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Plate Bending Analysis by Two-dimensional Non-linear Partial Differential Equations

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Abstract
A plate bending analysis is proposed and investigated, by reducing the above structural analysis problem to the solution of a two-dimensional non-linear partial differential equation. The plate is subjected to uniformly distributed load and then the transverse displacements, the bending moments and the transverse shear forces are computed. This computational plate bending method, consists to the analysis of the load to Fourier-double series. An application of plate bending problems is finally given to the determination of the transverse displacements, the bending moments and the transverse shear forces in an orthogonal plate consisting of reinforced concrete and subjected to uniform load.

Key Word and Phrases
Plate Bending, Non-linear Partial Differential Equation, Structural Analysis, Transverse Displacements, Bending Moments, Transverse Shear Forces, Fourier-double Series.

1. Introduction
A very interesting field of applied mechanics and structural analysis is the design of plates of several sizes and forms subjected under bending moments. These plates are made either by isotropic or orthotropic solids, or could be sandwich plates. They constitute basic units of both building frame structures and underground structures. Special attention should be therefore given when choosing the proper theory and evaluation method for the determination of the plate bending moments.

Over the past years the Boundary Element Method (BEM) and the Singular Integral Equation Methods have been successfully used for the numerical evaluation of plate bending problems. M.A.Jaswon and M.Maiti [1] were the first scientists who published an application of BEM to plate bending problems, while they reduced the plate bending to the solution of a biharmonic boundary-value problem. Beyond the above, C.M.Segedin and D.G.A.Brickell [2] proposed an integral method for the solution of plates with reentry corners.

Some time later M.Maiti and S.K.Chakrabarty [3] used an integral equation method for the numerical evaluation of simply supported polygonal plates. Furthermore, H.Fabre [4] proposed a numerical method of singular integral and integro-differential equations for the solution of plate bending problems. Also, N.J.Altiero and D.L.Sikarskie [5] applied the Boundary Integral Equation Method to plates of arbitrary forms. On the other hand, G.P.Bezine [6], [7] has further used the BEM for the solution of problems of plate flexure, with arbitrary boundary conditions.

M.A.Stern [8] made a study on applying the boundary element method to the numerical evaluation of plate bending problems. Some further studies of the boundary integral equation method to the classical plate theory were made by V.Sladek and J.Sladek [9]. In their monograph [10] C.A.Brebbia, J.C.F.Telles and L.C.Wrobel, made an extended research on the boundary integral equation method for the solution of general plate bending problems.

In addition, J.Balas, V.Sladek and J.Sladek [11] used the BEM for the numerical solution of plates resting on elastic foundation. G. Burgess and E.Mahajerin [12] applied the boundary integral equation method for the numerical evaluation of laterally loaded thin plates. On the other hand, J.A.Costa and C.A.Brebbia [13] made further improvements of the Boundary Element Method applied to plates on elastic foundation.

F.Paris and S.León [14], [15] applied the Boundary Element Method to the solution of thin bending plates. Furthermore, E.G.Ladopoulos [16]-[18] proposed a method of non-linear integro-differential equations for the numerical solution of bending plates of several forms. Also, B.Falter [19] proposed several numerical methods for the solution of bending plates. G.A.Hartley [20] made further improvements to the development of plate bending elements for frame analysis.

By the present research the plate bending problem will be reduced to the solution of a two-dimensional non-linear partial differential equation. Thus, the plate is subjected to uniformly distributed load and the transverse displacements, the bending moments and the transverse shear forces are computed. The derivation of the proposed computational plate bending method, consists to the analysis of the load to Fourier-double series.

Finally, an application of plate bending problems is given to the determination of the transverse displacements, the bending moments and the transverse shear forces in an orthogonal plate consisting of reinforced concrete and subjected to a uniform load.

2. Plate Bending Analysis by Partial Differential Equations

Consider a plate bending problem, where the plane *x-y* coincides with the mean surface of the plate and the thickness of the plate is defined by *t* (Figure 1). The applied forces are moments and trasverse shear forces.

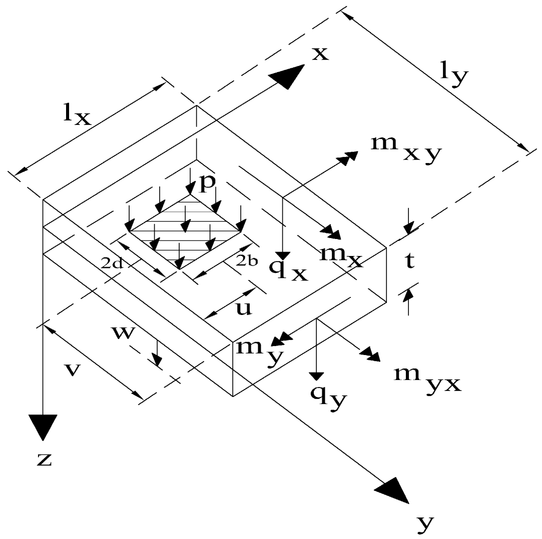

Fig. 1 Plate bending problem under uniformly distributed force *p*.

The plate non-linear partial differential equation is given by the following formula, with *w* the transverse displacement and *p* the uniformly distributed load:

$$\Delta^2 w(x,y) = \frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4} = \frac{p(x,y)}{k} \tag{2.1}$$

where $\Delta(\,.\,)$ denotes the Δ-operator:

$$\Delta(.) = \frac{\partial^2 (.)}{\partial x^2} + \frac{\partial^2 (.)}{\partial y^2} \tag{2.2}$$

and k is the bending rigidity of the plate:

$$k = \frac{Et^3}{12(1-\nu^2)} \tag{2.3}$$

with E the modulus of elasticity and ν Poisson's ratio.

The differential equation (2.1) is valid for elastic and isotropic plates.

For a plate bending problem as shown in Figure 1, then the load *p(x,y)* at any point *(x,y)* can be given by the following Fourier-double series:

$$p(x,y) = \sum_{i=1}^{n_x} \sum_{i=y}^{n_y} a_{ij} \sin\left(\frac{i\pi x}{l_x}\right) \sin\left(\frac{j\pi y}{l_y}\right) \tag{2.4}$$

with the coefficients:

$$a_{ij} = \frac{16p}{\pi^2 ij} \sin\left(\frac{i\pi u}{l_x}\right) \sin\left(\frac{i\pi b}{l_x}\right) \sin\left(\frac{j\pi v}{l_y}\right) \sin\left(\frac{j\pi d}{l_y}\right) \tag{2.5}$$

For the solution of the plate bending problem the following boundary condition is used:

$$w = \Delta w = 0 \tag{2.6}$$

from which follows:

$$w(x,y) = \sum_{i=1}^{n_x} \sum_{i=y}^{n_y} w_{ij} \sin\left(\frac{i\pi x}{l_x}\right) \sin\left(\frac{j\pi y}{l_y}\right) \tag{2.7}$$

and because of (2.1):

$$w_{ij} = \frac{a_{ij}}{k\pi^4 \left[(i/l_x)^2 + (i/l_y)^2 \right]^2} \tag{2.8}$$

3. Bending Moments and Transverse Shear Forces

The bending moments are given by the relations: [10]

$$m_x = -k\left(\frac{\partial w(x,y)}{\partial x} + \nu \frac{\partial w(x,y)}{\partial y} \right)$$

$$m_y = -k\left(\frac{\partial w(x,y)}{\partial y} + \nu \frac{\partial w(x,y)}{\partial x} \right) \tag{3.1}$$

$$m_{xy} = -k(1-\nu)\frac{\partial w(x,y)}{\partial x \partial y}$$

By using (3.1) and the formulas of previous section follows:

$$m_x = k\pi^2 \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{ij}\left[\left(\frac{i}{l_x}\right)^2 + \nu\left(\frac{j}{l_y}\right)^2\right]\sin\left(\frac{i\pi x}{l_x}\right)\sin\left(\frac{j\pi y}{l_y}\right) \tag{3.2}$$

$$m_y = k\pi^2 \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{ij}\left[\nu\left(\frac{i}{l_x}\right)^2 + \left(\frac{j}{l_y}\right)^2\right]\sin\left(\frac{i\pi x}{l_x}\right)\sin\left(\frac{j\pi y}{l_y}\right) \tag{3.3}$$

$$m_{xy} = -k(1-\nu)\pi^2 \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{ij}\,\frac{ij}{l_x l_y}\cos\left(\frac{i\pi x}{l_x}\right)\cos\left(\frac{j\pi y}{l_y}\right) \tag{3.4}$$

Furthermore, the transverse shear forces are given by the relations:

$$\begin{aligned} q_x &= \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} \\ q_y &= \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} \end{aligned} \tag{3.5}$$

Thus, from (3.2) to (3.5) follows:

$$q_x = k\pi^3 \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{ij}\left[\left(\frac{i}{l_x}\right)^2 + (2-\nu)\left(\frac{j}{l_y}\right)^2\right]\frac{i}{l_x}\cos\left(\frac{i\pi x}{l_x}\right)\sin\left(\frac{j\pi y}{l_y}\right) \tag{3.6}$$

$$q_y = k\pi^3 \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{ij}\left[(2-\nu)\left(\frac{i}{l_x}\right)^2 + \left(\frac{j}{l_y}\right)^2\right]\frac{j}{l_y}\sin\left(\frac{i\pi x}{l_x}\right)\cos\left(\frac{j\pi y}{l_y}\right) \tag{3.7}$$

Eqs (3.2) to (3.4) and (3.6), (3.7) will be used for the computations.

4. Application of Plate Bending Theory

As an application of the plate bending analysis by two-dimensional non-linear partial differential equations, the transverse displacement, the bending moments and the transverse shear forces will be computed for a plate of sizes $l_x = 5.0\,m$ and $l_y = 4.0\,m$ subjected under a uniform load $p = 1.0\,kN/m^2$.

The plate consists of reinforced concrete with elasticity modulus $E = 21\ GPa$ and Poisson's ratio $v = 0.20$. Furthermore, the thickness of the plate is $t = 0.20\,m$ and the sizes of the uniform load are $b = 0.50\,m$, $d = 0.75\,m$, $u = 1.50\,m$ and $v = 3.0\,m$ (Fig. 2).

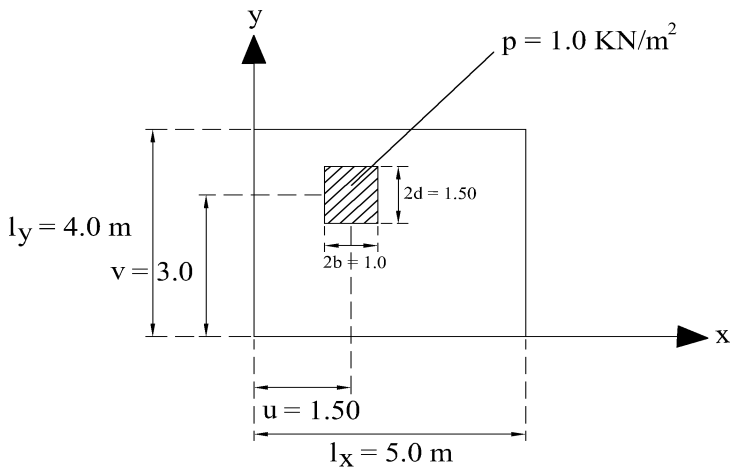

Fig. 2 Orthogonal plate of sizes 5.0x4.0 m, under a uniform load *p=1.0 KN/m* ^{2}.

Thus, by applying the method of the previous paragraphs, then the transverse displacements, the bending moments and the transverse shear forces are computed for several iterations.

Table 1 shows the results of the maximum transverse displacements w and the bending moments m_x and m_y for several iterations $n_x = n_y = 2, 5, 10, 20, 50$ and 100.

Table 1

Transverse Displacements w, Bending Moments m_x and m_Y for the plate of Fig.2

Iterations $n_x = n_y$	*2*	*5*	*10*	*20*	*50*	*100*
w(mm)	0.0090	0.0100	0.0100	0.0100	0.0100	0.0100
M_x *(kNm/m)*	0.1256	0.1580	0.1628	0.1649	0.1644	0.1644
M_y *(kNm/m)*	0.1552	0.1851	0.1812	0.1823	0.1824	0.1824

As it can be seen from Table 1 after 50 iterations then the values of the transverse displacements and the bending moments coincide to their final values.

Furthermore, Table 2 shows the values of the transverse displacements *w*, bending moments m_x, m_y and m_{xy} and transverse shear forces q_x and q_y for 100 iterations.

Table 2

Transverse Displacements w, Bending Moments m_x, m_Y, m_{xy} *and transverse shear forces* q_x, q_y *for the plate of Fig.2*

i	*j*	*w*	m_x	m_y	m_{xy}	q_x	q_y
1	1	0.000000	0.0000	0.0000	-0.0604	0.0000	0.0000
1	2	0.000000	0.0000	0.0000	-0.0589	0.0574	0.0000
1	3	0.000000	0.0000	0.0000	-0.0340	0.1464	0.0000
1	4	0.000000	0.0000	0.0000	0.0589	0.1804	0.0000
1	5	0.000000	0.0000	0.0000	0.1284	0.0000	0.0000
2	1	0.000000	0.0000	0.0000	-0.0325	0.0000	0.0520
2	2	0.000005	0.0523	0.0182	-0.0294	0.0260	0.0733
2	3	0.000010	0.1245	0.0841	-0.0177	0.0844	0.1846
2	4	0.000010	0.1644	0.1824	0.0294	0.1670	-0.0733
2	5	0.000000	0.0000	0.0000	0.0679	0.0000	-0.4212
3	1	0.000000	0.0000	0.0000	0.0138	0.0000	0.0533
3	2	0.000006	0.0368	0.0308	0.0182	-0.0224	0.0619
3	3	0.000010	0.0590	0.0904	0.0230	-0.0892	0.0677
3	4	0.000009	0.0429	0.1265	-0.0182	-0.1547	-0.0619
3	5	0.000000	0.0000	0.0000	-0.0598	0.0000	-0.1888
4	1	0.000000	0.0000	0.0000	0.0320	0.0000	0.0258
4	2	0.000003	0.0079	0.0222	0.0274	-0.0231	0.0235
4	3	0.000005	0.0066	0.0451	0.0082	-0.0443	0.0092
4	4	0.000004	-0.0001	0.0444	-0.0274	-0.0422	-0.0235
4	5	0.000000	0.0000	0.0000	-0.0485	0.0000	-0.0442
5	1	0.000000	0.0000	0.0000	0.0338	0.0000	0.0000
5	2	0.000000	0.0000	0.0000	0.0262	-0.0185	0.0000
5	3	0.000000	0.0000	0.0000	0.0035	-0.0296	0.0000
5	4	0.000000	0.0000	0.0000	-0.0262	-0.0237	0.0000
5	5	0.000000	0.0000	0.0000	-0.0408	0.0000	0.0000

From Table 2 follows that three intermediate points have been taken by the computer program, which are appropriate for the computation of the maximum values.

5. Conclusions

A two-dimensional plate bending analysis has been proposed and investigated by using the plate non-linear partial differential equation. Thus, the plate was subjected to uniformly distributed load and by using a special numerical method, the transverse displacements, the bending moments and the transverse shear forces were computed. According to the proposed structural analysis method the load was analyzed to Fourier-double series.

The proposed method of reducing the plate bending problem to the solution of two-dimensional non-linear partial differential equations, has many advantages in comparison to other numerical methods, like for example the boundary element method, as in many cases is much simpler, faster and giving more accurate results. Furthermore, the design of plates of several sizes and forms subjected to bending moments is very important for structural analysis and thus special attention should be given by the engineer to choose the proper numerical method for the solution of the plate bending problem. These types of plates, are used either in building frame structures, or in underground structures.

An application of the plate bending problem by using two-dimensional non-linear partial differential equations was finally given for the calculation of the transverse displacements, the bending moments and the transverse shear forces in an orthotropic plate consisting of reinforced concrete and subjected to a uniform load.

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