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Coupling Method of Singular Integral Operators Method with Finite Elements in 2-D Elasticity

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Abstract
Some “innovative” and “groundbreaking” aspects for the combined numerical method of the Singular Integral Operators Method (S.I.O.M.) with the Finite Element Method (F.E.M.) are proposed for the solution of two-dimensional elasticity problems. The above coupling evaluation method is usually used for structural analysis problems with unbounded domains or regions of high stress concentration, where strong singularities are present. Consequently, special solutions are determined in areas with infinite domain or when singularities occur and these then are combined with corresponding solutions by the Finite Element Method. Finally, an application of two-dimensional elasticity is given to the determination of the stress field in a cantilever truss beam subjected to concentrated end loads. For the solution of the above elasticity problem the combined method of Singular Integral Operators Method with Finite Elements is used.

Key Word and Phrases
Singular Integral Operators Method (S.I.O.M.), Finite Element Method (F.E.M.), Coupling Method, Two-dimensional Elasticity, Tractions, Stress Tensor, Strain Tensor, Stiffness Matrix.

1. Introduction

In general, many basic engineering problems of applied character like structural analysis, fracture mechanics, fluid mechanics, aerodynamics, elasticity, plasticity, thermoelastoplasticity, viscoelasticity, viscoplasticity and elastodynamics, are reduced to the solution of singular integral equations, or systems of such type of integral equations. For the solution of these singular integral equations some special numerical methods are used, as closed form solutions are only available in very seldom cases.

Besides, there is a big difficulty of solving the singular integral equations by using classical methods, because of the presence of singularities, in contrast to the regular forms of integral equations. Thus, if some singularities are present in an integral equation, then the existence and uniqueness of finite solutions are available only under special conditions, which should be specified in every special case. In addition, the regular integral equations are solved by using much simpler methods, than the corresponding singular integral equations. For applied character problems in engineering the singular integral equations are specially used in unbounded domains or regions of high stress concentration, where singularities are present. In the above cases normal methods, like Finite Elements, are not giving accurate results.

Consequently, E.G.Ladopoulos [1] - [13] proposed the Singular Integral Operators Method (S.I.O.M.) for the numerical evaluation of the singular integral equations in combination with the solution of several important problems of applied engineering character. By using the Singular Integral Operators Method problems of two - and three - dimensional elasto-plasticity of isotropic and anisotropic solids, structural analysis problems and crack problems in isotropic and anisotropic materials are solved.

On the other hand, the S.I.O.M. has many common elements with the Boundary Element Method (B.E.M.), as in both methods the bodies under study are divided into some boundary components. Recent studies on B.E.M. applied in several problems of civil and mechanical engineering were published by several scientists. [14] - [33]

Moreover, the Finite Element Method (F.E.M.) is a method of approximation to continuum problems so that the continuum is divided into a finite number of elements, the behavior of which is specified by a finite number of parameters. Over the past years several important studies on finite elements were published, with applications to many problems of applied engineering. Classical articles on Finite Elements are given in References [34] - [40]. Also, some studies have been published in the past dealing with numerical methods of coupling the Finite Elements with Boundary Element Solutions. Among them are mentioned some classical articles. [41] - [46]

By the present investigation the Singular Integral Operators Method (S.I.O.M.) is combined together with the Finite Element Method (F.E.M.) for the solution of several practical two-dimensional engineering problems, such as those with unbounded domains or regions of high stress concentration, which can be better represented by using singular integral equation methods. So, the coupling of S.I.O.M. and F.E.M. is done without loss of continuity.

In addition, the coupling of the above two numerical methods will be necessary in order to obtain more accurate results in several engineering problems. For example usually the Singular Integral Operators Method gives more accurate results than the Finite Element Method in regions of stress or flux concentrations. So, special solutions can be determined by the S.I.O.M. in regions with singularities and these to be combined with corresponding solutions by Finite Elements.

Normally, the S.I.O.M. is applied to problems extending to infinity as by this method the radiation conditions are satisfied, while such conditions are difficult to be represented by using the F.E.M. An important problem of the Finite Element is its inability to handle domains extending to infinity. On the contrary, the Singular Integral Operators Method is using fundamental solutions which generally obey the radiation condition.

An application of 2-dimensional elasticity is finally given to the determination of the stress field in a cantilever truss beam subjected to concentrated end loads. For the solution of the above engineering problem the coupling method of S.I.O.M. and Finite Elements is used.

2. Two-dimensional Isotropic Elasticity by Singular Integral Equations

Consider by Γ_1 the portion of the boundary of a body on which displacements are prescribed, Γ_2 the surface of the body on which the force tractions are employed and Γ the total surface of the body. Then, the principal virtual displacements can be written as: [7],[8],[11]

$$\int_{\Omega} (\sigma_{jk,j} + b_k) u_k^* \, \mathrm{d}\Omega = \int_{\Gamma_2} (p_k - \overline{p}_k) u_k^* \, \mathrm{d}\Gamma \tag{2.1}$$

in which u_k^* denote the virtual displacements identically satisfying the homogeneous boundary conditions $u_k^* \equiv 0$ on Γ_1, p_k are the surface forces or tractions, b_k the body forces and $\sigma_{jk,j}$ the stress tensor.

Hence, (2.1) may be further written as:

$$\int_{\Omega} (\sigma_{jk,j} + b_k) u_k^* \, \mathrm{d}\Omega = \int_{\Gamma_2} (p_k - \overline{p}_k) u_k^* \, \mathrm{d}\Gamma + \int_{\Gamma_1} (\overline{u}_k - u_k) p_k^* \, \mathrm{d}\Gamma \tag{2.2}$$

where $p_k^* = n_j \sigma_{jk}^*$ are the tractions corresponding to the u_k^* system.

By a first integration of (2.2) one has:

$$\int_{\Omega} b_k u_k^* \, \mathrm{d}\Omega - \int_{\Omega} \sigma_{jk} \varepsilon_{jk}^* \, \mathrm{d}\Omega = - \int_{\Gamma_2} \bar{p}_k u_k^* \, \mathrm{d}\Gamma - \int_{\Gamma_1} p_k u_k^* \, \mathrm{d}\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_k^* \, \mathrm{d}\Gamma \tag{2.3}$$

A second integration of (2.3) gives following result:

$$\int_{\Omega} b_k u_k^* \, \mathrm{d}\Omega + \int_{\Omega} \sigma_{jk,j}^* u_k \, \mathrm{d}\Omega = -\int_{\Gamma_2} \bar{p}_k u_k^* \, \mathrm{d}\Gamma - \int_{\Gamma_1} p_k u_k^* \, \mathrm{d}\Gamma + \int_{\Gamma_1} \bar{u}_k p_k^* \, \mathrm{d}\Gamma + \int_{\Gamma_2} u_k p_k^* \, \mathrm{d}\Gamma \tag{2.4}$$

By represanting a unit load at i in the "l" direction, then the solution may be written as:

$$u_l^i + \int_{\Gamma_1} \bar{u}_k p_k^* \, \mathrm{d}\Gamma + \int_{\Gamma_2} u_k p_k^* \, \mathrm{d}\Gamma = \int_{\Omega} b_k u_k^* \, \mathrm{d}\Omega + \int_{\Gamma_1} p_k u_k^* \, \mathrm{d}\Gamma + \int_{\Gamma_2} \bar{p}_k u_k^* \, \mathrm{d}\Gamma \tag{2.5}$$

in which u_l^i represents the displacement at i in the "l" direction.

For the general case, where $\Gamma = \Gamma_1 + \Gamma_2$, we have:

$$u_l^i + \int_{\Gamma} u_k p_{lk}^* \, \mathrm{d}\Gamma = \int_{\Gamma} p_k u_k^* \, \mathrm{d}\Gamma + \int_{\Omega} b_k u_k^* \, \mathrm{d}\Omega \tag{2.6}$$

Besides, by considering unit forces acting in the three directions, then (2.6) takes the form:

$$u_l^i + \int_{\Gamma} u_k p_{lk}^* \, \mathrm{d}\Gamma = \int_{\Gamma} p_k u_{lk}^* \, \mathrm{d}\Gamma + \int_{\Omega} b_k u_{lk}^* \, \mathrm{d}\Omega \tag{2.7}$$

where p_{lk}^* and u_{lk}^* are the surface tractions and displacements in the "k" direction, due to the unit forces acting in the "l" direction.

For a two-dimensional isotropic solid the fundamental solution is given by the relations: [11]

$$u_{lk}^* = \frac{1}{8\pi G(1-\nu)} \left[(3-4\nu) \ln\left(\frac{1}{r}\right) \Delta_{lk} + \frac{\vartheta r}{\vartheta x_l} \frac{\vartheta r}{\vartheta x_k} \right] \tag{2.8}$$

$$p_{lk}^* = -\frac{1}{4\pi(1-\nu) r} \left[\frac{\vartheta r}{\vartheta n} \left[(1-2\nu) \Delta_{kl} + 2 \frac{\vartheta r}{\vartheta x_k} \frac{\vartheta r}{\vartheta x_l} \right] - (1-2\nu) \left(\frac{\vartheta r}{\vartheta x_l} u_k - \frac{\vartheta r}{\vartheta x_k} u_l \right) \right] \tag{2.9}$$

in which n denotes a normal to the surface of the body, r the distance from the point of application of the load to point under consideration and n_j the direction cosines. (see: Figs. 2.1 and 2.2.)

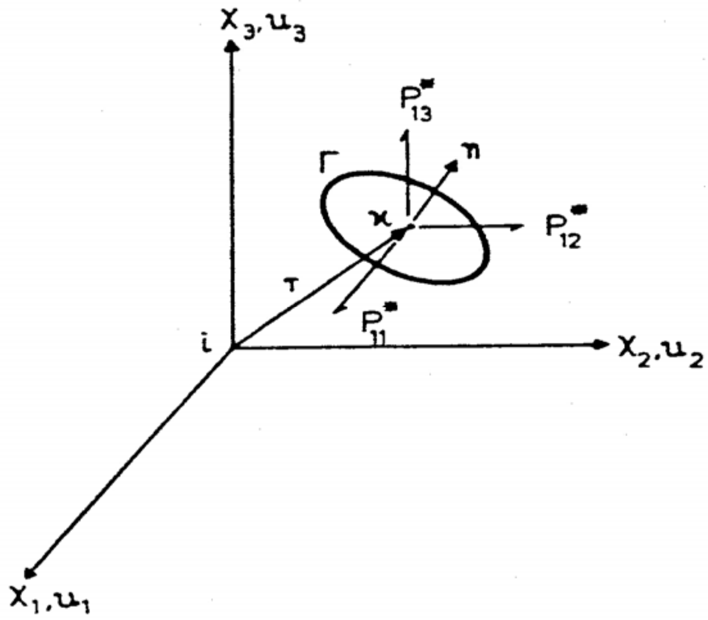

Fig. 2.1 Surface forces acting at point *k*, due to unit load at point *i* acting in the "*l*" direction.

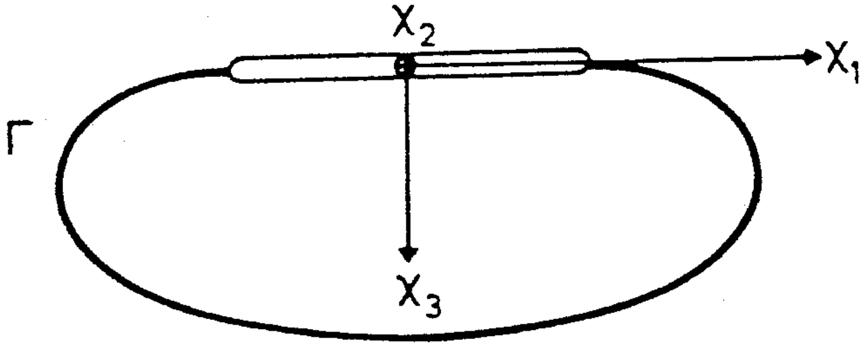

Fig. 2.2 Geometry of the cracked solid.

From (2.7) follow the displacements at a point for the "*l*" component:

$$u_l^i = \int_\Gamma u_{lk}^* p_k \, \mathrm{d}\Gamma - \int_\Gamma p_{lk}^* u_k \, \mathrm{d}\Gamma + \int_\Omega b_k u_{lk}^* \, \mathrm{d}\Omega \tag{2.10}$$

Beyond the above, the strain-displacement relations in the linear theory are:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\vartheta u_i}{\vartheta x_j} + \frac{\vartheta u_j}{\vartheta x_i} \right) \tag{2.11}$$

and the stress-strain relations:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \frac{2Gv}{1-2v} \Delta_{ij} \varepsilon_{ll} \tag{2.12}$$

in which Δ_{ij} is Kronecker's delta, G the shear modulus and v Poisson's ratio.

Thus, from (2.11) and (2.12) we obtain the formula for the stresses of an isotropic solid:

$$\sigma_{ij} = G\left(\frac{\vartheta u_i}{\vartheta x_j} + \frac{\vartheta u_j}{\vartheta x_i} \right) + \frac{2Gv}{1-2v} \Delta_{ij} \frac{\vartheta u_i}{\vartheta x_j} \tag{2.13}$$

By carrying out the differentiation one has:

$$\begin{aligned} \sigma_{ij} = & \int_\Gamma \left[\frac{2Gv}{1-2v} \Delta_{ij} \frac{\vartheta u_{lk}^*}{\vartheta x_l} + G\left(\frac{\vartheta u_{ik}^*}{\vartheta x_j} + \frac{\vartheta u_{jk}^*}{\vartheta x_i} \right) \right] p_k \, \mathrm{d}\Gamma \\ & + \int_\Omega \left[\frac{2Gv}{1-2v} \Delta_{ij} \frac{\vartheta u_{lk}^*}{\vartheta x_l} + G\left(\frac{\vartheta u_{ik}^*}{\vartheta x_j} + \frac{\vartheta u_{jk}^*}{\vartheta x_i} \right) \right] b_k \, \mathrm{d}\Omega \\ & - \int_\Gamma \left[\frac{2Gv}{1-2v} \Delta_{ij} \frac{\vartheta p_{lk}^*}{\vartheta x_l} + G\left(\frac{\vartheta p_{ik}^*}{\vartheta x_j} + \frac{\vartheta p_{jk}^*}{\vartheta x_i} \right) \right] u_k \, \mathrm{d}\Gamma \end{aligned} \tag{2.14}$$

Consequently, (2.14) can be reduced to the following formula:

$$\sigma_{ij} = \int_\Gamma A_{kij} p_k \, \mathrm{d}\Gamma - \int_\Gamma B_{kij} u_k \, \mathrm{d}\Gamma + \int_\Omega A_{kij} b_k \, \mathrm{d}\Omega \tag{2.15}$$

in which the components A_{kij} and B_{kij} are:

$$A_{kij} = \frac{1}{4\pi(1-v)r}[(1-2v)(\Delta_{ki}r_{,j} + \Delta_{kj}r_{,i} - \Delta_{ij}r_{,k}) + 2r_{,i}r_{,j}r_{,k}] \qquad (2.16)$$

$$B_{kij} = \frac{G}{2\pi(1-v)r^2}\left\{2\frac{\vartheta r}{\vartheta n}[(1-2v)\Delta_{ij}r_{,k} + v(\Delta_{ik}r_{,j} + \Delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}]\right.$$

$$\left. + 2v(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}) + (1-2v)(2n_k r_{,i}r_{,j} + n_j\Delta_{ik} + n_i\Delta_{jk}) - (1-4v)n_k\Delta_{ij}\right\} \qquad (2.17)$$

where:

$$r_{,i} = \frac{\vartheta r}{\vartheta x_i}$$

In general, the singular integral equations are applied in many important fields of engineering mechanics like elasticity, plasticity, thermoelastoplasticity, viscoelasticity, viscoplasticity and fracture mechanics theory. Furthermore, applications of singular integral equations are given for the solution of important problems of structural analysis, elastodynamics, fluid mechanics, hydraulics and aerodynamics. So, several important problems of applied mechanics, civil engineering, mechanical engineering and electrical engineering are reduced to the solution of systems of singular integral equations. These singular integral equations for the case of two-dimensional elasticity of isotropic solids, are solved by using the method previously described.

3. Two-dimensional Isotropic Elasticity by Finite Elements

By equating the external work with the total internal work obtained by integrating over the volume of the whole continuum Ω, then the principal of virtual displacements for the finite elements can be written as:

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}\,\Omega = \int_{\Gamma_2} \overline{p}_i\delta u_i\,\mathrm{d}\,\Gamma + \int_{\Omega_1} b_i\delta u_i\,\mathrm{d}\,\Omega \qquad (3.1)$$

in which σ_{ij} and ε_{ij} are the components of the stress and strain tensors, $\overline{p}_i$ the tractions applied on Γ_2, u_i the displacements and b_i the body forces. Moreover, the conditions on Γ_1 are identically satisfied, i.e., $u_i = \overline{u}_i$ and $\delta u_i \equiv 0$ on Γ_1.

When a finite element procedure is applied to (3.1), then follows a system of ordinary equations written in matrix form as:

$$\mathbf{K}\,\mathbf{U} = \mathbf{H} + \mathbf{R} \qquad (3.2)$$

where **K** is the stiffness matrix for the system, **U** is the matrix of the unknown nodal displacements, **H** the matrix of the equivalent nodal forces resulting from the first right - hand side integral of (3.1) and **R** denotes the vector due to the body forces.

Beyond the above, the vector **H** follows by weighting the applied tractions with the interpolation functions used for the virtual displacements. So, a distribution matrix **M** can be found such that **H** is given by the relation:

$$\mathbf{H} = \mathbf{M}\,\mathbf{Q} \tag{3.3}$$

in which $\mathbf{Q}$ is the matrix of the actual nodal values of tractions. Moreover, the coefficients of the matrix $\mathbf{M}$ will depend on the type of interpolation functions which will be used for the displacements and tractions.

By combining (3.2) and (3.3), then follow the finite element matrices:

$$\mathbf{K}\,\mathbf{U} = \mathbf{M}\,\mathbf{Q} + \mathbf{R} \tag{3.4}$$

which can be used for the computations.

Generally, the Singular Integral Operators Method is less expensive to apply than the Finite Elements by using some special factors, as less data are required to run the problem, and some time more accurate, especially for problems with large strain gradients. On the contrary, finite elements are usually accurate for the original variables under consideration, but when these variables are differentiated, the results are much less accurate and are usually discontinuous between elements. Also, a big advantage of the Singular Integral Equations, is that are usually able to represent regions of stress concentration in a better manner than finite elements, although this will depend on the type of the approximating function used.

4. Coupling Method of Finite Elements and Singular Integral Operators Method (S.I.O.M.)

The Finite Elements and the Singular Integral Equations will be combined together for the solution of 2-D isotropic elasticity problems. Hence, consider two regions Ω_1 and Ω_2 which are joined together at the interface Γ_1. Then Ω_1 is studied by using finite elements and Ω_2 by using the S.I.O.M.

In order parts Ω_1 and Ω_2 to be joined, the following assumptions must be satisfied:

i. The displacements at the Γ_1 interface for region Ω_1 (U_1) and region Ω_2 (U_2) should be equal:

$$U_1 = U_2 \quad \text{on} \quad \Gamma_1 \tag{4.1}$$

ii. The tractions at the Γ_1 interface for region Ω_1 (Q_1) and region Ω_2 (Q_2) should add to zero:

$$Q_1 + Q_2 = 0 \quad \text{on} \quad \Gamma_1 \tag{4.2}$$

If the above assumptions are fulfilled then the 2-D elasticity problem is studied by the coupling method of finite elements and singular integral equations.

5. Two-dimensional Elasticity Application to the Determination of the Stress Field in a Cantilever Truss Beam subjected to Concentrated End Loads

The coupling method of S.I.O.M. and Finite Elements will be applied to the determination of the stress field in a cantilever truss beam subjected to concentrated end loads. (Fig. 5.1)

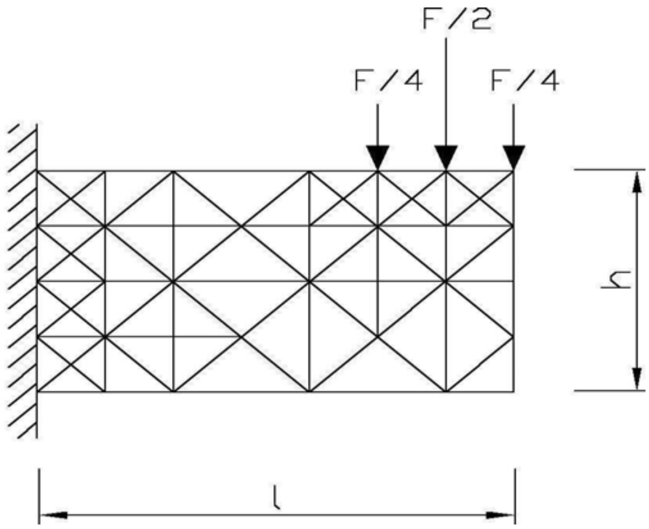

Fig. 5.1 A cantilever truss beam subjected to concentrated end loads.

The cantilever truss beam consists of steel with the following parameters: Modulus of Elasticity $E = 2.1 \times 10^8 \ KN/m^2$ and Poisson's ratio $\nu = 0.3$. Besides, the length of the beam is l = 3.50 *m*, the height h = 1.00 *m* and the width b = 0.10 *m* (Fig. 5.1). Moreover, the concentrated load is F = 100 *kN.*

So, by using the combined method of S.I.O.M. and Finite Elements then the normal σ_x, σ_y and shear stresses τ_{xy} are computed at internal points of the cantilever beam. Then, Fig. 5.2 shows the normal stresses σ_x at some elements of the cantilever truss beam.

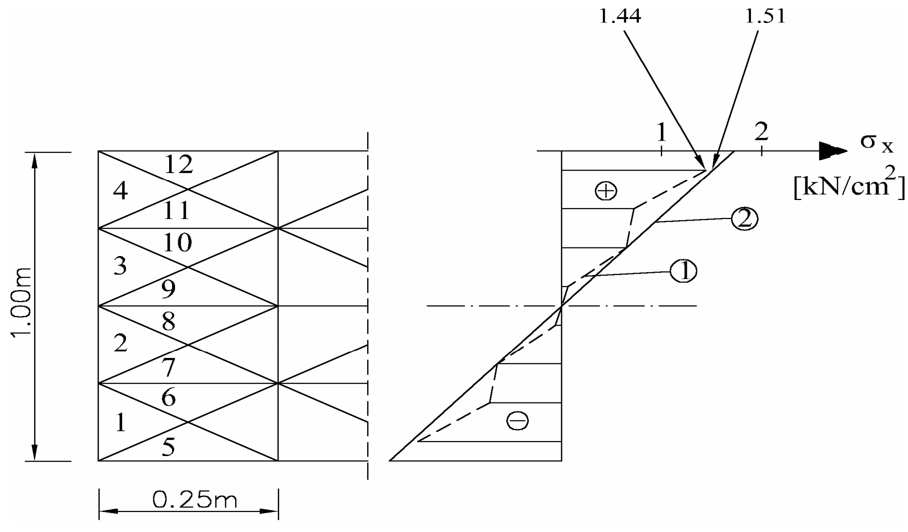

Fig. 5.2 Normal stresses σ_x at the cantilever truss beam, by using the combined method of Singular Integral Operators Method and Finite Elements.
1. SIOM & FEM
2. Theoretical

Finally, as it can be seen from Fig. 5.2 the maximum normal stress equal to 1.44 kN/cm^2 appears in element 12 at the up level of the beam. At the above position the theoretical value of the normal stress is 1.51 kN/cm^2. [40]

6. Conclusions

The Coupling Method of the Singular Integral Operators Method (S.I.O.M.) and the Finite Element Method (F.E.M.) was proposed for the solution of two-dimensional generalized elasticity problems. This coupling evaluation technique can be used for the solution of a big range of practical structural analysis problems, especially when unbounded domains or regions of high stress concentration with singularities are present.

In addition, elasticity problems extending to infinity are usually handled by the Singular Integral Equations, as by the above method the radiation conditions are satisfied, while such conditions are difficult to be represented by using the F.E.M. As the Singular Integral Equations use fundamental solutions, then the radiation conditions are normally satisfied. Furthermore, there is a big inability of the Finite Element Method to handle domains extending to infinity. So, such a problem is solved by the combined technique of the Singular Integral Equations and the F.E.M.

The Singular Integral Equations are giving further more accurate results than the Finite Element Method in regions of stress or flux concentrations, where singularities are present. Hence, special solutions are produced by the Singular Integral Equations in such regions with singularities, and these are then combined with corresponding solutions by the F.E.M.

Finally, an application of two-dimensional elasticity was given to the determination of the stress field in a cantilever truss beam subjected to concentrated end loads. For the solution of the above problem the coupling method of Singular Integral Operators Method and Finite Elements was used.

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