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Non-linear Singular Integral Equations for Unsteady Inviscid Flowfields of 2-D Airfoils

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Abstract
An “innovative” two-dimensional aerodynamics representation analysis is introduced for the investigation of inviscid flowfields of unsteady airfoils. The above problem of the unsteady flow of a two-dimensional NACA airfoil is therefore reduced to the solution of a non-linear multidimensional singular integral equation, when the form of the source and vortex strength distribution is dependent on the history of the above distribution on the NACA airfoil surface. An application is finally given to the determination of the velocity and pressure coefficient field around an aircraft by assuming constant source distribution.

Key Word and Phrases
Two-dimensional NACA airfoil, Non-linear Multidimensional Singular Integral Equations, Non-linear Aerodynamics, Constant Source Distribution, Aircraft, Velocity & Pressure Coefficient Field.

1. Introduction

Over the past years the non-linear singular integral equations have concentrated an increasing interest, because of their application to the solution of basic problems of aerodynamics and fluid mechanics, especially referring to unsteady flows. The theory and computational methods by non-linear singular integral equations consist of the latest high technology to the solution of generalized problems of solid and fluid mechanics. Consequently, there is a big interest to the continuous improvement of such computational methods.

The new design aerodynamic problems are reduced to the solution of a non-linear singular integral equation, which is used for the determination of the velocity and pressure coefficient field around a NACA airfoil. Such an aerodynamic behavior of the NACA airfoils is a very important element to the design of new generation aircrafts, with very high speeds. Hence, special attention should be given to the new technology computational methods concentrated to the solution of the before mentioned aerodynamic and fluid dynamic problem.

A.M.O.Smith and J.L.Hess [1], were the first scientists who investigated aerodynamic panel methods for studying airfoils with zero lift. According to them, the airfoil was modeled with distributed potential source panels for nonlifting flows, or vortex panels for flow with lift. This method was further extended by R.H.Djojodihardjo and S.E.Widnall [2], P.E.Robert and G.R.Saaris [3], J.M.Summa [4], D.R.Bristow [5], D.R.Bristow and J.D.Hawk [6] and R.J.Lewis [7], for studying three-dimensional steady and unsteady flows, by combining source and vortex singularities. Furthermore, the unsteady panel methods to the modeling of separated wakes using discrete vortices, were further extended by T.Sarpkaya and R.L.Schoaf [8].

In addition, N.D.Ham [9], F.D.Deffenbaugh and F.J.Marschall [10], M.Kiya and M.Arie [11] and T.Sarpkaya and H.K.Kline [12] investigated some other flow models. According to them, the separating boundary layers were represented by an array of discrete vortices, emanating from a known separation point location on the airfoil surface.

On the contrary, during the past years, several scientists made extensive calculations by using unsteady turbulent boundary layer methods. Among them we mention: R.E.Singleton and J.F.Nash [13], J.F.Nash, L.W.Carr and R.E.Singleton [14], A.A.Lyrio, J.H.Ferzinger and S.J.Kline [15], W.J.McCroskey and S.I.Pucci [16] and J.Kim, S.J.Kline and J.P.Johnston [17].

Recently, non-linear singular integral equation methods were proposed by E.G.Ladopoulos [18] - [22] for the solution of fluid mechanics and aerodynamic problems and by E.G.Ladopoulos and V.A.Zisis [23], [24] for two-dimensional fluid mechanics problems applied to turbomachines.

So, by the current research, the aerodynamic problem of the unsteady flow of a two-dimensional NACA airfoil moving by a velocity U_A , is reduced to the solution of a non-linear multidimensional singular integral equation. This nonlinearity results because the source and vortex strength distribution are dependent on the history of the vorticity and source distribution on the NACA airfoil surface. Moreover, a turbulent boundary layer model is further proposed, based on the formulation of the unsteady behavior of the momentum integral equation.

An application is finally given to the determination of the velocity and pressure coefficient field around an aircraft by assuming constant source distribution.

2. Non-linear Fluid Dynamics and Unsteady Aerodynamics

A new non-linear unsteady fluid mechanics representation analysis is introduced and studied, for the aerodynamic problem of a two-dimensional NACA airfoil. The proposed method consists to the generalization of all past methods, by reducing the problem to the solution of a non-linear multidimensional singular integral equation. The above nonlinearity results because of the general form given to the source and vortex strength distribution, while these are dependent on the history of the vorticity and source distribution on the NACA airfoil surface. In this case the airfoil is moving with a speed U_A . [18] – [22]

Consequently, consider a two-dimensional airfoil moving in an homogeneous and inviscid fluid. (Fig.1).

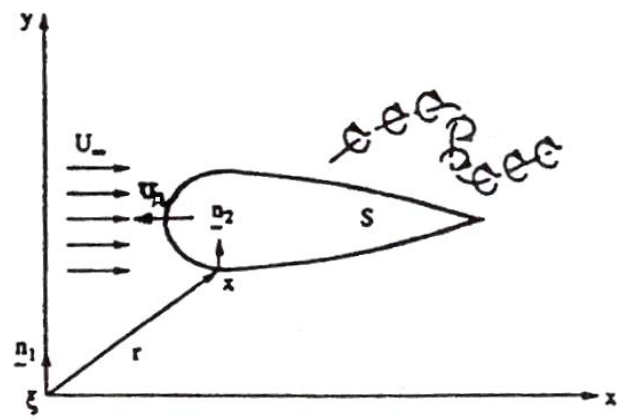

Fig. 1 A two-dimensional airfoil of surface S in an homogeneous and inviscid fluid.

The airfoil with the wake comprise s complete lifting system in an irrotational flow through the ideal fluid. Because of the existence of such an irrotationality, then for the local fluid velocity **U** one has:

$$\nabla \times \mathbf{U} = 0 \tag{2.1}$$

Also, by replacing the fluid velocity with the total velocity potential H one has:

$$\mathbf{U} = \nabla H \tag{2.2}$$

while (2.2) can be further written as:

$$\mathbf{U} = \mathbf{U}_{\infty} + \nabla h \tag{2.3}$$

with $\mathbf{U}_{\infty}$ the outward velocity (Fig. 1) and h the potential due to the presence of the airfoil.

In addition, by using Green's theorem [25] follows a basic relation for the velocity potential $h(\mathbf{x}, t)$, with t the time, at any point $\mathbf{x}$ in continuous, acyclic irrotational flow:

$$h(\mathbf{x}, t) = -1/2\pi \int_S \frac{g[\boldsymbol{\xi}, t, h]}{r} \mathrm{d}S + 1/2\pi \int_{S+W} \delta[\boldsymbol{\xi}, t, h] \frac{\partial}{\partial n_1}\left(\frac{1}{r}\right) \mathrm{d}S \tag{2.4}$$

where S is the surface of the airfoil (Fig. 1), W the surface of the wake, $\mathbf{n_1}$ the surface normal at the source point $\boldsymbol{\xi}$ (Fig. 1), $g[\boldsymbol{\xi}, t, h]$ the source strength distribution, $\delta[\boldsymbol{\xi}, t, h]$ the vortex strength distribution and r the distance equal to:

$$r = |\mathbf{x} - \boldsymbol{\xi}| \tag{2.5}$$

The velocity potential (2.4) can be also written as following, which denotes a two-dimensional non-linear singular integral equation:

$$h(\mathbf{x}, t) = -1/2\pi \int_S \frac{g[\boldsymbol{\xi}, t, h]}{r} \mathrm{d}S + 1/2\pi \int_{S+W} \frac{\delta[\boldsymbol{\xi}, t, h]}{r^2} \mathrm{d}S \tag{2.6}$$

The kinematical surface tangency condition on the surface of the airfoil can be written as following: [26]

$$\left(1/|\nabla S(\mathbf{x}, t)|\right) \frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{\partial h}{\partial n_2} + \mathbf{U}_{\infty} \cdot \mathbf{n}_2 = 0 \tag{2.7}$$

where $\mathbf{n}_2$ denotes the surface normal at the field point $\mathbf{x}$ (Fig. 1).

The above condition can be further written as following, for a body fixed coordinate system:

$$\left(1/|\nabla S(\mathbf{x}, t)|\right) \frac{\partial S(\mathbf{x}, t)}{\partial t} = -\left(\mathbf{U}_A + \boldsymbol{\omega}_{\mathbf{A}} \times \mathbf{x}\right) \cdot \mathbf{n}_2 \tag{2.8}$$

in which $\mathbf{U}_A$ denotes the airfoil translation velocity and $\boldsymbol{\omega}_A$ the airfoil angular rotation.

From eqs (2.7) and (2.8) follows:

$$\frac{\partial h}{\partial n_2} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \mathbf{n}_2 = 0 \tag{2.9}$$

Beyond the above, by inserting (2.9) into (2.6) results the following two-dimensional non-linear singular integral equation:

$$\begin{gathered} 1/2\pi \int_S g[\xi, t, h] \frac{\partial}{\partial n_2}\left(\frac{1}{r}\right) \mathrm{d}\,S + 1/2\pi \int_{S+W} \delta[\xi, t, h] \frac{\partial}{\partial n_2}\left(\frac{1}{r^2}\right) \mathrm{d}\,S = \\ -\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \mathbf{n}_2 \end{gathered} \tag{2.10}$$

The non-linear singular integral equation (2.10) can be further written as:

$$\begin{gathered} 1/2\pi \int_S \frac{g[\xi, t, h]}{r^2} \mathrm{d}\,S + 1/\pi \int_{S+W} \frac{\delta[\xi, t, h]}{r^3} \mathrm{d}\,S = \\ \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_\mathbf{A} \times \mathbf{x}\right) \cdot \mathbf{n_2} \end{gathered} \tag{2.11}$$

Consequently, by solving the non-linear integral equation (2.11) with the corresponding boundary conditions, then the velocity at any field point will be determined through (2.7).

3. Non-linear Pressure Distribution Analysis

The pressure distribution on the airfoil may be obtained by the unsteady Bernoulli equation, valid at any point in an irrotational, ideal flow:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + 1/2\left(\nabla H\right)^2\right] \tag{3.1}$$

where ρ denotes the fluid density.

In addition, by using the derivation of the previous section, then (3.1) will be written as:

$$P = P_\infty - \rho\left[\frac{\partial h}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \nabla h + 1/2\left(\nabla h\right)^2\right] \tag{3.2}$$

Also, (3.2) reduces to the following form:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla_S H + \right.$$
$$\left. + \frac{\partial H}{\partial n_1}\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \mathbf{n}_1 + 1/2\left(\nabla_S H\right)^2 + 1/2\left(\frac{\partial H}{\partial n_1}\right)^2\right] \tag{3.3}$$

if we replace the ∇f , by the surface gradient $\nabla_S h$:

$$\nabla h = \nabla_S h + \frac{\partial h}{\partial n_1}\varepsilon_{n_1} \tag{3.4}$$

Hence, because of (2.9), then (3.3) can be written as:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla_S H - \right.$$
$$\left. - 1/2\left\{\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \mathbf{n}_1\right\}^2 + 1/2\left(\nabla_S H\right)^2\right] \tag{3.5}$$

which will be used for the computations.

4. Laminar and Turbulent Boundary Layer Models

Several boundary layer models can be used for the laminar, the turbulent parts of the flow and the transition region between them, in order to determine the aerodynamic behavior of the airfoils. These boundary layer models are the finite difference, finite element or integral models.

The turbulent boundary layer model which is proposed by the present research is based on the formulation of the unsteady behavior of the momentum integral equation [15]. The major extension of the above method by the present research is the inclusion of unsteady terms in the momentum integral equation.

The unsteady momentum integral equation, which is valid for both laminar and turbulent flow can be therefore written as: (Fig. 2)

$$\frac{1}{u_B^2}\frac{\partial}{\partial t}(u_B d_1) + \frac{\partial d_2}{\partial S} + \frac{1}{u_B}\frac{\partial u_B}{\partial S}(2d_2 + S) = \frac{c_F}{2} \tag{4.1}$$

in which u_B is the boundary layer edge velocity, t the time, d_1 the displacement thickness, d_2 the momentum thickness, S the surface distance and c_F the friction factor.

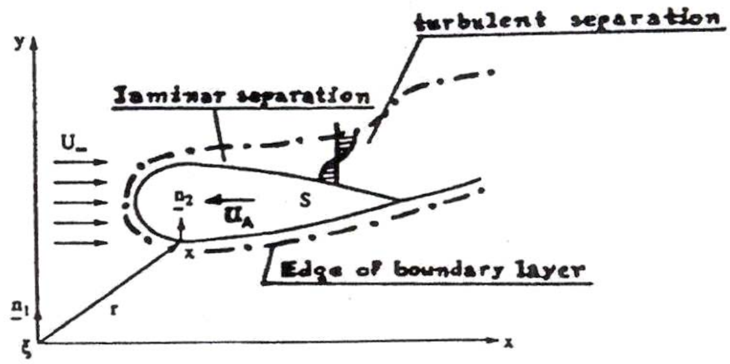

Fig. 2 Laminar and Turbulent Boundary Layer Model for Aerodynamics.

In addition consider the case for the laminar layer, then the pressure gradient parameter μ is given by the relation:

$$\mu = \frac{d_2}{u_B} R_d \left(\frac{\partial u_B}{\partial S} + \frac{1}{u_B} \frac{\partial u_B}{\partial t} \right) \tag{4.2}$$

where R_d is the Reynolds number based on u_B and d_2.

Moreover, by considering some special relations between the parameters $c_F/2$, d_2 and d_1, then a solution for the laminar formulation may be obtained. For the wedge flow solutions following relations are valid: [27]

$$\begin{aligned} \frac{c_F}{2} &= \frac{1.91 - 4.13D}{R_d} \\ N &= (0.68 - 0.922D)^{-1} \\ D &= 0.325 - 0.13kN^2 \end{aligned} \tag{4.3}$$

where N is the shape parameter, D the blockage factor d_1/d_B with d_B the boundary layer thickness and R_d the Reynolds number based on u_d and d.

On the contrary, for the turbulent layer model following formula is valid:

$$\frac{1}{u_B} \frac{\partial}{\partial S} [u_B (d_B - d_1)] = \Lambda \tag{4.4}$$

and the function Λ is obtained by the formulas:

$$\begin{aligned}
&\frac{\mathrm{d}\Lambda}{\mathrm{d}S} = 0.025(\Lambda_B - \Lambda)d_B \\
&\Lambda_B = 4.24K_B(\tfrac{B}{1-B})^{0.916} \\
&K_B = 0.013 + 0.0038e^{-\varphi/15} \\
&\varphi = \frac{d_1}{\tau_w}\frac{\mathrm{d}P}{\mathrm{d}x}
\end{aligned} \tag{4.5}$$

where τ_w is the wall shear stress and dp/dx the streamwise pressure gradient.

Also, the shape factor relationships are obtained by following relations:

$$\begin{aligned}
&\frac{u}{u_B} = 1 + \xi\ln(\frac{y}{d_B}) - f\cos^2(\frac{\pi y}{2d_B}) \\
&\xi = \frac{1}{0.41}(\mathrm{sgn}\frac{c_F}{2})(\frac{c_F}{2})^{1/2} \\
&f = 2(B - \xi) \\
&\frac{c_F}{2} = \frac{\tau_w}{\rho {u_B}^2}
\end{aligned} \tag{4.6}$$

with u the velocity in the boundary layer at a distance y from the wall and ρ the fluid density.

Finally, the skin friction law is valid as:

$$\frac{c_F}{2} = 0.051|1-2B|^{1.732}\left(\frac{R_d}{B}\right)^{-0.268}\mathrm{sgn}(1-2B) \tag{4.7}$$

Additional details concerning the entrainment, the wall shear stress and the skink friction relations can be found in [15].

5. Velocity and Pressure Coefficient Field for Constant Source Distribution (Airfoil with Velocity)

Let us consider the special case of a constant source distribution g. In this case the general non-linear problem presented in previous paragraphs, is much more simplified and is solved as a linear problem. The geometrical representation of the problem is shown in Fig. 3.

For constant source distribution g, then the fluid velocity **U**, is determined by the formula:

$$\mathbf{U} = \int_{-A/2}^{A/2}\frac{g\,\mathrm{d}r}{2\pi r}(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) \tag{5.1}$$

where **i**, **j** are the unit vectors on the x and y axes, respectively, and A denotes the separating wake (Fig. 3).

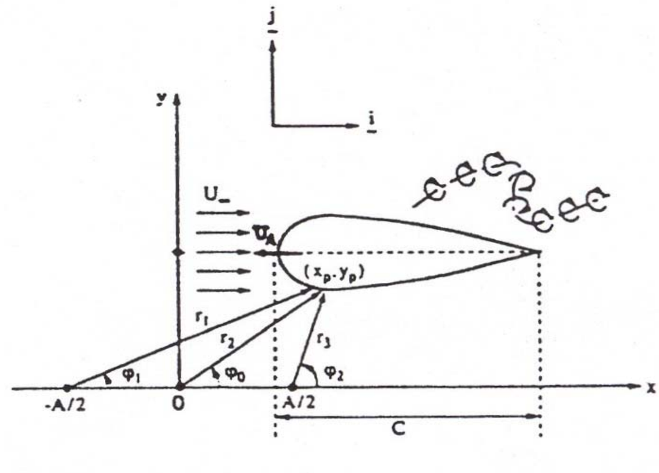

Fig. 3 Coordinate system for the 2D airfoil of an aircraft.

So, when $y_p \neq 0$ and $y_p = 0$, then the fluid velocity **U** will be computed by the following formulas:

$$\mathbf{U} = \begin{cases} g/2\pi\left[\ln\left|\frac{r_1}{r_2}\right|\mathbf{i} - (\varphi_1 - \varphi_2)\mathbf{j}\right], & y_p \neq 0 \\ g/2\pi \ln\left|\frac{r_1}{r_2}\right|\mathbf{i}, & y_p = 0 \end{cases} \tag{5.2}$$

Moreover, we consider the pressure coefficient C_p:

$$C_p = (P - P_\infty)/[1/2\,\rho(U_\infty - U_A)^2] \tag{5.3}$$

where ρ denotes the fluid density and P_∞ the stream pressure.

By using further the unsteady equation of Bernoulli, then the pressure coefficient will be simplified through the relation:

$$C_p = -U^2/(U_\infty - U_A)^2 \tag{5.4}$$

which will be used for the computations.

6. Application of Unsteady Aerodynamics to New Generation Aircraft

As an application of the previous mentioned two-dimensional unsteady aerodynamics theory, we will calculate the velocity field presented around an aircraft. The construction of new generation turbojet engines makes possible the design of very fast big jets. Beyond the above, the increasing evolution of aeroelasticity in aircraft turbomachines continues to be still improved, according to the needs of aircraft powerplant and turbine designers. Consequently, the Aeronautical Industries should achieve a competitive technological advantage in several strategic areas of new and fast developing advanced technologies, by which a bigger market share can be achieved, in the medium and longer terms. Such an increasing big market share includes the design of new generation large aircrafts with very high speeds.

In the present application the length of the aircraft under consideration is c=50.0m and the airfoil section NACA 0021 (Fig. 3).

It was supposed unit vortex distribution and hence, the velocity field on the boundary and around of the airfoil was computed by (5.2). Also, the pressure coefficients C_p were calculated through (5.4) for several aircraft velocities U_A and wind velocity $U_\infty = 15 m/sec$.

Figures 4, 5, 6 and 7 show the pressure distribution on the turbojet presented, for aircraft speeds U_A = 1,2,3,4 Mach respectively (1 Mach=332 *m/sec*). Also, Figs. 4a to 7a show the same pressure distribution on the airfoil, in three dimensional form.

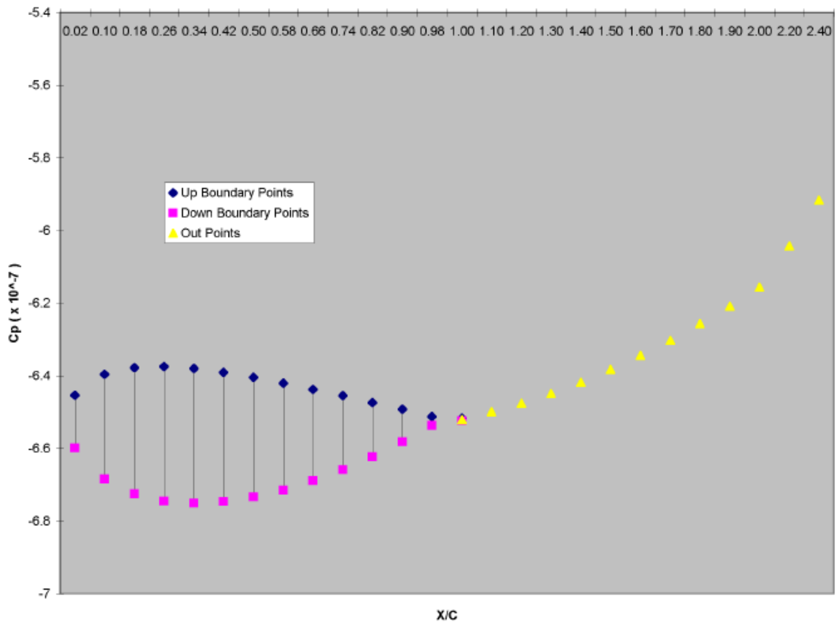

Fig. 4 Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 1 Mach.

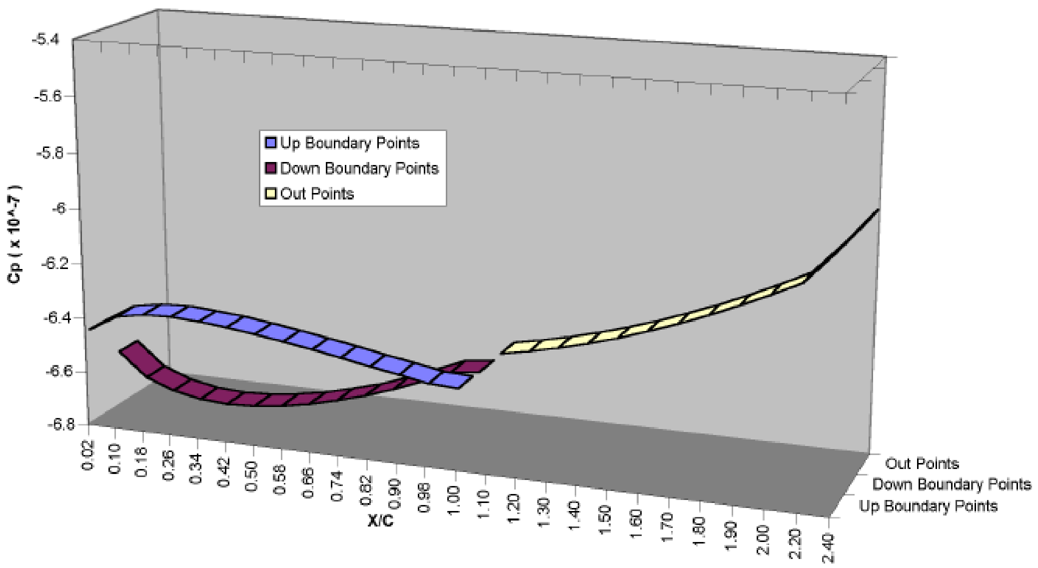

Fig. 4a Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 1 Mach – 3D form.

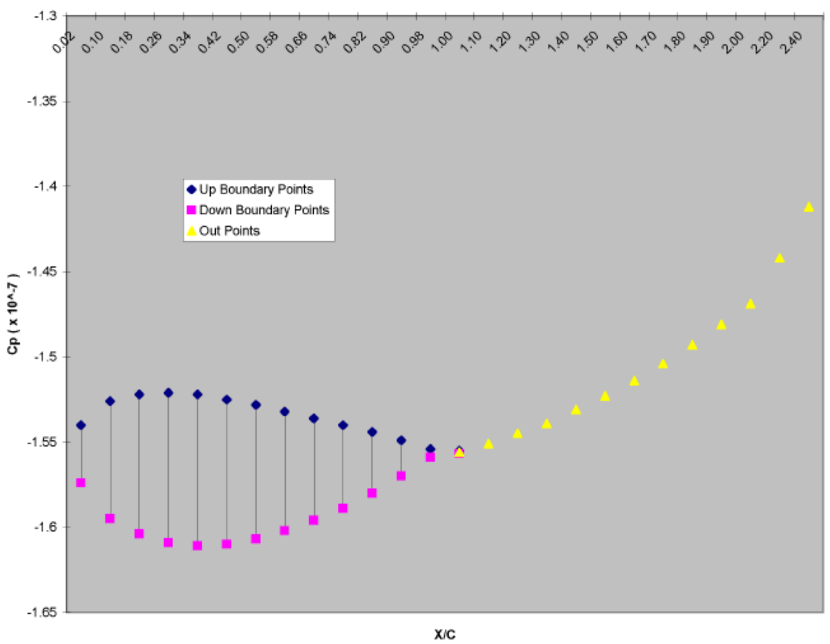

Fig. 5 Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 2 Mach.

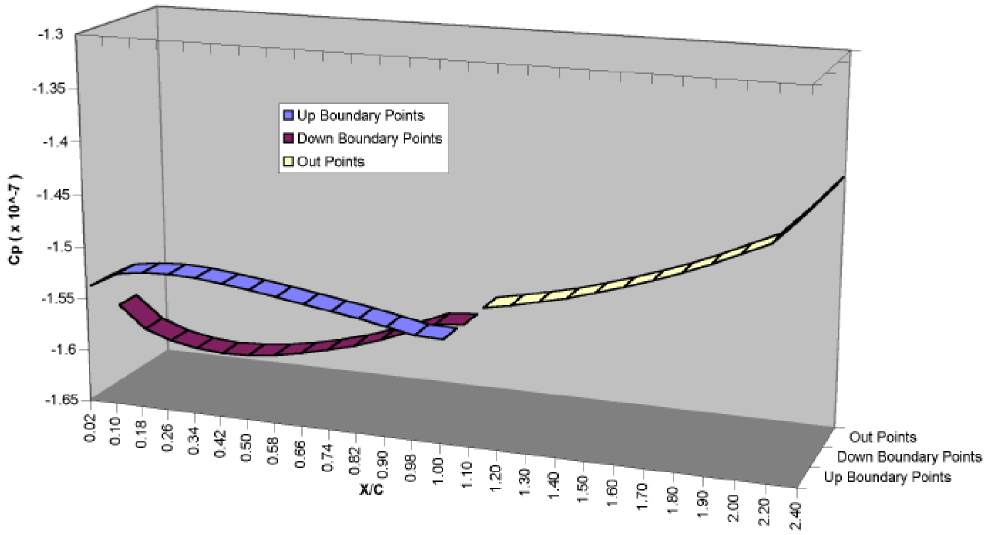

Fig. 5a :Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 2 Mach – 3D form.

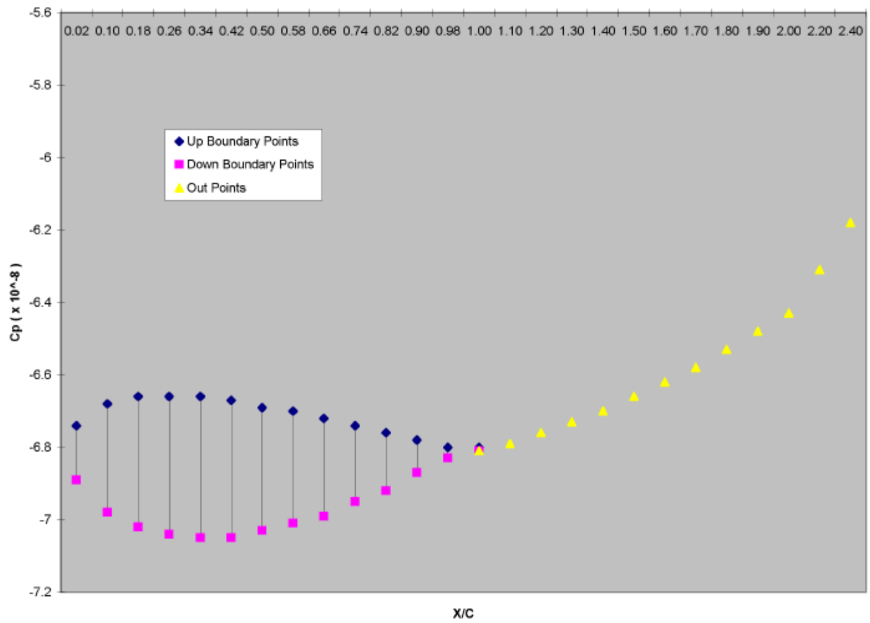

Fig. 6 Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 3 Mach.

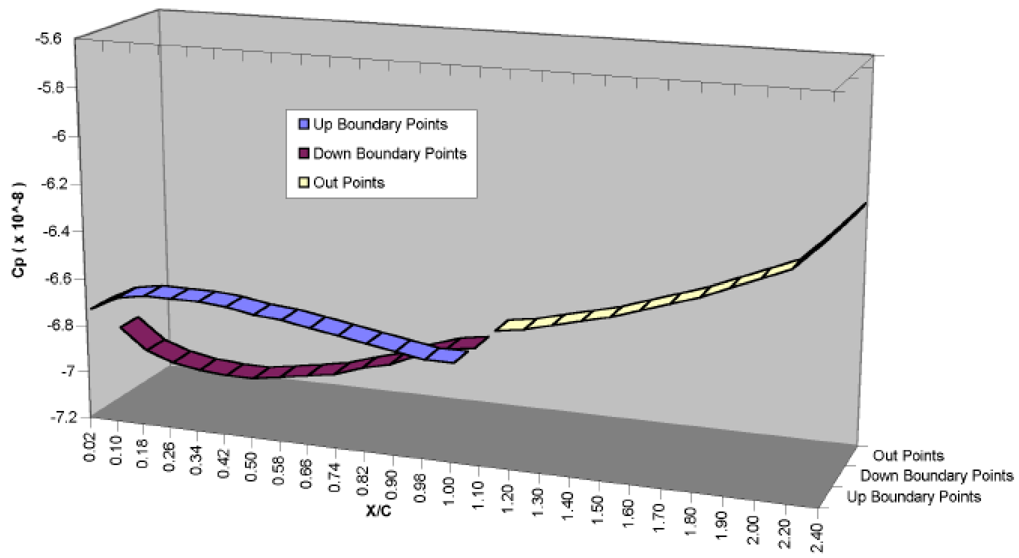

Fig. 6a Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 3 Mach – 3D form.

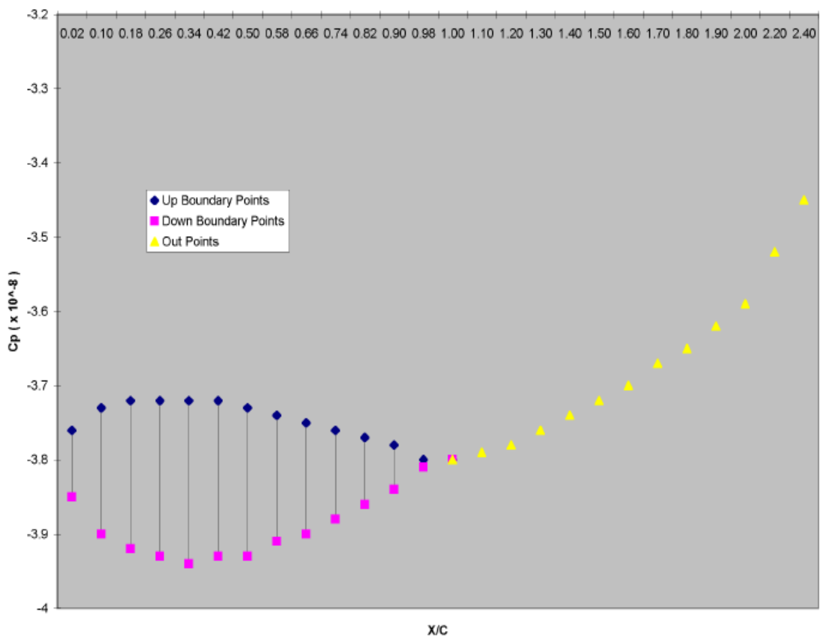

Fig. 7 Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 4 Mach.

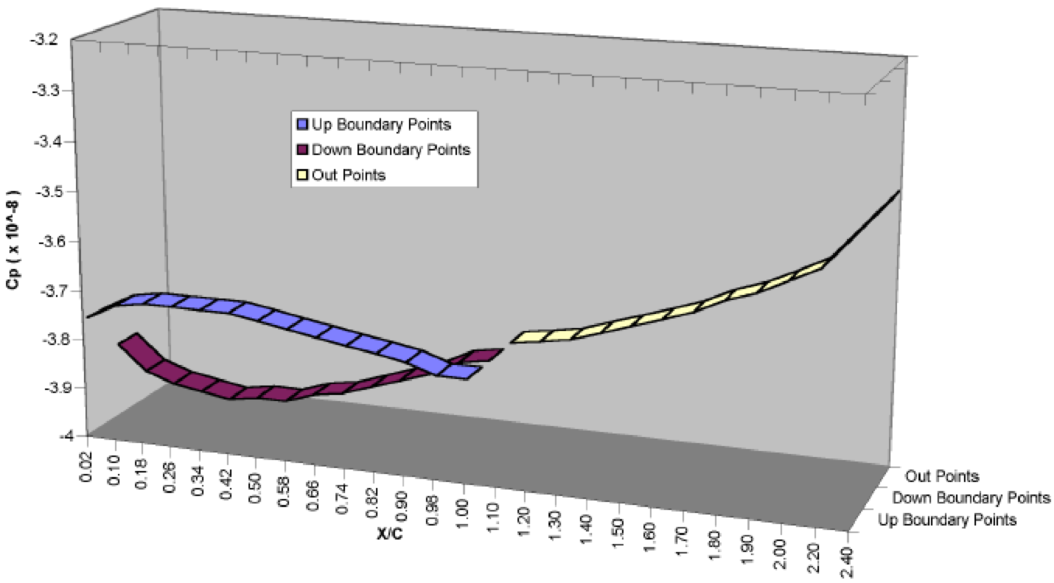

Fig. 7a Pressure distribution around the aircraft of Fig.3, for constant source distribution and speed 4 Mach – 3D form.

As it is shown in the above Figures, for the up boundary points of the NACA airfoil the values of the pressure coefficient are increasing approximately up to x/c = 0.25, while they decreasing again up to x/c =1. On the other hand, for the down boundary points the values of C_p are decreasing up to x/c = 0.35, and then increasing up to x/c = 1.

7. Conclusions

A general non-linear model has been proposed for the determination of the velocity and pressure coefficient field around a NACA airfoil moving by a velocity U_A in two-dimensional unsteady flow. Such a problem was reduced to the solution of a two-dimensional non-linear singular integral equation, which has to be solved by computational methods. The nonlinearity resulted because of the form of the general type of the source and vortex strength distribution.

Also, a boundary layer model was proposed based on the formulation of the unsteady behavior of the momentum integral equation. Such a boundary layer model is valid for both laminar and turbulent flow, and was proposed as a general method for the study of the aerodynamic behavior of the airfoils.

On the other hand, by supposing constant source distribution, then the velocity and pressure coefficient field around an aircraft moving with several velocities, was determined. This method should be applied for the design of new generation large aircrafts with very high speeds.

Consequently, the non-linear singular integral equation methods, will be in future of continuously increasing interest, as such methods will be very important for the solution of the generalized solid and fluid mechanics problems. Special attention should be therefore given to the amelioration of the non-linear singular integral equation methods, as many modern solid and fluid mechanics problems with considerable complicated forms, are recently reduced to non-linear forms.

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