ij,i} + \dot{a}_{j} = 0 \tag{3.7}$$

in which $\dot{a}_j$ stand for the body force rates per unit volume. On the boundary of the body, the equilibrium conditions are: [5]

$$\dot{p}_{i} - \dot{\sigma}_{ij} n_{j} = 0 \tag{3.8}$$

where n_j are components of the outward unit normal vector applied to the boundary of the body and $\dot{p}_i$ are the traction rates per unit area.

From eqs. (3.1), (3.2), (3.4), (3.7) and (3.8) the Navier's equations in three-dimensions are found:

$$\dot{u}_{j,ll} + \frac{1}{1-2v}\dot{u}_{l,lj} = -\frac{\dot{a}_j}{G} + 2\left(\dot{\varepsilon}^{p}_{ij,i} + \frac{v}{1-2v}\dot{\varepsilon}^{p}_{kk,j}\right) \tag{3.9}$$

a solution of which is: [4]

$$\dot{u}_i(y) = \int_S \left[U_{ij}(y,X)\dot{r}_j(X) - T_{ij}(y,X)\dot{u}_j(X) \right] \mathrm{d}S_x$$

$$+ \int_V U_{ij}(y,x)\dot{a}_j(x)\,\mathrm{d}V_x + \int_V \Sigma_{jki}(y,x)\dot{\varepsilon}^p_{jk}(x)\,\mathrm{d}V_x \tag{3.10}$$

where S is the surface, V the volume, τ_i the traction vector, X and Y are surface points and x and y are interior points. By differentiating (3.10) at a load point and by using (3.3), the following formula for the stress rates is obtained:

$$\dot{\sigma}_{ij}(y) = \int_S \left[-\Sigma_{ijk}(y,X)\dot{\tau}_k(X) - T_{ijk}(y,X)\dot{u}_k(X) \right] \mathrm{d}S_x$$

$$+ \int_V \left(-\Sigma_{ijk}(y,x) \right) \cdot \dot{a}_k(x)\,\mathrm{d}V_x - \frac{G(8-10v)}{15(1-v)}\dot{\varepsilon}^p_{ij}(y) + \int_V \Sigma_{ijkl}(y,x)\dot{\varepsilon}_{kl}(x)\,\mathrm{d}V_x \tag{3.11}$$

In (3.11), Σ_{ijkl} are given by

$$\Sigma_{ijkl} = \frac{G}{4\pi(1-v)r^3} \left[3(1-2v)(\delta_{ij} r_{,k} r_{,l} + \delta_{kl} r_{,i} r_{,j}) + \right.$$

$$+ 3v(\delta_{li} r_{,j} r_{,k} + \delta_{jk} r_{,l} r_{,i} + \delta_{ik} r_{,l} r_{,j} + \delta_{jl} r_{,i} r_{,k})$$

$$\left. -15 r_{,i} r_{,j} r_{,k} r_{,l} + (1-2v)(\delta_{ik}\delta_{lj} + \delta_{jk}\delta_{li}) - (1-4v)\delta_{ij}\delta_{kl} \right] \tag{3.12}$$

where:

$$T_{ijk} = \Sigma_{ijk} n_l \tag{3.13}$$

$$r = |x - y| \tag{3.14}$$

Fundamental solutions

In accordance with the method in [41] for the elliptic systems of linear partial differential equations, the delta function for scalars can be introduced:

$$\delta(r) = -\frac{1}{8\pi^2}\Delta_y \int_{|\zeta|=1} \delta(\mathbf{r}\cdot\boldsymbol{\zeta})\,\mathrm{d}\,\omega_\zeta \tag{3.15}$$

with Δy being the Laplacean with respect to y_i. With the aid of (3.15), a fundamental solution is found such that:

$$U_{ij}(x,y) = \frac{1}{8\pi^2}\Delta_y \int_{|\zeta|=1} V_{ij}(x,y,\zeta)\,\mathrm{d}\,\omega_\zeta \tag{3.16}$$

where $V_{ij}(x,y,\zeta)$ is given by:

$$V_{ij}(x,y,\zeta) = \begin{cases} u_{ij}(x,y,\zeta) = A_{ij}(\zeta)(x_k - y_k)\zeta_k\,, & (x-y)\cdot\zeta > 0 \\ 0, & (x-y)\cdot\zeta \le 0 \end{cases} \tag{3.17}$$

in which:

$$A_{ij}(\zeta) = \frac{\frac{1}{2}\varepsilon_{imn}\varepsilon_{jrs}B_{mr}(\zeta)Q_{ns}(\zeta)}{\det B} \tag{3.18}$$

and:

$$B_{ik}(\zeta) = C_{ijkl}\zeta_j\zeta_l \tag{3.19}$$

From eqs. (3.16) and (3.17), a fundamental solution of the form:

$$U_{ij}(x,y) = \frac{1}{8\pi^2}\Delta_y \int_{\substack{|\zeta|=1 \\ r\cdot\zeta=0}} A_{ij}(\zeta)(x_k - y_k)\zeta_k\,\mathrm{d}\,\omega_\zeta \tag{3.20}$$

is obtained. Here, the Laplacean Δ_y is given by:

$$\Delta_y \cdot r = \frac{2}{r} \tag{3.21}$$

Consequently, eqs. (3.20) and (3.21) lead to:

$$U_{ij}(x,y)=\frac{1}{8\pi^2 r}\oint\limits_{|\zeta|=1} A_{ij}(\zeta)\,\mathrm{d}\,s \tag{3.22}$$

where $\mathrm{d}s$ is an element of arc length.

For the special case for an isotropic solid, c_{ijkl} is given by:

$$c_{ijkl}=\frac{2Gv}{(1-2v)}\delta_{ij}\delta_{kl}+G\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right) \tag{3.23}$$

In addition, eqs. (3.18), (3.19) and (3.23) can be used to yield:

$$A_{ij}=\frac{1}{G}\left[\delta_{ij}-\frac{1}{2(1-v)}\zeta_i\zeta_j\right]\mathrm{d}\,s \tag{3.24}$$

Consider further the direction cosines of r to be denoted by β_{ij}. These result:

$$\zeta_i=\beta_{ij}\lambda_j \tag{3.25}$$

where λ_j is the new axes, so that the plane $\lambda_3 = 0$ is perpendicular to the vector **r**. By the use of (3.24), then (3.22) becomes:

$$U_{ij}(x,y)=\frac{1}{8\pi^2 Gr}\oint\limits_{|\zeta|=1}\left[\delta_{ij}-\frac{1}{2(1-v)}\zeta_i\zeta_j\right]\mathrm{d}\,s \tag{3.26}$$

Further reduction as a result of (3.25) renders:

$$U_{ij}(x,y)=\frac{1}{8\pi^2 Gr}\int_0^{2\pi}\left[\delta_{ij}-\frac{1}{2(1-v)}\beta_{ik}\beta_{jl}n_k n_l\right]\mathrm{d}\,\varphi \tag{3.27}$$

where:

$$n_1=\cos\varphi,\;\; n_2=\sin\varphi \;\; and \;\; n_3=0$$

So (3.27) can be integrated to give:

$$U_{ij} = \frac{1}{8\pi G r}\left[\left(2 - \frac{1}{2(1-v)}\right)\delta_{ij} + \frac{1}{2(1-v)}\beta_{i3}\beta_{j3}\right] \tag{3.28}$$

The three-dimensional fundamental solution finally takes the form:

$$U_{ij} = \frac{1}{16\pi(1-v)Gr}\left[(3-4v)\delta_{ij} + r_{,i}r_{,j}\right] \tag{3.29}$$

where:

$$\delta_{ij} = \beta_{ki}\beta_{kj} \tag{3.30}$$

The corresponding boundary tractions of (3.27) are:

$$T_{im} = c_{ijkl}U_{km,l}n_j \tag{3.31}$$

By means of (3.23), then (3.31) becomes:

$$T_{ij} = \frac{2Gv}{(1-2v)}U_{ki,k}n_i + G(U_{ij,k} + U_{kj,i})n_k \tag{3.32}$$

Inserting (3.29) into (3.32) yields the relation:

$$T_{ij} = -\frac{(1-2v)}{8\pi(1-v)r^2}\left[\left(\delta_{ij} + \frac{3}{(1-2v)}\beta_{i3}\beta_{j3}\right)\beta_{k3}n_k + \beta_{i3}n_j - \beta_{j3}n_i\right] \tag{3.33}$$

This is the expression for the boundary tractions in three dimensions.

4. Three-dimensional Thermoelastoplastic Stress Analysis of Isotropic Solids

Consider the total strain rate $\dot{\varepsilon}_{ij}$, which is the sum of the elastic $\dot{\varepsilon}_{ij}^{e}$, the plastic $\dot{\varepsilon}_{ij}^{p}$ and the thermal $\dot{\varepsilon}_{ij}^{T}$ strains: [10]

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p} + \dot{\varepsilon}_{ij}^{T}, \; i, j = 1,2,3 \tag{4.1}$$

where:

$$\dot{\varepsilon}_{ij}^{T} = a\dot{T}\delta_{ij} \tag{4.2}$$

and where a is the coefficient of linear thermal expansion, T the temperature and δ_{ij} Kronecker's delta.

By assuming the plastic strains $\dot{\varepsilon}_{ij}^{p}$ to be deviatoric one has:

$$\dot{\varepsilon}_{ii}^{p} = \dot{\varepsilon}_{11}^{p} + \dot{\varepsilon}_{22}^{p} + \dot{\varepsilon}_{33}^{p} = 0 \tag{4.3}$$

Moreover, the relation between the total strains $\dot{\varepsilon}_{ij}$ and displacements $\dot{u}_i$ is valid as:

$$\dot{\varepsilon}_{ij} = \left(\dot{u}_{i,j} + \dot{u}_{j,i}\right)/2 \tag{4.4}$$

By combining eqs. (4.1) - (4.4), Hooke's law and the equations of equilibrium, we obtain Navier's equation for the three-dimensional thermoelastoplastic problem:

$$\dot{u}_{i,jj} + \frac{1}{1-2v}\dot{u}_{k,ki} = -\frac{\dot{B}_i}{G} + 2\dot{\varepsilon}_{ij,j}^{p} + \frac{2(1+v)}{1-2v}a\dot{T}_{,i} \tag{4.5}$$

where v denotes Poisson's ratio, G the shear modulus and B_i the prescribed body force per unit volume.

A solution of the Navier's equation (4.5) is given by the Somigliana identity: [2],[4]

$$\dot{u}_i(x) = \int_S \left[U_{ij}(x,Y)\dot{\tau}_j(Y) - T_{ij}(x,Y)\dot{u}_j(X)\right]\mathrm{d}S_Y$$

$$+\int_V U_{ij}(x,y)\dot{B}_j(y)\,\mathrm{d}V_y + \int_V \Sigma_{jki}(x,y)\left[\dot{\varepsilon}_{jk}^{p}(y) + \delta_{jk}a\dot{T}(y)\right]\mathrm{d}V_y \tag{4.6}$$

in which V denotes the volume of the body, S its surface, X and Y are surface points, x and y interior points and $\dot{\tau}_i$ the traction vector:

$$\dot{\tau}_i = \dot{\sigma}_{ij}m_j \tag{4.7}$$

with $\dot{\sigma}_{ij}$ the stress rate tensor and m_j the outward unit normal to the surface S.

In eq. (4.6) $U_{ij}(x,y)$ denotes the Kelvin-Somigliana tensor:

$$U_{ij}(x,y)=\frac{1}{16\pi G(1-v)r}\left\{(3-4v)\delta_{ij}+\frac{(x_i-y_i)(x_j-y_j)}{r^2}\right\} \tag{4.8}$$

$\Sigma_{jki}(x,y)$ is the stress tensor corresponding to $U_{ij}(x,y)$:

$$\Sigma_{jki}(x,y)=-\frac{1}{8\pi(1-v)r^2}\left[\begin{array}{l}(1-2v)\left(\delta_{jk}\dfrac{\vartheta r}{\vartheta y_i}+\delta_{ji}\dfrac{\vartheta r}{\vartheta y_k}-\delta_{ki}\dfrac{\vartheta r}{\vartheta y_j}\right)+\\ +3\dfrac{\vartheta r}{\vartheta y_i}\dfrac{\vartheta r}{\vartheta y_k}\dfrac{\vartheta r}{\vartheta y_i}\end{array}\right] \tag{4.9}$$

and T_{ij} the traction vector corresponding to Σ_{jki}:

$$T_{ij}=\Sigma_{jki}m_k \tag{4.10}$$

with r the distance between the points x, y:

$$r=|x-y| \tag{4.11}$$

Furthermore, the stress rate tensor $\dot{\sigma}_{ij}$ is valid as:

$$\dot{\sigma}_{ij}=G(\dot{u}_{i,j}+\dot{u}_{j,i})+\frac{2Gv}{1-2v}\dot{u}_{k,k}\delta_{ij}-2\left(G\dot{\varepsilon}_{ij}^{p}+G\left(\frac{1+v}{1-2v}\right)a\dot{T}\delta_{ij}\right) \tag{4.12}$$
$$i,j,k=1,2,3$$

By differentiating Somigliana's identity (4.6) at a load point and using (4.12) we obtain:

$$\dot{\sigma}_{ij}(x)=-\int_S\left[\Sigma_{ijk}(x,Y)\dot{\tau}_k(Y)+T_{ijk}(x,Y)\dot{u}_k(Y)\right]\mathrm{d}S_Y$$

$$-\int_V\Sigma_{ijk}(x,y)\dot{B}_k(y)\,\mathrm{d}V_y+2G\dot{\varepsilon}_{ij}^{p}(x)+3Ka\dot{T}(x)\delta_{ij}$$

$$+\int_V\Sigma_{ijkn}(x,y)\left[\dot{\varepsilon}_{kn}^{p}(y)+\delta_{kn}a\dot{T}(y)\right]\mathrm{d}V_y,\quad i,j,k,n=1,2,3 \tag{4.13}$$

where K denotes the bulk modulus and Σ_{ijkn} is valid as:

$$\Sigma_{ijkn} = \frac{G}{4\pi(1-v)r^3}\Big[3(1-2v)(\delta_{ij} r_{,k} r_{,n} + \delta_{kn} r_{,i} r_{,j}) +$$

$$3v(\delta_{ni} r_{,j} r_{,k} + \delta_{jk} r_{,n} r_{,i} + \delta_{ik} r_{,n} r_{,j} + \delta_{jn} r_{,i} r_{,k}) - 15 r_{,i} r_{,j} r_{,k} r_{,n} + \qquad (4.14)$$

$$(1-2v)(\delta_{ik}\delta_{nj} + \delta_{jk}\delta_{ni}) - (1-4v)\delta_{ij}\delta_{kn} \Big]$$
$$i, j, k, n = 1,2,3$$

In eq. (4.13) T_{ij} denotes the traction vector corresponding to Σ_{ijkn}:

$$T_{ijk} = \Sigma_{ijkn} m_n \qquad (4.15)$$

where m_n is the outward unit normal to the surface S.

Finally, the stress tensor (4.13) gives the complete three-dimensional thermoelastoplastic stress analysis for every isotropic body.

5. Conclusions

From the previous described analysis, it is clear that plasticity and non-linearity provide great numerical difficulties. So the stress field for the two – and three – dimensional plasticity has been reduced to a system of multidimensional singular integral equations.

In addition an extensive report of the three – dimensional thermoelastoplastic stress analysis was presented by using Navier's equation and Somigliana identities.

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