Universal Journal of Aerodynamics **1 (2013), 1-14**
www.papersciences.com

Two-dimensional Aerodynamics by Finite-Part Singular Integro-Differential Equations

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Abstract
An innovative and groundbreaking method is proposed for the solution of the finite-part singular integro-differential equations, applied in several areas of engineering mechanics and mathematical physics and especially in fluid dynamics and aerodynamics. The method consists in the reduction to a system of linear equations, by applying the singular integro–differential equation at properly selected collocation points. Beyond the above, an application is given to the numerical solution of the generalized airfoil equation, which presents the pressure acting on a planar airfoil undergoing simple amplitude oscillations about the central plane of a two-dimensional ventilated wind tunnel.

Key Word and Phrases
Finite-Part Singular Integro-differential Equation, Two-dimensional Aerodynamics, Airfoil Equation, Planar Airfoil, Two-Dimensional Ventilated Wind Tunnel, Gauss – Chebyshev Integration Rule.

1. Introduction

Finite-part singular integro-differential equations are used in a major field of engineering mechanics and mathematical physics and mostly in fluid dynamics and aerodynamics. Such singular integral equations are evaluated only through computational methods because of the big complication of their form, as closed form solutions are available only in very seldom cases.

So, a very effective method for solving numerically such type of singular integro-differential equations is the direct method which consists in reducing such an equation (or systems of equations) to a system of linear algebraic equations, by using an appropriate numerical integration rule on a properly selected set of collocation points.

Several studies of the generalized two-dimensional aerodynamics began very early, in the 30's. Some general aerodynamic problems were studied by V.V.Golubev [1], T.von Karman and J.M.Burgers [2], H.Schmidt [3] and K.Schröder [4], [5].

During the 40's the two-dimensional aerodynamic problems were advanced by the work of J.Weissinger [6], H.Küssner and L.Schwarz [7], L.G.Magnaradze [9], [10], I.N.Vekua [11], [12] and H.Schöngen [13].

Furthermore, N.I.Muskhelishvili [14] investigated the integro-differential equation of the aircraft wings of finite span and V.V.Ivanov [15] obtained some new numerical methods for the evaluation of the integro-differential equations.

Over the past years several studies have been published on the application of the singular integral equations in aerodynamics. Among them we shall mention the following authors : S.R.Bland et al. [16], [17], J.Blackwell and G.Pounds [18], J.A.Fromme and M.A.Golberg [19] - [24], M.A.Golberg, M.Lea and G.Miel [25], M.A.Golberg [26], E.G.Ladopoulos [27], W.F.Moss [28], [29], D.J.Salmond [30], M.H.Williams [31], E.Kraft and C.Lo [32], M.Mokry [33] and E.Nissim and I.Lottati [34].

By the current research an innovative and groundbreaking method is proposed for the numerical solution of the finite-part singular integro-differential equation. Furthermore, the generalized airfoil equation is investigated and solved, which presents the pressure acting on a planar airfoil undergoing simple amplitude oscillations about the central plane of a two-dimensional ventilated wind tunnel.

The method presented in the present study consists in the generalization of the finite-part singular integral equation methods introduced and investigated by E.G.Ladopoulos [35] – [40] and used in structural analysis and fracture mechanics problems.

2. Finite-Part Singular Integral Equations

Integral equations encountered in several boundary value problems of applied science can be written in the form:

$$f_i(x,\mu)=\sum_{j=1}^{N} C_{ij}(x)\Gamma(\mu)\,⨎_L \frac{\varphi_j(t)}{(t-x)^{\mu}}\,\mathrm{d}t+\sum_{j=1}^{N}\int_L K_{ij}(t,x)\varphi_j(t)\,\mathrm{d}t \qquad (2.1)$$
$$(i=1,2,3,\dots,N)$$

where L denotes the interval $[a,b]$ of the real axis, $\varphi_j(t)$ is an analytic function of t in any plane domain S containing the interval L, $\Gamma(\mu)$ the Gamma function and the functions $C_{ij}(x), K_{ij}(t,x)$ and $f_i(x,\mu)\,(i,j=1,2,3,\dots N)$ are known.

A general form of a system of finite-part singular integral equations, in which the dominant part has a generalized kernel, is as follows :

$$A\varphi(x)+\Gamma(\mu)\,⨎_L B\varphi(t)\frac{\mathrm{d}t}{(t-x)^{\mu}}+$$
$$\Gamma(\mu)\,⨎_L \sum_{0}^{k} C_k\varphi(t)(x-a)^k\,\frac{\mathrm{d}^k}{\mathrm{d}x^k}(t-z_1)^{-\mu}\,\mathrm{d}t$$
$$(2.2)$$
$$+\,\Gamma(\mu)\,⨎_L \sum_{0}^{j} D_j\varphi(t)(b-x)^j\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}(t-z_2)^{-\mu}\,\mathrm{d}t+$$
$$\int_L K(x,t)\varphi(t)\,\mathrm{d}t=f(x,\mu),\; x\in L$$

where A, B, C_k and D_j are $(N\times N)$ matrices which are generally constant, the matrix $K(x,t)$ consists of Fredholm kernels $K_{ij}(x,t)$ $(i,j=1,\dots,N)$ and $f=f(x,\mu)$, $(i=1,\dots,N)$ is the input vector which satisfies a Hölder-condition in L.

Furthermore, the variables z_1 and z_2 are given by:

$$z_1=a+(x-a)e^{i\Theta_1}$$
$$(2.3)$$
$$z_2=a+(b-x)e^{i\Theta_2}$$

in which Θ_1, Θ_2 are known constants with $0<\Theta_1<2\pi$ and $-\pi<\Theta_2<\pi$.

Also, consider the finite-part singular integral equation of the first kind:

$$\Gamma(\mu)\,⨎_L \frac{\varphi(t)}{(t-x)^{\mu}}\,\mathrm{d}t + \int_L K(x,t)\varphi(t)\,\mathrm{d}t = f(x,\mu), \quad \mu = 1,2,3,... \tag{2.4}$$

where $\varphi(x)$ is the unknown, $\Gamma(\mu)$ the Gamma function and $f(x,\mu)$, $K(x,t)$ are known functions which are H-continuous in the closed interval L.

The finite-part singular integral equation of the second kind, with constant coefficients can be written as:

$$a\varphi(x) + b\Gamma(\mu)\,⨎_L \frac{\varphi(t)}{(t-x)^{\mu}}\,\mathrm{d}t + \int_L K(x,t)\varphi(t)\,\mathrm{d}t = f(x,\mu), \; \mu = 1,2,3,... \tag{2.5}$$

where the interval is again normalized to be L without any loss in generality. It will also be assumed that a, b are constants and the known functions f and K are H-continuous. The functions φ, k, f and the constants a, b may be real or complex.

3. Finite-Part Singular Integro-differential Equations

A large class of problems, in mathematical physics, can be reduced to the solution of a singular integro-differential equation of the form:

$$\sum_{j=0}^{m}\left(a_j(t)\varphi^{(j)}(t) + \Gamma(\mu)\,⨎_L \frac{K_j(t,\tau)\varphi^{(j)}(\tau)\,\mathrm{d}\tau}{(\tau-t)^{\mu}} \right) = f(t), \; \mu = 1,2,3,... \tag{3.1}$$

$t \in L$, and a_i, K_i, $f(t)$ are given functions and $\varphi^{(j)}$ denotes the j-th derivative of φ. Assuming that a_i, K_j and f are sufficiently differentiable and that L is a simple, closed, sufficiently smooth contour, we can reduce (3.1) to an equivalent singular or regular integral equation. Consequently, let us give a method for the reduction of (3.1) to a singular integral equation.

Let $L \equiv (a,b)$ an open smooth curve. By writing:

$$\varphi^{(m)}(t) = g(t) \tag{3.2}$$

one obtains:

$$\varphi^{(k)}(t) \int_L \omega_{m-k-1}(t,t_1)g(t_1)\,\mathrm{d}t_1 + \sum_{i=0}^{m-k-1} C_{m-k-i}\frac{t^i}{i!} \tag{3.3}$$

for $k = 0,1,\ldots, m$-1, where:

$$\begin{aligned} \omega_0(t,t_1) &= 1,\ \textit{if}\ t_1 \in (a,t) \\ \omega_0(t,t_1) &= 0,\ \textit{if}\ t_1 \notin (a,t) \\ \omega_{k-1}(t,t_1) &= \int_L \omega_0(t,t_2)\omega_{k-2}(t_2,t_1)\,\mathrm{d}t_2,\ k = 2,3,\ldots,m \end{aligned} \tag{3.4}$$

and $C_1, C_2, \ldots, C_m$ are arbitrary constants.

Substituting into (3.1) we obtain a finite-part singular integral equation for $\mu = 1$ of the form:

$$a_m(t)g(t) + \Gamma(\mu)\int_L \frac{K_m(t,\tau)g(\tau)}{\tau - t}\,\mathrm{d}t + \int_L K(t,\tau)g(\tau)\,\mathrm{d}\tau = f(t) - \sum_{k=1}^{m} C_k X_K(t) \tag{3.5}$$

where:

$$K(t,\tau) = \sum_{j=0}^{m-1}\left(a_j(t)\omega_{m-j-1}(t,\tau) + \Gamma(\mu)\int_L \frac{K_j(t,u)\omega_{m-j-1}(u,\tau)}{u-t}\,\mathrm{d}u \right) \tag{3.6}$$

and:

$$X_k(t) = \sum_{j=0}^{m-k}\left(a_j t^{m-j-k} + \Gamma(\mu)\int_L \frac{K_j(t,\tau)\tau^{m-j-k}}{\tau - t}\,\mathrm{d}\tau \right) \Big/ (m-j-k)! \tag{3.7}$$

It follows that if for any values of the constants $C_1, C_2, \ldots, C_m$ the function g is the solution of (3.5), then the function φ as given by (3.3) will be a solution of the original equation (3.1). So, it is obvious that if φ is the solution of (3.1), then $\varphi^{(m)}(t) = g(t)$ gives the solution of eq. (3.5) for the specific values of $C_1, C_2, \ldots, C_m$.

Also, it is also possible to apply this method to the case where L is a closed smooth curve. Thus, it is necessary to consider the function $\varphi^{(k)}$, $k = 0,1,\ldots,m$-1 as defined by (3.3), since in general they will not be unique.

It can also be seen that for the Cauchy problem, when the values of $\varphi^{(k)}(a)$, $k = 0, 1,\ldots, m$-1 are given, we have:

$$C_k = \sum_{j=0}^{k-1} (-1)^j \varphi^{(m-j)}(a)\big/ j!\ \ \textit{for}\, k = 1,2,\ldots,m \tag{3.8}$$

By the same way for the case where $\mu > 1$, we have the following finite-part singular integral equation:

$$a_m(t)g(t)+\Gamma(\mu)\oint_L \frac{K_m(t,\tau)g(\tau)}{(\tau-t)^{\mu}}\,\mathrm{d}t+\int_L K(t,\tau)g(\tau)\,\mathrm{d}\tau = f(t)-\sum_{k=1}^{m} C_k X_K(t) \tag{3.9}$$

where:

$$K(t,\tau)=\sum_{j=0}^{m-1}\left(a_j(t)\omega_{m-j-1}(t,\tau)+\Gamma(\mu)\oint_L \frac{K_j(t,u)\omega_{m-j-1}(u,\tau)}{(u-t)^{\mu}}\,\mathrm{d}u \right) \tag{3.10}$$

and also:

$$X_k(t)=\sum_{j=0}^{m-k}\left(a_j t^{m-j-k}+\Gamma(\mu)\oint_L \frac{K_j(t,\tau)\tau^{m-j-k}}{(\tau-t)^{\mu}}\,\mathrm{d}\tau \right)\Big/ (m-j-k)! \tag{3.11}$$

Equation (3.9) gives the general use of the reduction of a finite-part singular integro-differential equation to a singular integral equation.

4. Two-dimensional Aerodynamics Application of Planar Airfoils

Let a planar airfoil undergoing simple amplitude oscillations about a central plane of a two-dimensional ventilated wind tunnel (Fig. 1). Then, by removing the walls to infinity, a very important special case exists which gives free air conditions.

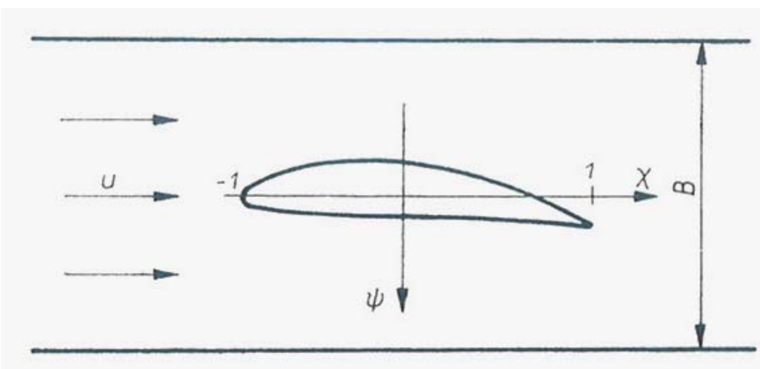

Fig. 1 A planar airfoil in a two-dimensional ventilated wind tunnel.

Beyond the above, the flow is assumed to be inviscid and strictly subsonic and thus the following unsteady wave equation is valid: [21], [27]

$$\nabla^2 \xi - M^2 \left(\frac{\vartheta}{\vartheta x} + ik \right)^2 \xi = 0 \tag{4.1}$$

in which ξ denotes the perturbation velocity potential, M the freestream Mach number and k the reduced frequency:

$$k = \frac{\omega d}{u} \tag{4.2}$$

where ω is the frequency of the simple harmonic motion of the airfoil, d its semi-chord and u the free stream velocity.

Moreover, the nondimensional pertubation pressure p is given by the following relation:

$$p = -2\left(\frac{\vartheta}{\vartheta x} + ik \right)\xi \tag{4.3}$$

with the boundary conditions:

$$p(x,0) = \begin{cases} 0, & |x| \geq 1 \\ -1/2\Delta p(x), & |x| < 1 \end{cases} \tag{4.4}$$

in which Δp denotes the lifting pressure jump across the airfoil. The relation between the downwash velocity w and the pressure potential ξ is equal to:

$$w(x) = \left. \frac{\vartheta \xi}{\vartheta y} \right|_{y=0}, \quad |x| < 1 \tag{4.5}$$

Hence, the downwash velocity w is related to the potential ξ as follows: [16]

$$w(x, y, t) = \frac{1}{u} \int_{-\infty}^{x} \xi_y \left(\mu, y, t - d\,\frac{x - \mu}{u} \right) \mathrm{d}\,\mu \tag{4.6}$$

where t denotes the time and μ the ventilation coefficient.

By using the Fourier transforms:

$$\Xi(s,y)=\int_{-\infty}^{\infty} e^{-ixt}\xi(x,y)\,\mathrm{d}\,x$$
$$\xi(x,y)=\int_{-\infty}^{\infty} e^{ixs}\Xi(s,y)\,\mathrm{d}\,s \tag{4.7}$$

the pressure potential will be given by the following relation:

$$\xi(x,y)=\frac{1}{4\pi\rho_0}\int_{-\infty}^{\infty} e^{ixs} f(s)\int_{-1}^{1} e^{-is\zeta}\,\Delta p(\zeta)\,\mathrm{d}\,\zeta\,\mathrm{d}\,s \tag{4.8}$$

where:

$$f(s)=\frac{\sin ha(B/2-y)+ca\cos ha(B/2-y)}{\sin h(a\,B/2)+ca\cos h(a\,B/2)} \tag{4.9}$$

in which c denotes the porosity coefficient, B the tunnel height and ρ_0 the free stream density.

Furthermore, in (4.9) the parameter a is valid as:

$$a(s)=(\beta^2 s^2-2M^2 gs-M^2 g^2)^{1/2} \tag{4.10}$$

where g is the complex reduced frequency and $\beta=\sqrt{1-M^2}$.

By combining therefore eqs (4.6) and (4.8), one has:

$$\frac{w(x,y)}{u}=\frac{1}{4\pi\rho_0 u^2}\int_{-\infty}^{x} e^{-ig(x-\mu)}\,\frac{\vartheta}{\vartheta y}\int_{-\infty}^{\infty} e^{i\mu t} f(s)\int_{-1}^{1} e^{-is\zeta}\,\Delta p(\zeta)\,\mathrm{d}\,\zeta\,\mathrm{d}\,s\,\mathrm{d}\,\mu \tag{4.11}$$

By taking the derivative and interchanging the orders of integration, we obtain:

$$w(x,y)=\frac{2}{\rho_0 u}\int_{-1}^{1}\Delta p(\zeta)K(M,g,x-\zeta,y,B,c)\,\mathrm{d}\,\zeta \tag{4.12}$$

in which the kernel function K is given by the formula:

$$K = -\frac{1}{8\pi}\int_{-\infty}^{\infty} a e^{-ig(x-\zeta)} \frac{\cos ha(B/2-y)+ca\sin ha(B/2-y)}{\sin h(a\,B/2)+ca\cos h(a\,B/2)} \times \\ \times \int_{-\infty}^{x-\zeta} e^{i(s+g)\mu}\,\mathrm{d}\mu\,\mathrm{d}s \tag{4.13}$$

For steady (g = 0), incompressible (M = 0) flow and in free air (no tunnel walls $B = \infty$), the kernel takes the simple form:

$$K(x) = 1/x \tag{4.14}$$

For this case, with y = 0, from eq. (4.12) results the following singular integral equation:

$$w(x) = \frac{1}{2\pi\rho_0}\int_{-1}^{1}\frac{\Delta p(\zeta)}{\zeta - x}\,\mathrm{d}\zeta \tag{4.15}$$

By using further the Kutta boundary condition of a smooth flow at the airfoil trailing edge:

$$\lim_{x\to 1}\frac{2\Delta p(x,t)}{\rho_0 u^2} = 0 \tag{4.16}$$

then (4.15) has the following closed form solution:

$$\Delta p(\zeta) = -\frac{2\rho_0 u}{\pi}\left(\frac{1-\zeta}{1+\zeta}\right)^{1/2}\int_{-1}^{1}\frac{w^*(x)w(x)}{x-\zeta}\,\mathrm{d}x \tag{4.17}$$

with the weight function $w^*(x) = (1+x)^{1/2}(1-x)^{-1/2}$.

Hence, by putting the pressure factor:

$$p(\zeta) = -\frac{4}{\pi}\int_{-1}^{1} w^*(x)\frac{w(x)}{u}\frac{\mathrm{d}x}{x-\zeta} \tag{4.18}$$

then (4.17) can be written as follows:

$$\Delta p(\zeta) = \frac{1}{2}\rho_0 u^2\left(\frac{1-\zeta}{1+\zeta}\right)^{1/2} p(\zeta) \tag{4.19}$$

The pressure factor $p(\zeta)$ in (4.18) is continuous on [-1,1] if $w(x)/u$ is also continuous.

5. Numerical Solution of the Airfoil Equation

For the numerical evaluation of the airfoil equation (4.18) the Gauss-Chebyshev numerical integration rule will be used, while solving the same problem, S. R. Bland [14] has used the Gauss-Jacobi rule.

Consider the following singular integral:

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{w^*(x)\varphi(x)}{x-\zeta}\,\mathrm{d}\,x \tag{5.1}$$

in which $w^*(x)$ is the weight function defined in the interval [-1,1], $\varphi(x)$ is an analytic function without poles in a domain Ω containing the interval [-1,1] and $\Phi(\zeta)$ is a sectionally analytic function in the whole complex plane except [-1,1].

In order to evaluate numerically the singular integral (5.1), we consider the following contour integral on a curve C surrounding the interval [-1,1]:

$$\Phi_0 = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta')}{(\zeta'-x)(\zeta'-\zeta)m_n(\zeta')}\,\mathrm{d}\,\zeta' \tag{5.2}$$

where:

$$m_n(\zeta) = \prod_{k=1}^{n}(\zeta - x_k) \tag{5.3}$$

in which x_k are the abscissae.

Consequently, by applying the Cauchy residue theorem to the integral (5.2), we obtain:

$$\begin{aligned} 2\pi i\Phi(\zeta) &= \int_{-1}^{1} \frac{w^*(x)\varphi(x)}{x-\zeta}\,\mathrm{d}\,x = \\ &= \sum_{k=1}^{n} A_k \frac{\varphi(x_k)}{x_k-\zeta} - 2\varphi(\zeta)\frac{d_n(\zeta)}{m_n(\zeta)} + E_n \end{aligned} \tag{5.4}$$

where the error function E_n is equal to:

$$E_n = \frac{1}{\pi i}\int_C \frac{\varphi(\zeta')}{\zeta'-\zeta}\frac{d_n(\zeta')}{m_n(\zeta')}\,\mathrm{d}\,\zeta' \tag{5.5}$$

A_k are the weights and $d_n(\zeta)$ is given by the relation:

$$d_n(\zeta) = -\frac{1}{2}\int_{-1}^{1} w^*(x)\frac{m_n(x)}{x-\zeta}\,\mathrm{d}\,x \tag{5.6}$$

By using further the Gauss-Chebyshev numerical integration rule with the weight function $w^*(x) = (1+x)^{\pm 1/2}(1-x)^{\pm 1/2}$, then (5.4) can be written as:

$$\int_{-1}^{1} \frac{w^*(x)\varphi(x)}{x-\zeta}\,\mathrm{d}\,x = \sum_{k=1}^{n} A_k\,\frac{\varphi(x_k)}{x_k-\zeta} - 2\varphi(\zeta)R_n(\zeta) + E_n \tag{5.7}$$

for $\zeta \neq x_m$, $m = 1, 2, \ldots, n$, and:

$$\int_{-1}^{1} \frac{w^*(x)\varphi(x)}{x-\zeta}\,\mathrm{d}\,x = \sum_{\substack{k=1\\ k\neq m}}^{n} A_k\,\frac{\varphi(x_k)}{x_k-\zeta} + A_m\varphi'(\zeta) - 2\varphi(\zeta)G_n(\zeta) + E_n \tag{5.8}$$

for $\zeta = x_m$, $m = 1, 2, \ldots, n$, where:

$$R_n(\zeta) = -\frac{\pi U_{n-1}(\zeta)}{nT_n(\zeta)},\ \zeta \neq x_m,\ m = 1,2,\ldots,n \tag{5.9}$$

and:

$$G_n(\zeta) = -\frac{\pi}{2}\frac{U_{n-2}(\zeta)}{T_{n-1}(\zeta)} + \frac{2n-1}{4}A_m\,\frac{\zeta}{1-\zeta^2},\ \zeta = x_m,\ m = 1,2,\ldots,n \tag{5.10}$$

in which $T_n(\zeta)$ and $U_n(\zeta)$ denote the Chebyshev polynomials of the first and the second kind and degree n, respectively, expressible in terms of trigonometric functions as follows:

$$\begin{gathered} T_n(\zeta) = \cos n\vartheta \\ U_{n-1}(\zeta) = \frac{\sin n\vartheta}{\sin\vartheta} \\ \zeta = \cos\vartheta \end{gathered} \tag{5.11}$$

In eqs. (5.7) and (5.8) ζ is not permitted to coincide with the endpoints -1 or 1 of the integration interval.

As an application of the airfoil equation (4.18), we consider the case where the downwash is valid as:

$$\frac{w(x)}{u}=\begin{cases}0, & x\le 0\\ x, & x>0\end{cases} \tag{5.12}$$

So, by using the Gauss-Chebyshev numerical integration rule given by eqs. (5.7) and (5.8), it is possible to compute the airfoil equation (4.18). The same equation was computed by S. R. Bland [16], while using the Gauss-Jacobi rule.

Figure 2 shows the pressure distribution $p(\zeta)$ for downwash given by (5.12).

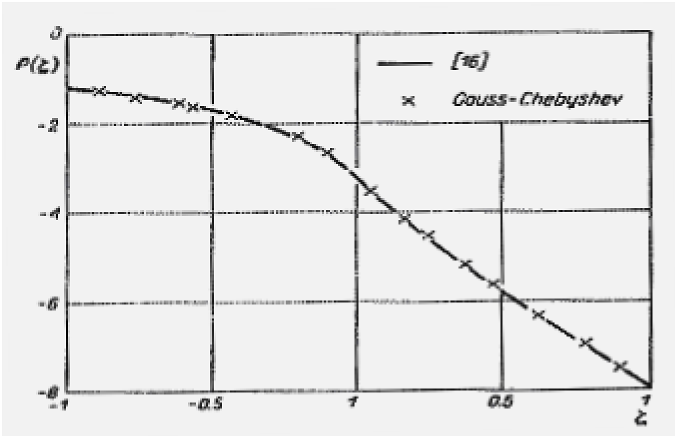

Fig. 2 Pressure distribution $p(\zeta)$ for downwash $w(x)/u=\begin{cases}0, x\le 0\\ x, x>0\end{cases}$ for the planar airfoil of Fig.1.

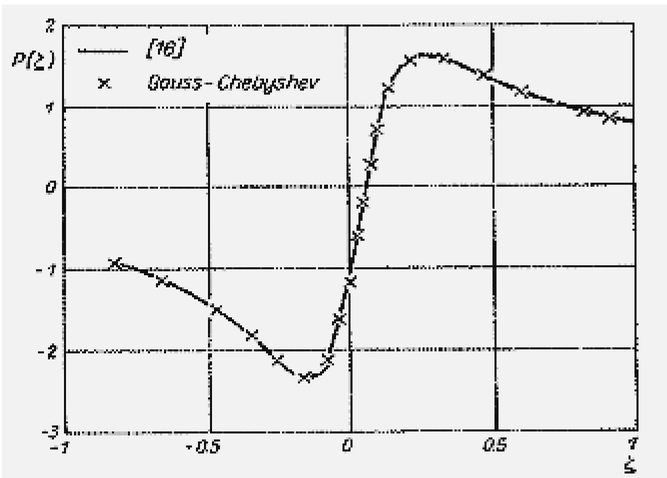

Fig. 3 Pressure distribution $p(\zeta)$ for downwash $w(x)/u = 1/(1{+}25x^2)$ for the planar airfoil of Fig.1.

As a second application of the airfoil equation, let us consider the following downwash function:

$$\frac{w(x)}{u} = \frac{1}{1 + 25x^2} \tag{5.13}$$

Figure 3 shows the pressure distribution $p(\zeta)$ for downwash given by (5.13).

Finally, as it is easily seen from Figs.2 and 3, the two different numerical rules, the Gauss-Chebyshev and Gauss-Jacobi numerical integration rules coincide very well.

6. Conclusions

A new method has been proposed for the numerical evaluation of the finite-part singular integro-differential equation by reducing this equation to a system of linear equations and then the integrals are approximated by sums and this equation is applied at the abscissas used in the numerical integration rule.

Furthermore, for the cases of finite-part singular integro-differential equations with complex singularities, it is possible that the points of their application lie outside the integration rule. In these cases the methods which have been used for finite-part singular integro-differential equations with real singularities, can be extended to corresponding cases of finite-part singular integro-differential equations with complex singularities.

One form of the finite-part singular integro-differential equation has been numerically solved, by using the Gauss-Chebyshev integration rule. Such an equation presents the pressure factor of a planar airfoil undergoing simple amplitude oscillations about the central plane of a two-dimensional ventilated wind tunnel.

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