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Risk Management Analysis by Non-linear Integro-differential Equations

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Abstract

The probability of an operational risk model, when given an initial reserve, is estimated by using an “innovative”modern method. The above problem is reduced to the solution of a non-linear Volterra partial integro-differential equation. Consequently, such a non-linear integro-differential equation is numerically evaluated by using Lagrange polynomials approximation solution. Also, the new model can be applied to the estimation of the risk for every risk business like hedge funds, bond loans, insurance companies, etc. Hence, risk analysis is a technique used to identify and assess factors that may jeopardize the success of a project or achieving a goal. This technique also helps to define preventive measures to reduce the probability of these factors from occurring and identify countermeasures to successfully deal with these constraints when they develop to avert possible negative effects on the competitiveness of the company.

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Key Word and Phrases

Non-linear Volterra Partial Integro-differential Equations, Lagrange Polynomials Approximation Method, Hedge Funds, Bond Loans, Risk Management Analysis, Risk Business.

1. Introduction

Generally, risk analysis is the process of defining and analyzing the dangers to individuals, businesses and government agencies posed by potential natural and human-caused adverse events. Besides, a risk analysis report can be used to align technology-related objectives with a company's business objectives. So, a risk analysis report can be either quantitative or qualitative. In quantitative risk analysis, an attempt is made to numerically determine the probabilities of various adverse events and the likely extent of the losses if a particular event takes place. Qualitative risk analysis, which is used more often, does not involve numerical probabilities or predictions of loss. Instead, the qualitative method involves defining the various threats, determining the extent of vulnerabilities and devising countermeasures should an attack occur.

Risk management analysis is the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and the impact of unfortunate eventsor to maximize the realization of opportunities. Beyond the above, risks can come from uncertainty in financial markets, threats from project failures (at any phase in design, development, production, or sustainment life-cycles), legal liabilities, credit risk, accidents, natural causes and disasters as well as deliberate attack from an adversary, or events of uncertain or unpredictable root-cause.

On the contrary, the strategies to manage threats (uncertainties with negative consequences) typically include transferring the threat to another party, avoiding the threat, reducing the negative effect or probability of the threat, or even accepting some or all of the potential or actual consequences of a particular threat, and the opposites for opportunities (uncertain future states with benefits).

Hence, risk analysis is the science of risks and their probability and evaluation. Furthermore, risk analysis should be performed as part of the risk management process for each project. The data of which would be based on risk discussion workshops to identify potential issues and risks ahead of time before these were to pose cost and schedule negative impacts. Consequently, the process of identifying risks, assessing risks and developing strategies to manage risks is known as risk management. A risk management plan and a business impact analysis are important parts of your business continuity plan. By understanding potential risks to your business and finding ways to minimize their impacts, you will help your business recover quickly if an incident occurs.

Types of risk vary from business to business, but preparing a risk management plan involves a common process. The risk management plan should detail company's strategy for dealing with risks specific to your business. It is important to allocate some time, budget and resources for preparing a risk management plan and a business impact analysis. This will help to meet the legal obligations for providing a safe workplace and can reduce the likelihood of an incident negatively impacting on the business.

Besides, the goal of risk management is to measure and assess risk, with the ultimate goal of managing that risk. Risk management falls into the arena of Project Planning. Over time, specific standards and methods have been developed with respect to risk management. These methods of analysis help those that practice risk management to use established ways of identifying risk. It also helps them manage risk by either avoiding it, transferring it, reducing the impact of the risk, or by various other alternative solutions. Risk management requires to identify potential risks; risk being anything that can possibly harm or have a negative impact on the project. Risk managers generally approach the search for potential risk from two distinct angles: source analysis and problem analysis. So, source analysis seeks to look at the potential sources of risk whereas problem analysis looks at specific individual problems that could arise.

Risk management analysis methods were proposed and investigated by several authors during the past years. [1]-[18] Such methods include Volterra Partial Integro-differential Equations models and several other integral equations methods. These methods include both analytical solutions for simpler cases of the integral equations, as well as computational methods for more complicated cases. Mostly the above numerical methods include polynomial approximations, which are usually the most suitable for the computational recipes of risk management analysis.

On the other hand, E.G.Ladopoulos [19] - [30] used non-linear integral equations methods for the solution of problems of fluid mechanics, aerodynamics, structural analysis and petroleum engineering. The above non-linear methods will be extended by the current research to the solution of risk management analysis problems.

Hence, a new method is presented for the estimation of the probability of an operational risk model, when given an initial reserve. Consequently, this problem is reduced to the solution of a non-linear Volterra partial integro-differential equation. The above non-linear integro-differential equation is further numerically evaluated by using Lagrange polynomials approximation solution. Then the new model can be applied to the estimation of the risk for every risk business like hedge funds, bond loans, insurance companies, etc.

2. Risk Management by Non-linear Volterra Partial Integro-differential Equations

Very important in modelling a risk business such as a hedge fund, or a bond loan is the problem of estimating the probability when an initial reserve exists. Then the following probability should be determined:

$$R(z,t) = P\left[Z(s) > 0,\ 0 \le s \le t,\ Z(0) = z\right] \tag{2.1}$$

in which $Z(s)$ denotes the risk reserve at time s and z is the initial risk reserve.

Beyond the above, the model obtained for the risk management business of a hedge fund, bond loan, etc., assumes the following relation:

$$\textbf{\textit{Risk Reserve}} = \textit{Initial Reserve} + \textit{Total Premiums} - \textit{Total Claims} \tag{2.2}$$

Claim sizes at time t which are denoted by X_t are assumed to have distribution function $D(x) = P\left[X_t \leq x\right]$ with corresponding density function $d(x)$ and are assumed to arrive according to a Poisson process N_t with parameter μ.

Consequently, we have:

$$p_n(t) = P\left[N_t = n\right] = \frac{e^{-\mu t}(\mu t)^n}{n!}\ , \quad n = 0,1,.... \tag{2.3}$$

So, the accumulated claims process is a compound Poisson process:

$$S_t = \sum_{i=1}^{N_t} X_{t_i} \tag{2.4}$$

It is also assumed that the premium is received at a continuous rate $a(r)$ which depends on the current reserve $Z(t) = r$.

Furthermore, in the absence of claims, it is assumed that the reserve satisfies the following deterministic equation:

$$\frac{dZ(t)}{dt} = a\left(Z(t)\right) \tag{2.5}$$

From the above described analysis follows that the problem is reduced to the solution of a non-linear Volterra partial integro-differential equation of the following form:

$$\frac{\partial R(z,t)}{\partial t} = -\mu R(z,t) + a(z)\frac{\partial R(z,t)}{\partial z} + \mu \int_0^z R(z-y,t)dD(y) \tag{2.6}$$

with the following conditions:

$$\begin{aligned} \lim_{z\to\infty} R(z,t) = 1, \quad R(z,0) = 1, \quad z \geq 0 \\ R(z,t) = 0, \quad z < 0 \end{aligned} \tag{2.7}$$

So, analytical solutions of the non-linear integro-differential equation (2.6) are not available and therefore the above equation has to be solved only by computational methods.

3. Computational Method by Lagrange Polynomials Approximation for Risk Management

In order to solve numerically the non-linear Volterra partial integro-differential equation (2.6)÷(2.7), let us rewrite the above non-linear integro-differential equation as following:

$$\frac{\partial R(z,t)}{\partial t} = -\mu R(z,t) + a(z)\frac{\partial R(z,t)}{\partial z} - \mu \int_0^z b(z-y)R(y,t)dy \tag{3.1}$$

$$R(\infty,t) = 1, \quad t \geq 0, \quad R(z,0) = 1, \quad z \geq 0 \tag{3.2}$$

For the numerical evaluation of the non-linear Volterra partial integro-differential equation (3.1) ÷(3.2), Lagrange polynomials approximations will be used.

Besides, for fixed *t,* the following approximation for the unknown function *R(z,t)* of the non-linear integro-differential equation (3.1) is used:

$$R(z,t) = \sum_{j=1}^{n+1} c_j(t)F_j(t) \ , \qquad t \geq 0, \quad 0 \leq z \leq \tilde{z} \tag{3.3}$$

$$F_j(z) = \prod_{k=1,\, k\neq j}^{n+1} \frac{(z - z_k)}{(z_j - z_k)} \ , \qquad 0 = z_1 < \ldots.. < z_n < z_{n+1} = \tilde{z} \tag{3.4}$$

The boundary condition $g(t) = R(\tilde{z}, t)$ was further found by using the true solution. Then since :

$$F_{n+1}(z_{n+1}) = 1 \tag{3.5}$$

the above boundary condition was used to express $c_{n+1}(t)$ in (2.10) in terms of $c_1(t), \ldots\ldots.., c_n(t)$, as following:

$$c_{n+1}(t) = g(t) - \sum_{j=1}^{n} c_j(t)F_j(\tilde{z}) \tag{3.6}$$

On the contrary, for practical use the proper value of $\tilde{z}$ has to be chosen in order to represent the infinity, so that the following boundary condition can be used instead:

$$\lim_{z\to\infty} R(z,t) = 1 \tag{3.7}$$

Beyond the above, inserting (3.6) in (3.3) follows:

$$R(z,t) = \sum_{j=1}^{n} c_j(t)F_j(z) + F_{n+1}(z)\left[g(t) - \sum_{j=1}^{n} c_j(t)F_j(\tilde{z}) \right] \tag{3.8}$$

which further can be written as:

$$R(z,t) = \sum_{j=1}^{n} c_j(t) P_j(t) + H(z,t) \tag{3.9}$$

in which:

$$\begin{aligned} P_j(z) &= F_j(z) - F_{n+1}(z) F_j(\tilde{z}) \;, \qquad j = 1,\dots\dots,n \\ H(z,t) &= g(t) F_{n+1}(z) \end{aligned} \tag{3.10}$$

Furthermore, as $F_j(z)$ is given by (3.4), then from (3.10) follows:

$$P_j(z) = F_j(z) \tag{3.11}$$

Then from (3.9) we have:

$$\frac{\partial R(z,t)}{\partial t} = \sum_{j=1}^{n} \frac{dc_j(t)}{dt} P_j(z) + \frac{\partial H(z,t)}{\partial t} \tag{3.12}$$

$$\frac{\partial R(z,t)}{\partial z} = \sum_{j=1}^{n} c_j(t) \frac{dP_j(t)}{dz} + \frac{\partial H(z,t)}{\partial z} \tag{3.13}$$

Besides, by replacing (3.9), (3.12) and (3.13) in (3.1) follows that the coefficients $c_j(t)$ in (3.9) satisfy a first order system of ordinary differential equations of the form:

$$\frac{d\mathbf{c}(t)}{dt} = S^{-1} M \mathbf{c} + S^{-1} \mathbf{b}(t) \tag{3.14}$$

where *S,M* denote *n* x *n* matrices with elements:

$$\begin{aligned} S_{kj} &= P_j(z_k), \qquad k, j = 1,2,\dots\dots,n \\ M_{kj} &= -\mu P_j(z_k) + a(z_k) \frac{dP_j(z_k)}{dz} + \mu J_j(z_k) \end{aligned} \tag{3.15}$$

Also, $\mathbf{b}(t)$ is given by the relations:

$$b_k(t) = -\frac{\partial H(z_k,t)}{\partial t} - \mu H(z_k,t) + a(z_k)\frac{\partial H(z_k,t)}{\partial z} + \mu\, I(z_k,t) \tag{3.16}$$

in which $J_j(z_k)$, $I(z_k,t)$ are approximations to the following integrals by quadrature rules:

$$\int_0^{z_k} b(z_k - y)P_j(y)dy \;,\; \int_0^{z_k} b(z_k - y)H(y,t)dy \tag{3.17}$$

and $P_j(z)$, $H(z,t)$ are given by (3.10).

Consequently, in order the ordinary differential equations (3.14) some staring values of **a**(0). Consequently, the requested values are found by using the second of the conditions (3.2) for R and eqn (3.9).

Thus, the following linear system of equations in $c_j(0)$, $j = 1,.......,n$ is obtained:

$$\sum_{j=1}^{n} c_j(0)P_j(z_k) = 1 - H(z_k,0) \;,\; k = 1,2,..........,n \tag{3.18}$$

Then after solving the system of ordinary differential equations (3.14), the function $R(z,t)$ is numerically evaluated by (3.9).

4. Conclusions

A modern method has been presented for the estimation of the probability of an operational risk model, when given an initial reserve. The above problem has been reduced to the solution of a non-linear Volterra partial integro-differential equation. Hence, the above non-linear integro-differential equation was numerically evaluated by using Lagrange polynomials approximation solution. So, the new model can be applied to the estimation of the risk for every risk business like hedge funds, bond loans, insurance companies, etc.

Risk management analysis is very helpful in examining the risks and following a well planned process to hedge the risk. At the same time, the effectiveness of the process and the financial factors related to the process are also discussed through this analysis. The business sector always faces some kind of risk. So, the risk management initiatives are becoming all the more important with the growing competition in the global market. In the highly competitive global market there is hardly any scope to afford any kind of loss. As a result of this, the concept of risk management has gained considerable importance over the passage of time.

The risk management analysis is very important for proper application of the risk management policies. This analysis is necessary because the demand of the market and the trends are changing constantly and only proper analysis of risks can help the businesses to achieve the set targets.

Beyond the above, the risk management process refers to the different types of methods and procedures that are utilized for risk management. The process of risk management involves a number of steps and it helps in the betterment in decision-making on a continuous basis. Hence, Risk management is a function that incorporates identification of risk, evaluation of risk, formulation of schemes to handle risks and reduction or elimination of risk utilizing a number of methods. Risk management process is an important task for the managers of an organization.

A risk management strategy delineates in what manner the risks are going to be handled. Risk management strategy acts as a major device for the higher management of a company because with the formulation of a risk management strategy, a number of risks can be averted easily. Thus, Risk management strategy delineates a technique for analyzing and handling various types of risks. The principles of risk management can be applicable for a number of situations. The risk strategy and the plan, which is backing it up should recognize the real and probable threats to the productive delivery of a project and ascertain the functions that are necessary to reduce or get rid of the risks.

Finally, Risk Management Policy devices a back up plan for mitigating the negative effects arising out of unforeseen events. Risks can sometimes be partially phased out to capable third parties. This is done through waivers, insurance policies and contracts. The remaining risk profile needs to be strategically managed.

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