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Planar Airfoils in Two-dimensional Aerodynamics by Non-linear Multidimensional Singular Integral Equations

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Abstract
Planar airfoils are studied for the unsteady flow motion in two-dimensional aerodynamics. Such problems are reduced to the solution of a non-linear multidimensional singular integral equation, when the form of the source and vortex strength distribution is dependent on the history of these distributions on the NACA airfoil surface. A turbulent boundary layer model is also proposed, based on the formulation of the unsteady behavior of the momentum integral equation. Finally, an application is given to the determination of the velocity and pressure coefficient field around an aircraft by assuming constant vortex distribution.

Key Word and Phrases
Non-linear Aerodynamics, Two-dimensional NACA airfoil, Non-linear Multidimensional Singular Integral Equations, Constant Vortex Distribution, Aircraft, Velocity & Pressure Coefficient Field.

1. Introduction

Over the last years a continuously increasing interest has been given to the non-linear singular integral equations by which are solved very important problems of aerodynamics and fluid mechanics, especially these referred to unsteady flows. The computational methods which are used for the numerical evaluation of the non-linear singular integral equations consist of the latest high technology to the solution of general problems of solid and fluid mechanics. For this reason such computational methods are continuously improved.

The aerodynamic characteristics of the NACA airfoils are too important for the design of the new generation aircrafts, with very high speeds. This new technology aerodynamic problems are therefore reduced to the solution of non-linear singular integral equations, which are used for the determination of the velocity and pressure coefficient field around the NACA airfoils. Hence, special attention should be concentrated to such computational methods used for the solution of the above mentioned aerodynamic and fluid mechanics problems of unsteady flows.

The first scientists who investigated aerodynamic panel methods for studying airfoils with zero lift, were A.M.O.Smith and J.L.Hess [1]. They modeled the airfoil with either distributed potential source panels for nonlifting flows, or vortex panels for flow with lift. This method was further extended by R.H.Djojodihardjo and S.E.Widnall [2], P.E.Robert and G.R.Saaris [3], J.M.Summa [4], D.R.Bristow [5], D.R.Bristow and J.D.Hawk [6] and R.J.Lewis [7], when studying three-dimensional steady and unsteady flows, by combining source and vortex singularities. The unsteady panel methods were also extended to the modeling of separated wakes using discrete vortices, by T.Sarpkaya and R.L.Schoaf [8].

Beyond the above, N.D.Ham [9], F.D.Deffenbaugh and F.J.Marschall [10], M.Kiya and M.Arie [11] and T.Sarpkaya and H.K.Kline [12] studied some other potential flow models, and the separating boundary layers were represented by an array of discrete vortices, emanating from a known separation point location on the airfoil surface.

Over the last years, several other scientists made extensive calculations by using unsteady turbulent boundary layer methods. Among them we shall mention: R.E.Singleton and J.F.Nash [13], J.F.Nash, L.W.Carr and R.E.Singleton [14], A.A.Lyrio, J.H.Ferzinger and S.J.Kline [15], W.J.McCroskey and S.I.Pucci [16] and J.Kim, S.J.Kline and J.P.Johnston [17].

Non-linear singular integral equation methods were recently proposed by E.G.Ladopoulos [18] - [22] for the solution of fluid mechanics problems and by E.G.Ladopoulos and V.A.Zisis [23], [24] for two-dimensional fluid mechanics problems applied to turbomachines.

By the current research, the aerodynamic problem of the unsteady flow of a two-dimensional NACA airfoil which is moving by a velocity U_A, is reduced to the solution of a non-linear multidimensional singular integral equation. Such a nonlinearity is valid, because the source and vortex strength distribution are dependent on the history of the vorticity and source distribution on the NACA airfoil surface. A turbulent boundary layer model is further proposed, based on the formulation of the unsteady behavior of the momentum integral equation.

An application is finally given to the determination of the velocity and pressure coefficient field around an aircraft by assuming constant vortex distribution.

2. Non-linear Unsteady Aerodynamics and Fluid Dynamics

A general non-linear unsteady aerodynamics and fluid dynamics representation analysis is investigated, for the unsteady flow of a two-dimensional NACA airfoil. The method presented consists to the generalization of all past methods, by reducing the problem to the solution of a non-linear multidimensional singular integral equation. Hence, such a nonlinearity results because of the general form given to the source and vortex strength distribution, as these functions are dependent on the history of the vorticity and source distribution on the NACA airfoil surface. In this case the airfoil is moving with a speed U_A.

Consider a two-dimensional airfoil moving in an homogeneous and inviscid fluid. (Fig.1).

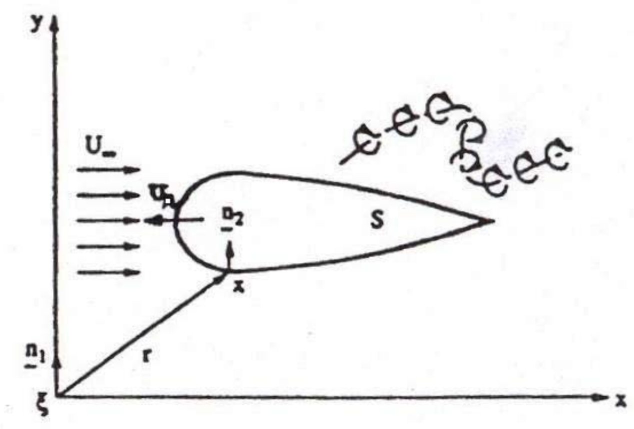

Fig. 1 A two-dimensional airfoil of surface S in an homogeneous and inviscid fluid.

A complete lifting system in an irrotational flow through the ideal fluid is comprised by the airfoil with the wake. Because of this irrotationality, then for the local fluid velocity **U** is valid:

$$\nabla \times \mathbf{U} = 0 \tag{2.1}$$

Moreover, by replacing the fluid velocity with the total velocity potential H we have:

$$\mathbf{U} = \nabla H \tag{2.2}$$

while (2.2) can be further written as:

$$\mathbf{U} = \mathbf{U}_\infty + \nabla h \tag{2.3}$$

with $\mathbf{U}_\infty$ the outward velocity (Fig. 1) and h the potential due to the presence of the airfoil.

Furthermore, by using Green's theorem [25] follows a basic relation for the velocity potential $h(\mathbf{x},t)$, with t the time, at any point $\mathbf{x}$ in continuous, acyclic irrotational flow:

$$h(\mathbf{x},t) = -1/2\pi \int_S \frac{\lambda[\xi,t,h]}{r}\,\mathrm{d}S + 1/2\pi \int_{S+W} \delta[\xi,t,h]\frac{\partial}{\partial n_1}\left(\frac{1}{r}\right)\mathrm{d}S \tag{2.4}$$

in which S is the surface of the airfoil (Fig. 1), W the surface of the wake, $\mathbf{n_1}$ the surface normal at the source point ξ (Fig. 1), $\lambda[\xi,t,h]$ the source strength distribution, $\delta[\xi,t,h]$ the vortex strength distribution and r the distance equal to:

$$r = |\mathbf{x} - \xi| \tag{2.5}$$

The velocity potential (2.4) can be also written as following, which denotes a two-dimensional non-linear singular integral equation:

$$h(\mathbf{x},t) = -1/2\pi \int_S \frac{\lambda[\xi,t,h]}{r}\,\mathrm{d}S + 1/2\pi \int_{S+W} \frac{\delta[\xi,t,h]}{r^2}\,\mathrm{d}S \tag{2.6}$$

The kinematical surface tangency condition on the surface of the airfoil can be written as following: [26]

$$\left(1/|\nabla S(\mathbf{x},t)|\right)\frac{\partial S(\mathbf{x},t)}{\partial t} + \frac{\partial h}{\partial n_2} + \mathbf{U}_\infty \cdot \mathbf{n}_2 = 0 \tag{2.7}$$

in which $\mathbf{n}_2$ denotes the surface normal at the field point $\mathbf{x}$ (Fig. 1).

The above condition can be further written in the following form, for a body fixed coordinate system:

$$\left(1/|\nabla S(\mathbf{x},t)|\right)\frac{\partial S(\mathbf{x},t)}{\partial t} = -\left(\mathbf{U}_A + \boldsymbol{\omega}_\mathbf{A} \times \mathbf{x}\right)\cdot \mathbf{n}_2 \tag{2.8}$$

where $\mathbf{U}_A$ denotes the airfoil translation velocity and $\boldsymbol{\omega}_A$ the airfoil angular rotation.

From eqs (2.7) and (2.8) follows:

$$\frac{\partial h}{\partial n_2} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \mathbf{n}_2 = 0 \tag{2.9}$$

Furthermore, by inserting (2.9) into (2.6) results the following two-dimensional non-linear singular integral equation:

$$\begin{gathered} 1/2\pi \int_S \lambda[\xi, t, h] \frac{\partial}{\partial n_2}\left(\frac{1}{r}\right) \mathrm{d}S + 1/2\pi \int_{S+W} \delta[\xi, t, h] \frac{\partial}{\partial n_2}\left(\frac{1}{r^2}\right) \mathrm{d}S = \\ -\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \mathbf{n}_2 \end{gathered} \tag{2.10}$$

The non-linear singular integral equation (2.10) can be further written as:

$$\begin{gathered} 1/2\pi \int_S \frac{\lambda[\xi, t, h]}{r^2} \mathrm{d}S + 1/\pi \int_{S+W} \frac{\delta[\xi, t, h]}{r^3} \mathrm{d}S = \\ \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_\mathbf{A} \times \mathbf{x}\right) \cdot \mathbf{n}_\mathbf{2} \end{gathered} \tag{2.11}$$

Hence, by solving the non-linear integral equation (2.11) with the corresponding boundary conditions, then the velocity at any field point will be determined through (2.7).

3. Non-linear Pressure Analysis

The pressure distribution on the airfoil may be obtained by the unsteady Bernoulli equation, valid at any point in an irrotational, ideal flow:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + 1/2\left(\nabla H\right)^2\right] \tag{3.1}$$

in which ρ denotes the fluid density.

Moreover, by using the derivation of the previous section, then (3.1) will be written as:

$$P = P_\infty - \rho\left[\frac{\partial h}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right) \cdot \nabla h + 1/2\left(\nabla h\right)^2\right] \tag{3.2}$$

Also, (3.2) reduces to the following form:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla_S H + \right. \\ \left. + \frac{\partial H}{\partial n_1}\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \mathbf{n}_1 + 1/2\left(\nabla_S H\right)^2 + 1/2\left(\frac{\partial H}{\partial n_1}\right)^2\right] \tag{3.3}$$

if we replace the ∇f , by the surface gradient $\nabla_S h$:

$$\nabla h = \nabla_S h + \frac{\partial h}{\partial n_1}\varepsilon_{n_1} \tag{3.4}$$

Therefore, because of (2.9), then (3.3) can be written as:

$$P = P_\infty - \rho\left[\frac{\partial H}{\partial t} + \left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \nabla_S H - \right. \\ \left. - 1/2\left\{\left(\mathbf{U}_\infty - \mathbf{U}_A - \boldsymbol{\omega}_A \times \mathbf{x}\right)\cdot \mathbf{n}_1\right\}^2 + 1/2\left(\nabla_S H\right)^2\right] \tag{3.5}$$

which will be used for the computations.

4. Turbulent Boundary Layer Models

There are several boundary layer models which can be used in order to determine the aerodynamic behavior of the airfoils, for the laminar and turbulent flow, as well as the transition region between them. Such boundary layer models are the finite difference, finite element or integral models.

The turbulent boundary layer model which is proposed by the present research is based on the formulation of the unsteady behavior of the momentum integral equation [15]. Furthermore, the formulation for the laminar portion of the boundary layer is based on Thwaites method [27]. The major extension of the above method by the present research is the inclusion of unsteady terms in the momentum integral equation.

The unsteady momentum integral equation, which is valid for both laminar and turbulent flow can be therefore written as: (Fig. 2)

$$\frac{1}{u_B^2}\frac{\partial}{\partial t}(u_B\delta) + \frac{\partial d}{\partial S} + \frac{1}{u_B}\frac{\partial u_B}{\partial S}(2d + S) = \frac{c_F}{2} \tag{4.1}$$

in which u_B is the boundary layer edge velocity, t the time, δ the displacement thickness, d the momentum thickness, S the surface distance and c_F the friction factor.

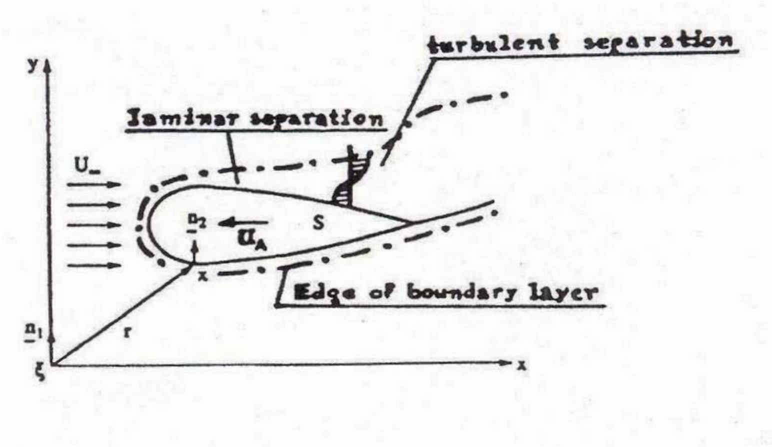

Fig. 2 Laminar and Turbulent Boundary Layer Model for Aerodynamics.

Considering firstly the case for the laminar layer, then the pressure gradient parameter μ is given by the relation:

$$\mu = \frac{d}{u_B} R_d \left(\frac{\partial u_B}{\partial S} + \frac{1}{u_B}\frac{\partial u_B}{\partial t}\right) \tag{4.2}$$

where R_d is the Reynolds number based on u_B and d.

Furthermore, by considering some special relations between the parameters $c_F/2$, d and δ, then a solution for the laminar formulation may be obtained. For the wedge flow solutions following relations are valid: [27]

$$\begin{aligned} \frac{c_F}{2} &= \frac{1.91 - 4.13A}{R_\delta} \\ M &= (0.68 - 0.922A)^{-1} \\ B &= 0.325 - 0.13\varphi M^2 \end{aligned} \tag{4.3}$$

in which M is the shape parameter, B the blockage factor δ / δ_B, with δ_B the boundary layer thickness and R_δ the Reynolds number based on u_δ and δ.

On the other hand, for the turbulent layer model following formula is valid:

$$\frac{1}{u_B}\frac{\partial}{\partial S}[u_B(\delta_B - \delta)] = E \tag{4.4}$$

and the function E is obtained as following:

$$\begin{aligned}
&\frac{\mathrm{d}E}{\mathrm{d}S} = 0.025(E_B - E)\delta_B \\
&E_B = 4.24 K_B \left(\tfrac{B}{1-B}\right)^{0.916} \\
&K_B = 0.013 + 0.0038 e^{-\beta/15} \\
&\beta = \frac{\delta}{\tau_w}\frac{\mathrm{d}P}{\mathrm{d}x}
\end{aligned} \tag{4.5}$$

where τ_w is the wall shear stress and dp/dx the streamwise pressure gradient.

Also the shape factor relationships are obtained by following relations:

$$\begin{aligned}
&\frac{u}{u_B} = 1 + \lambda \ln\left(\frac{y}{\delta_B}\right) - J\cos^2\left(\frac{\pi y}{2\delta_B}\right) \\
&\lambda = \frac{1}{0.41}\left(\mathrm{sgn}\frac{c_F}{2}\right)\left(\frac{c_F}{2}\right)^{1/2} \\
&J = 2(1-\lambda) \\
&\frac{c_F}{2} = \frac{\tau_w}{\rho {u_B}^2}
\end{aligned} \tag{4.6}$$

with u the velocity in the boundary layer at a distance y from the wall and ρ the fluid density.

Finally, the skin friction law is valid as:

$$\frac{c_F}{2} = 0.051\left|1-2B\right|^{1.732}\left(\frac{R_\delta}{B}\right)^{-0.268} \mathrm{sgn}(1-2B) \tag{4.7}$$

Additional details concerning the entrainment, the wall shear stress and the skink friction relations can be found in [15].

5. Velocity and Pressure Coefficient Field for Constant Vortex Distribution (Airfoil with Velocity)

Consider the special case of a constant vortex distribution δ. Then the non-linear problem presented in previous paragraphs, is greatly simplified and is solved as a linear problem. Also, in Fig. 3 is shown the geometrical representation of the above problem.

For constant vortex distribution δ, then the fluid velocity **U**, is determined by the formula:

$$\mathbf{U} = \int_{-A/2}^{A/2} \frac{\delta\,\mathrm{d}r}{2\pi r}\left(-\sin\varphi\mathbf{i} + cos\varphi\mathbf{j}\right) \tag{5.1}$$

where A is the separating wake (Fig. 3) and $\mathbf{i}$, $\mathbf{j}$ are the unit vectors on the x and y axes, respectively.

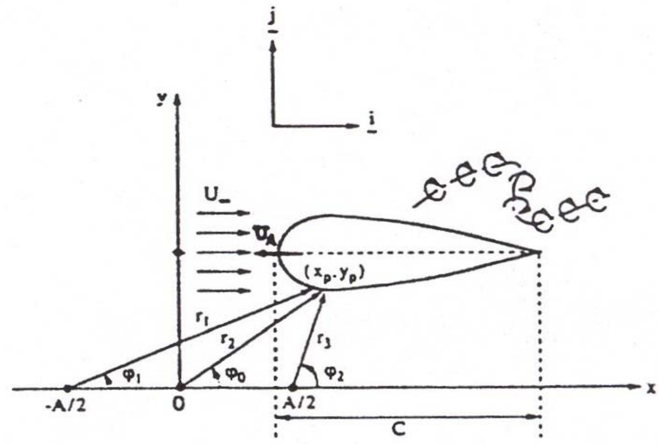

Fig. 3 Coordinate system for the 2D airfoil of an aircraft.

Hence, when $y_p \neq 0$ and $y_p = 0$, then the fluid velocity $\mathbf{U}$ will be computed by the following formulas:

$$\mathbf{U} = \begin{cases} \delta/2\pi\left[(\varphi_1 - \varphi_2)\mathbf{i} + \ln\left|\frac{r_1}{r_2}\right|\mathbf{j}\right], & y_p \neq 0 \\ \delta/2\pi \ln\left|\frac{r_1}{r_2}\right|\mathbf{j}, & y_p = 0 \end{cases} \tag{5.2}$$

Moreover, we consider the pressure coefficient C_p:

$$C_p = (P - P_\infty)/[1/2\,\rho(U_\infty - U_A)^2] \tag{5.3}$$

in which ρ denotes the fluid density and P_∞ the stream pressure.

By using further the unsteady equation of Bernoulli, then the pressure coefficient will be simplified through the relation:

$$C_p = -U^2/(U_\infty - U_A)^2 \tag{5.4}$$

which will be used for the computations.

6. Unsteady Aerodynamics for Aircraft Application

The analytical theory of 2-D unsteady inviscid flowfields will be applied for the computation of the velocity and pressure coefficient field around an aircraft. The application of new generation turbojet engines makes possible the design of very fast big jets. Beyond the above, the increasing evolution of aeroelasticity in aircraft turbomachines continues to be under active investigation, driven by the needs of aircraft powerplant and turbine designers. The target of Aeronautical Industries is therefore to achieve a competitive technological advantage in certain strategic areas of new and rapidly developing advanced technologies, by which increased market share can be achieved, in the medium and longer terms. This considerably big market share consists to the design of new generation large aircrafts with very high speeds.

In the present application the length of the aircraft under consideration is c=50.0m and the airfoil section NACA 0021 (Fig. 3).

Beyond the above, it was supposed unit vortex distribution and hence, the velocity field on the boundary and around of the airfoil was computed by (5.2). Also, the pressure coefficients C_p were calculated through (5.4) for several aircraft velocities U_A and wind velocity $U_\infty = 15m/sec$.

Figures 4, 5, 6 and 7 show the pressure distribution on the turbojet presented, for aircraft speeds $U_A = 1,2,3,4$ Mach respectively (1 Mach=332 m/sec). Also, Figs. 4a to 7a show the same pressure distribution on the airfoil, in three dimensional form.

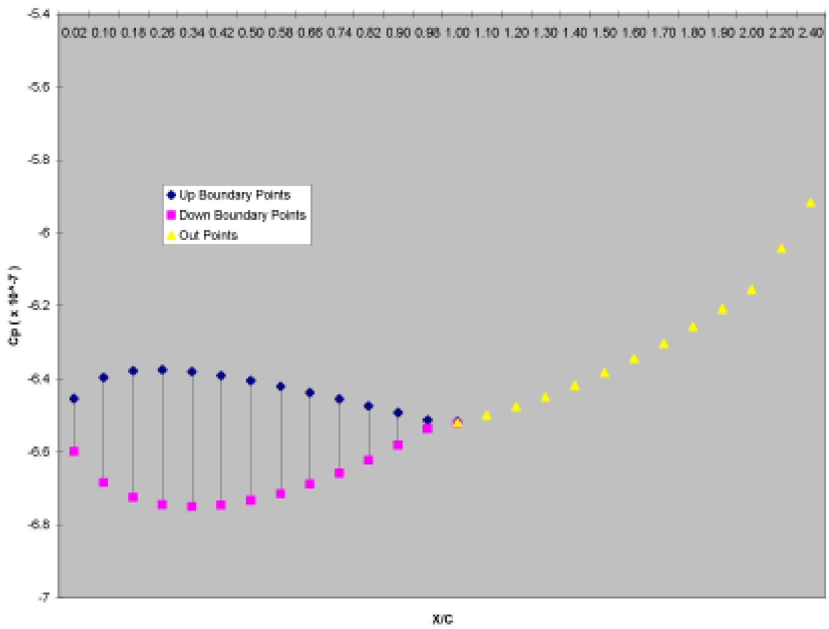

Fig. 4 Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 1 Mach.

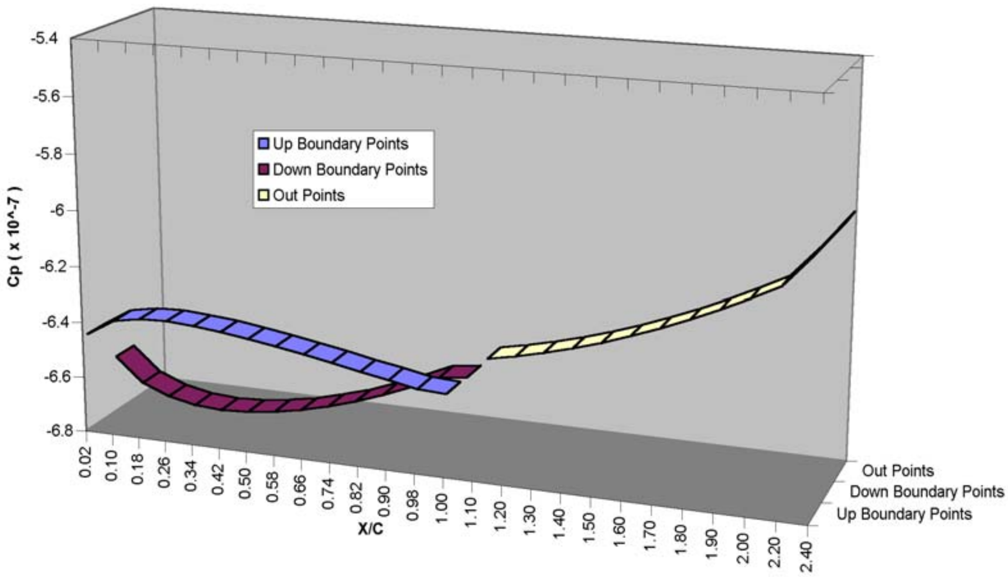

Fig. 4a Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 1 Mach – 3D form.

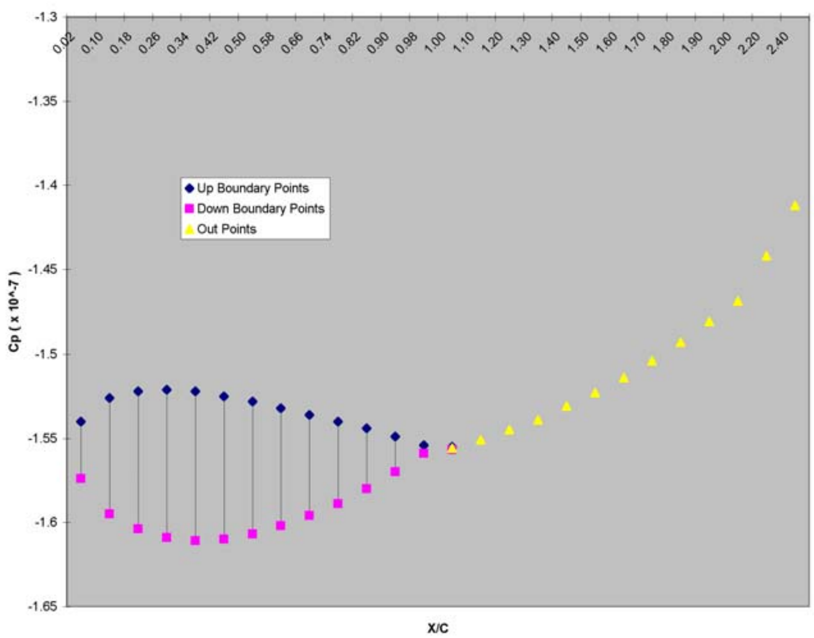

Fig. 5 Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 2 Mach.

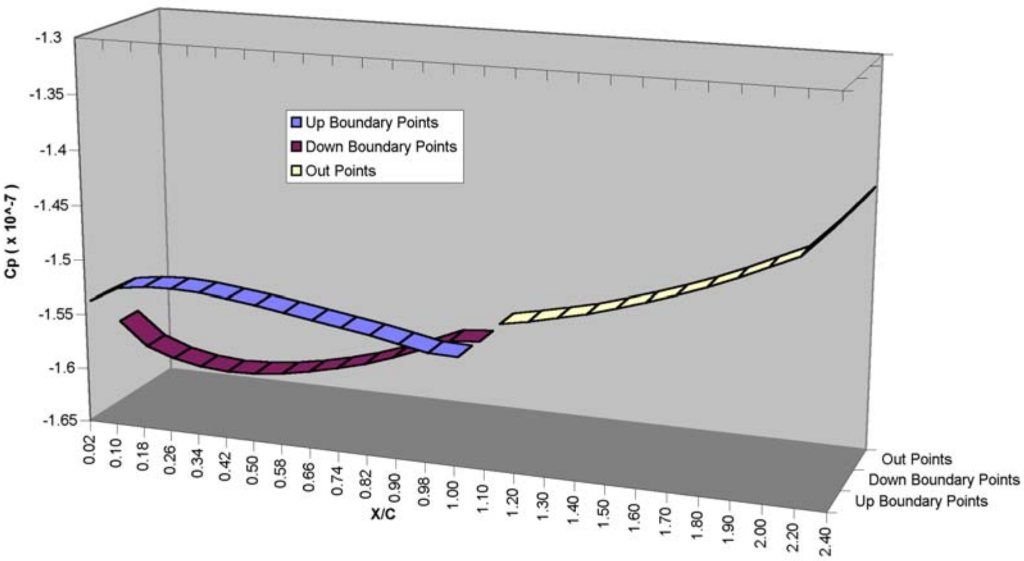

Fig. 5a :Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 2 Mach – 3D form.

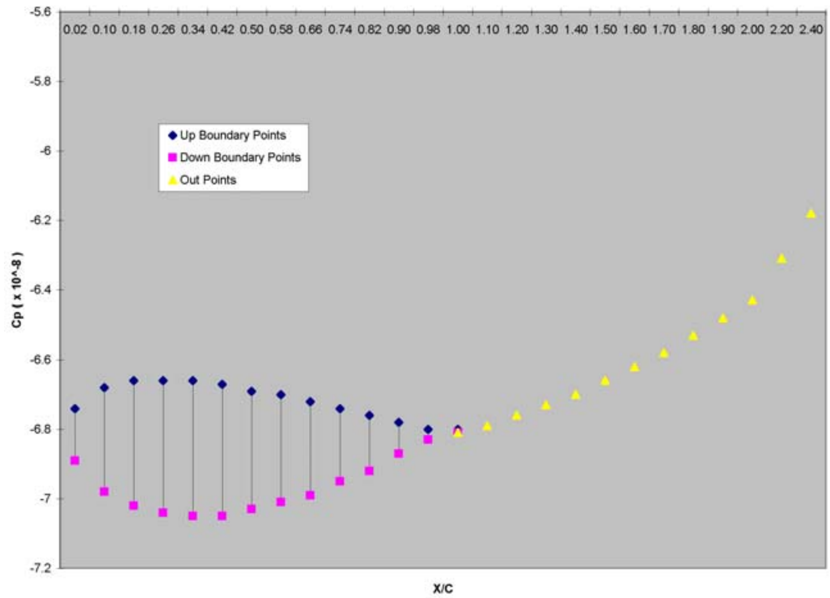

Fig. 6 Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 3 Mach.

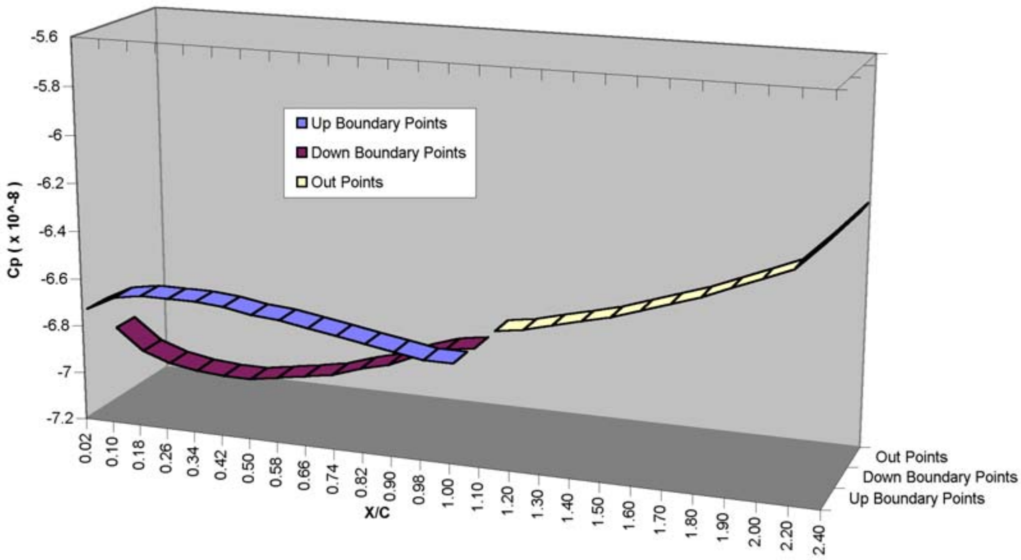

Fig. 6a Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 3 Mach – 3D form.

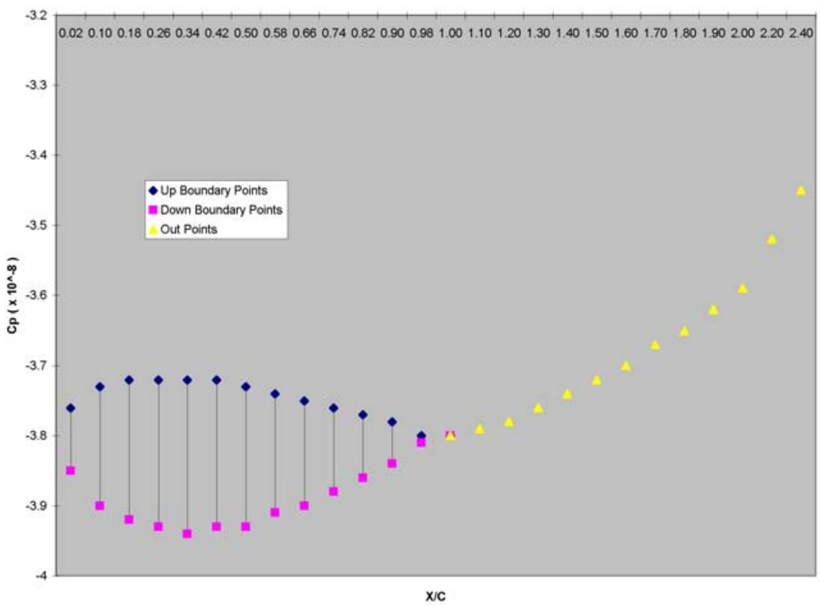

Fig. 7 Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 4 Mach.

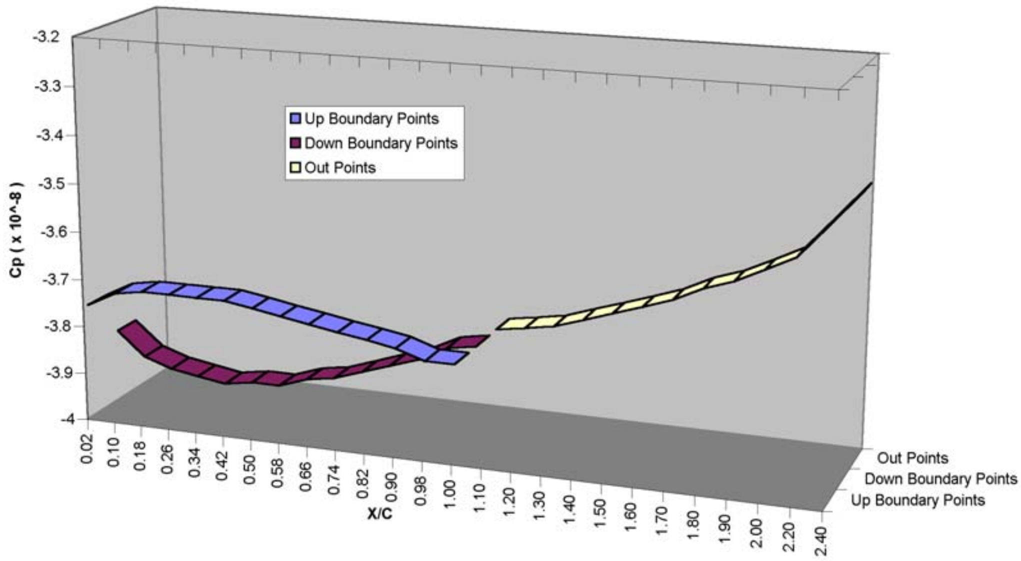

Fig. 7a Pressure distribution around the aircraft of Fig.3, for constant vortex distribution and speed 4 Mach – 3D form.

From the above Figures, it is shown, that for the up boundary points of the airfoil the values of the pressure coefficient are increasing approximately up to x/c = 0.25, while then decreasing again up to x/c =1. On the other hand, for the down boundary points the values of the coefficient are decreasing up to x/c = 0.35, and then increasing again.

7. Conclusions

By the present research a non-linear model has been proposed for the determination of the velocity and pressure coefficient field around a NACA airfoil moving by a velocity U_A in three-dimensional unsteady flow. This problem was reduced to the solution of a non-linear multidimensional singular integral equation, for which closed form solutions are not possible to be determined and hence, has to be sold only by computational methods. Such a nonlinearity resulted because the source and vortex strength distributions are dependent on their history on the NACA airfoil surface.

A boundary layer model was proposed based on the formulation of the unsteady behavior of the momentum integral equation. Such a boundary layer model is valid for both laminar and turbulent flow, and was proposed as a general method for the study of the aerodynamic behavior of the airfoils.

Beyond the above, the velocity and pressure coefficient field around an aircraft moving with several velocities, was determined for constant vortex distribution. Such a method will be applied for the design of new generation large aircrafts with very high speeds.

The non-linear singular integral equation methods, as proposed in the present research, will be in future of continuously increasing interest for the solution of generalized solid and fluid mechanics problems. Hence, special attention should be given to the improvement of the non-linear singular integral equation methods, as many modern problems of aerodynamics and structural analysis with considerable complicated forms, are recently reduced to non-linear forms.

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