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Further Developments of Non-linear Semigroups in Hilbert Spaces used in Oil & Gas Engineering

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Abstract
A new mathematical approach is investigated by using non-linear semigroups in order to prove the existence and uniqueness of solutions for the non-linear partial differential equation defined in Hilbert Spaces and derived from the general porous medium analysis. Such an equation is used in well test analysis in petroleum reservoir engineering for the determination of the properties of the reservoir materials. Hence, by the new method is estimated the size of the oil reserves after their exploration. Additionally, the existence and uniqueness of solutions for the non-linear porous medium equation is proved, by presenting some general boundary conditions. Finally, some properties of the solutions for the above non-linear partial differential equation are finally proved.

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Key Word and Phrases
Non-linear Semigroups, Hilbert Spaces, Non-linear Partial Differential Equation, Porous Medium Equation, Petroleum Reservoir Engineering, Well Test Analysis.

1. Introduction

Over the past years an increasing interest was realized on studying non-linear semigroups in general Banach spaces associated with the existence and uniqueness theory of partial differential equations arising in a big level of problems of mathematical physics and engineering. Hence, the study of the non-linear semigroups was derived directly from the examination of non-linear parabolic equations and from various non-linear boundary value problems.

As a beginning the first work on semigroups was published by A.V.Balakrishnan [1], when studying fractional powers of closed operators. Besides, some years later T.Kato [2] studied non-linear semigroups in connection with evolution equations, while Y.Komura [3], [4] studied non-linear semigroups defined in Hilbert spaces.

On the other hand, K.Sato [5] investigated non-negative contraction semigroups in Banach spaces, while a general theory of non-linear semigroups was investigated by M.G. Crandall et al. [6] - [8]. Additionally, J.Watanabe [9], [10] studied semigroups of non-linear operators on closed convex sets and H.Brezis et al. [11], [12] introduced a general semigroups formulation.

At the same time, M.Iannelli [13] studied non-linear semigroups on cones of a non-reflexive Banach space, while J.Mermin [14] and S.Oharu [15] investigated general theories of non-linear semigroups. Furthermore, I.Miyadera [16] studied semigroups of non-linear operators and B.K.Quinn [17] investigated semigroups in L_1 spaces.

Besides, Y.Konishi [18] studied non-linear semigroups associated with some partial differential equations and U.Westphal [19] and S.Aizawa [20] investigated some formulations for non-linear semigroups. On the contrary, T.Kurtz [21] studied semigroups of non-linear operators applied to gas kinetics, while R.Bruck [22] investigated asymptotic convergence of non-linear contraction semigroups in Hilbert spaces.

The theory of non-linear semigroups was generated by Y.Kobayashi [23], [24] and a monograph on the above subject was written by V. Barbu [25]. Also, J.M. Ball [26] studied strongly continuous semigroups, while B.C.Burch [27] investigated a semigroup treatment of the Hamilton - Jacobi equations in several space variables.

At the same time, J.H.Lightbourne and R.H.Martin [28] investigated relatively continuous perturbations of analytic semigroups, when A.T.Plant [29] studied non-linear semigroups of translations in Banach spaces generated by functional differential equations.

Beyond the above, the theory of non-linear semigroups on general Banach spaces was further investigated by J.B.Baillon [30] and A.Pazy [31], [32], while J.A.Goldstein [33] wrote a monograph on semigroups of linear operators with some general applications. Besides, a monograph on non-linear evolution operators and semigroups was written by N.H.Pavel [34].

By the present research the non-linear semigroups are used in order to prove the existence and uniqueness of solutions for the non-linear partial differential equation defined in Hilbert spaces. This differential equation is derived from the general theory of porous medium analysis. The porous medium equation was recently used by E.G.Ladopoulos [35] - [37] in well test analysis in the petroleum reservoir engineering for the determination of the properties of the reservoir materials. The above theory together with *"Non-linear Real-time Expert Seismology"* was used for the exploration of on-shore and off-shore oil and gas reserves, as an extension of the non-linear theories investigated by E.G.Ladopoulos et al. during the last two decades. [38] - [49].

Besides, by the current investigation the existence and uniqueness of solutions for the non-linear porous medium equation is studied, by using a method of non-linear semigroups. Finally, some properties of the solution for the above non-linear differential equation are proved.

2. Non-linear Porous Medium Investigation

Theorem 2.1

Suppose that oil flows through a porous medium, that occupies the domain *B,* which is bounded in R^3. Denote by $u=u(x,t)$ the density of the oil and by $p=p(x,t)$ its pressure at the point $x=(x_1,x_2,x_3)\in B$ at time *t.* (Fig.1)

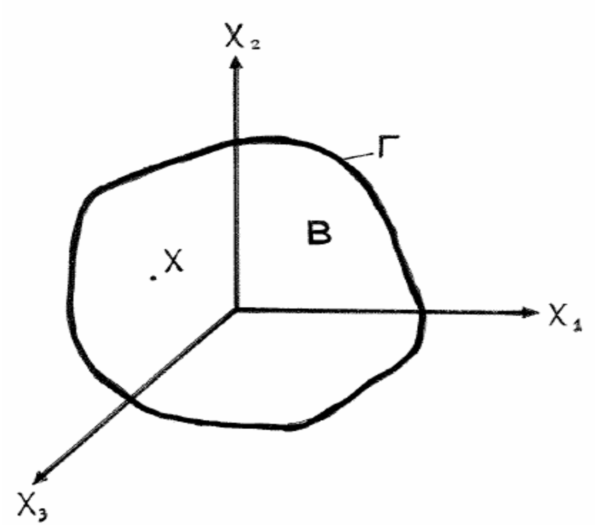

Fig. 1 A bounded domain B in R^3 with small boundary Γ inside which flows a gas with density $u=u(x,t)$ and pressure $p=p(x,t)$ at the point $x=(x_1,x_2,x_3)\in B$ at time *t.*

Then, the porous medium equation is equal to:

$$\frac{\partial u(x,t)}{\partial t}=\mu\nabla^2\left[u(t,x)\right]^m \tag{2.1}$$

where ∇^2 denotes the Laplace operator:

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\nabla x_3^2} \tag{2.2}$$

and μ is equal to:

$$\mu = a\lambda p_0 / \xi(a+1) \tag{2.3}$$

with p_0 a constant, $a \geq 1, \xi > 0$ the viscosity of the medium, λ>0 its permeability and $m = a + 1 \geq 2$.

Proof
We have:

$$p = p_0 u^a \tag{2.4}$$

where p_0 denotes a constant and $a \geq 1$.

Besides, if $v = (v, t, u)$ denotes the velocity of the oil, then Darcy's law gives:

$$\xi \mathbf{v} = -\lambda \nabla p \tag{2.5}$$

where ξ denotes the viscosity and λ the permeability of the medium.

Hence, by combining eqs (2.4) and (2.5) we obtain:

$$\xi \mathbf{v} = -\lambda \nabla p_0 u^a \tag{2.6}$$

Additionally, the dynamic of gas is given by the following conservation law:

$$f \frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{v}) = 0 \tag{2.7}$$

in which f denotes the porosity of the solid, $0 < f < 1$.

Furthermore, it is well known that:

$$\nabla \cdot (u \nabla u^a) = \frac{a}{a+1} \nabla^2 u^{a+1} \tag{2.8}$$

Hence, by combining eqs (2.4), (2.5), (2.6), (2.7) and (2.8) we obtain the required formula (2.1).

3. Non-linear Semigroups used for the Existence and Uniqueness Theorems for Non-linear Partial Differential Equations in Hilbert Spaces

Definition 3.1
Let *F* a nonempty subset of a Banach space *B*. Then, a semigroup on *F* is a function *S* on R_0^+ such that $S(t): F \to F$ for each $t \geq 0$ with the following properties:

$$S(0) = 1 \tag{3.1}$$

$$S(t+s) = S(t)S(s), \qquad t, s \geq 0 \tag{3.2}$$

$$\lim_{t \to 0} S(t)x = x, \qquad \forall x \in F \tag{3.3}$$

Theorem 3.1
Consider by B a bounded domain in R^3, with smooth boundary Γ and $u=u(x,t)$ the temperature function at the point $x=(x_1,x_2,x_3)\in B$ at time t. (Fig.1)

Then, the porous medium equation:

$$\frac{\partial u(x,t)}{\partial t}=\mu\nabla^2\left[u(t,x)\right]^m \quad \text{in} \quad]0,\ +\infty[\ ,\ x\in B \tag{3.4}$$

with the boundary conditions:

$$u(t,x)=0\ , \qquad \text{on} \quad]0,\ +\infty[\ ,\ x\in\Gamma \tag{3.5}$$

$$u(0,x)=u_0(x)\ , \quad \text{in}\ B \tag{3.6}$$

where μ is given by (2.3), $m\geq 2$ and $u_0\in L_2(B)$, has a unique solution $u(t,x)=(S(t)u_0)(x)$, $x\in B$, $t\geq 0$, $u\in C^n\{]0.\infty[\ ;\ L_2(B)\}$, $n=1,2,........$

Proof
As it is well known, the Laplace operator ∇^2 is self-adjoint in $L_2(B)$.

So, for $u,w\in D(\nabla^2)$ one has:

$$<\nabla^2 u,w>=\int_B(\nabla^2 u)wdx \tag{3.7}$$

which is equal to:

$$\int_B\{\nabla^2 u)wdx=-\int_B(\nabla u)(\nabla w)dx \tag{3.8}$$

and finally to:

$$-\int_B(\nabla u)(\nabla w)dx=\int_B u\nabla^2 wdx=<u,\nabla^2 w> \tag{3.9}$$

Consequently, from eqs (3.7), (3.8) and (3.9) one obtains:

$$<\nabla^2 u,w>=<u,\nabla^2 w> \tag{3.10}$$

from which follows that ∇^2 is symmetric and maximal monotone in L_2 and thus $\nabla^2=\nabla^{2^*}$ and $u(t)\in D(\nabla^2)^n$ $,n=0,1,2,.........., t>0$.

Thus, the solution $u(t,x)=(S(t)u_0)(x)$ has the property $u\in C^n\{]0,\infty[;\ D(\nabla^2)^r\}$ for every n, with $r=0,1,2,\ldots\ldots$ and finally $u\in C^n\{]0,\infty[;\ C^n(B)\}$, $n=0,1,2,.........$

Furthermore, as the solution $u(t,x)=(S(t)u_0)(x)$ tends in $L_2(B)$ to an equilibrium point, then finally follows that: $u\in C^n\{]0,\infty[;L_2(B)\}$, $n=1,2,........$

Theorem 3.2
Consider the non-linear partial differential porous medium equation (3.4) with the boundary conditions (3.5) and (3.6). Then, the solution $u(t,x)=(S(t)u_0)(x)$ of the above problem, with $u_0 \in L_2(B)$, satisfies the following properties:

$$\frac{1}{2}\|u(T)\|_{L_2}^2 + \int_0^T \|\nabla u(t)\|_{L_2}^2 dt = \frac{1}{2}\|u_0\|_{L_2}^2 , \quad \forall T>0 \tag{3.11}$$

and:
$$\|S(t)u_0\|_{L_p} \le \|u_0\|_{L_p} , \quad \forall u_0 \in L_p(B), \quad 1 \le p < \infty \tag{3.12}$$

Proof
Consider the following function:

$$g(t) = \frac{1}{2}\|u(t)\|_{L_2}^2 \tag{3.13}$$

Then, one has:

$$g'(t) = < u(t), u'(t) > = \int_B u \nabla^2 u dx \tag{3.14}$$

from which follows:

$$\int_B u \nabla^2 u dx = -\int_B \left| \nabla u(t,x)^2 \right| dx \tag{3.15}$$

Hence, by integrating over $[\varepsilon, T]$ with $0<\varepsilon<T$ and letting $\varepsilon \to 0$, then we obtain (3.11).

Besides, by multiplying (3.4) by $|u(t,x)|^{p-2} u(t,x)$ for the case m =1 and taking into account that:

$$\nabla(|u|^{p-2} u) = (p-1)|u|^{p-2} \nabla u \tag{3.16}$$

then, one has:

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_{L_p}^p = \int_B (\nabla^2 u) u |u|^{p-2} dx \tag{3.17}$$

which finally is equal to:

$$\int_B (\nabla^2 u) u |u|^{p-2} dx = -(p-1)\int_B \left|\nabla^2 u\right|^2 |u(t,x)|^{p-2} dx \le 0 \tag{3.18}$$

from which follows the required (3.12).

4. Conclusions

A general porous medium analysis was studied, by considering an oil which flows through a porous medium which occupies a closed domain, bounded in R^3. Such problem was reduced to the solution of a non-linear partial differential equation, under some general boundary conditions.

Consequently, the existence and uniqueness of solutions was proved, for the above non-linear porous medium equation defined in Hilbert Spaces, by using a technique based on non-linear semigroups. This method has clearly simplified the proofs of the existence and uniqueness theorems and has clarified the simpleness to other methods of mathematical analysis.

Finally, some properties for the solutions of the non-linear partial differential equations were proved, when these are defined in Hilbert Spaces. These properties were further generalized in L_p spaces, by proving a corresponding inequality for the solution of the non-linear porous medium equation.

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