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# On a New Proof of Feuerbach's Theorem

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## Abstract
In this article we present a new proof of the Feuerbach's Theorem by using a metric relation of Nine Point Center. So, by using a new and very modern method the above well know theorem is proved.

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**Key Word and Phrases**
Feuerbach's Theorem, Nine Point Center, Nine Point Circle, Medial Triangle.

## 1. Introduction
It is well known and proven fact that "Euler is master of us" and he was the first scientist (in 1765) to show that the midpoints of the sides of a triangle and the feet of the altitudes determine a unique circle [3]. It was not until 1820 that Brianchon and Poncelet showed that the three midpoints of the segments from the orthocentre to the vertices also lie on the same circle [3], hence its name, the nine-point circle [13]. The nine-point circle is often also referred to as the Euler or Feuerbach circle. A result closely associated with the nine-point circle is that of the Euler line, namely that the orthocentre (H), centroid (G), circumcentre (O) and nine-point centre (N) are collinear. In the geometry of triangles, the theorem commonly known as Feuerbach's theorem, published by Feuerbach in 1822 [8], states that "The Nine point circle is tangent to the three excircles of the triangle as well as its incircle".

By the current paper we prove this theorem using an idea intuitively given by Euler master of us. Many proofs have been given (Elder 1960) [5], with the simplest being the one presented by M'Clelland (1891, p. 225) [11] and Lachlan (1893, p. 74) [10], Over the recent years many other proofs are also available in the literature (some of them can be found in [1], [5], [7], [8], [9], [12] and [14]).

Especially the proof which was dealt by the article [1] is a peculiar proof, different from the available proofs which is actually based on a metric relation of incenter, but now in the present paper by the inspiration of the proof presented in [1] we will try to prove this theorem using a metric relation on Nine point center. Our proof actually follows from the well known result given by Euler that nine point center is the circumcenter of its medial triangle [3].

## 2. Notation and Background
Let ABC be a nonequilateral triangle. We denote its side-lengths by a, b, c, its semiperimeter by $s=\frac{1}{2}(a+b+c)$, and its area by $\Delta$. Its *classical centers* are the circumcenter S, the incenter I, the nine point center N, the centroid G, and the orthocenter O. The nine-point center N is the midpoint of SO. The Euler Line Theorem states that G

lies on SO with OG : GS = 2 : 1 and ON : NG : GS = 3 : 1 : 2. We write $I_1$, $I_2$, $I_3$ for the excenters opposite A, B, C, respectively, these are points where one internal angle bisector meets two external angle bisectors. Like I, the points $I_1$, $I_2$, $I_3$ are equidistant from the lines AB, BC, and CA, and thus are centers of three circles each tangent to the three lines. These are the excircles. The *classical radii* are the circumradius R (= SA = SB = SC), the inradius r, and the exradii $r_1$, $r_2$, $r_3$.

The following formulae are well known:

(a). $\Delta = \frac{abc}{4R} = rs = r_1(s-a) = r_2(s-b) = r_3(s-c) = \sqrt{s(s-a)(s-b)(s-c)}$

(b). $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$

(c). $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C = \frac{abc}{2R^3} = \frac{2\Delta}{R^2}$

(d). $\cos A \cos B \cos C = \frac{a^2+b^2+c^2-8R^2}{8R^2}$

(e). $AI = \sqrt{r^2+(s-a)^2} = \frac{r}{\sin\frac{A}{2}}, \mathrm{BI} = \sqrt{r^2+(s-b)^2} = \frac{r}{\sin\frac{B}{2}} \, and \, \mathrm{CI} = \sqrt{r^2+(s-c)^2} = \frac{r}{\sin\frac{C}{2}}$

(f). $AI_1 = \sqrt{r_1^2+s^2} = \frac{r_1}{\sin\frac{A}{2}}, \mathrm{B}I_1 = \sqrt{r_1^2+(s-c)^2} = \frac{r_1}{\cos\frac{B}{2}} \, and \, \mathrm{C}I_1 = \sqrt{r_1^2+(s-b)^2} = \frac{r_1}{\cos\frac{C}{2}}$

(g). $Tan\frac{A}{2} = \frac{(s-b)(s-c)}{\Delta} = \frac{\Delta}{s(s-a)}$

## 3. Basic Lemma's

***Lemma – 1***

If a, b, c are the sides of the triangle ABC, and if s, R, r and Δ are semi perimeter, Circumradius, Inradius and area of the triangle ABC respectively the following:

(1.1) $ab+bc+ca = r^2+s^2+4Rr$

(1.2) $a^2+b^2+c^2 = 2\left(s^2-r^2-4Rr\right) = 8R^2+8R^2\cos A\cos B\cos C$

(1.3) $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc = 2s(r^2+s^2+2Rr)$

***Proof***

The proofs of (1.1), (1.2) are available in [4] and by using (1.1), (1.2) and (b) we can prove (1.3).

***Lemma - 2***

For any triangle ABC, $\sum_{a,b,c} a^3\cos(B-C) = 3abc$

***Proof***
The proof of above lemma can be available in any academic trigonometry book.

***Lemma -3***
For any triangle ABC,
(3.1) $a^2 sin2A + b^2 sin2B + c^2 sin2C = \frac{2\Delta}{R^2}\left(a^2 + b^2 + c^2 - 6R^2\right)$

(3.2) $acos(B - C) + bcos(C - A) + ccos(A - B) = \frac{4\Delta}{R}$

(3.3) $cos(B - C) + cos(C - A) + cos(A - B) = \frac{1}{2R^2}\left(r^2 + s^2 + 2Rr - 2R^2\right)$

***Proof***
Consider $a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C$

$$= \sum_{a,b,c} a^2 \sin 2\mathrm{A} = (a^2 + b^2 + c^2)(\sin 2A + \sin 2B + \sin 2C) - \sum_{a,b,c} a^2(\sin 2B + \sin 2C)$$

It can be further simplified by replacing $\sum_{a,b,c} a^2(\sin 2B + \sin 2C)$ as $\frac{1}{R}\sum_{a,b,c} a^3 \cos(\mathrm{B-C})$

Now applying lemma-2 and using (a), (c) we can prove the required conclusion (3.1).

By the same way, we can prove conclusion (3.2).

Besides, for (3.3), consider $\frac{a+b}{c} = \frac{\cos\left(\frac{\mathrm{A-B}}{2}\right)}{\sin\frac{C}{2}}$

So, it is easy to verify that $\sum \cos(\mathrm{A-B}) = 4\cos\left(\frac{\mathrm{A-B}}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right) - 1$

Using (a), (1.3) and by little algebra we can arrive at the conclusion (3.3).

***Lemma -4***
For any triangle ABC,
(4.1)
$$cot\frac{A}{2}cos(B - C) + cot\frac{B}{2}cos(C - A) + cot\frac{C}{2}cos(A - B) = \frac{s^2}{2\Delta R^2}\left(r^2 + s^2 - 6Rr - 2R^2\right)$$
(4.2)
$$cot\frac{A}{2}cos(B - C) + tan\frac{B}{2}cos(C - A) + tan\frac{C}{2}cos(A - B) = \frac{2\Delta}{r_1^2 R^2}\left(2r_1^2 - r^2 + s^2 - 4Rr + 2Rr_1 - 2R^2\right)$$

(4.3)
$$tan\frac{A}{2}cos(B - C) + Cot\frac{B}{2}cos(C - A) + tan\frac{C}{2}cos(A - B) = \frac{2\Delta}{r_2^2 R^2}\left(2r_2^2 - r^2 + s^2 - 4Rr + 2Rr_2 - 2R^2\right)$$

(4.4)

$$\tan\frac{A}{2}\cos(B-C)+\tan\frac{B}{2}\cos(C-A)+\cot\frac{C}{2}\cos(A-B)=\frac{2\Delta}{r_3^2R^2}\left(2r_3^2-r^2+s^2-4Rr+2Rr_3-2R^2\right)$$

***Proof***

We have,

$$\sum\cot\frac{A}{2}\cos(B-C)=\sum\frac{s(s-a)}{\Delta}\cos(B-C)=\frac{s}{\Delta}\left[\sum s\cos(B-C)-\sum a\cos(B-C)\right]$$

By using (3.2), (3.3), and by further simplification gives the conclusion (4.1).

In the similar manner we can prove (4.2), (4.3) and (4.4).

***Lemma - 5***

If S is the Circumcenter of the triangle ABC and M be any point in the plane then:

$$SM^2=\frac{R^2}{2\Delta}\left(\sin 2A.AM^2+\sin 2B.BM^2+\sin 2C.CM^2-2\Delta\right)$$

***Proof***

The proof of above lemma can be found in [2].

## 4. Main Results

***Theorem-1***

If N be the center of the nine point circle of triangle ABC and M be any point in the plane then:

$$4NM^2=\frac{R^2}{\Delta}\left[(\sin 2B+\sin 2C)AM^2+(\sin 2A+\sin 2C)BM^2+(\sin 2A+\sin 2B)CM^2\right]-\left(a^2+b^2+c^2-5R^2\right)$$

***Proof***

For proving the above said result we will make use of lemma-3.

Since Nine Point Circle acts as a circumcircle of medial triangle DEF where D, E and F are the mid points of sides BC, CA and AB respectively with circum radius $\frac{R}{2}$, area $\frac{\Delta}{4}$ and the angles of medial triangle DEF are A, B and C (since the lengths of sides of medial triangle DEF are $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{c}{2}$ and they are parallel to the sides of the triangle ABC) [3].

Also, by using lemma-4, by replacing S as N, R as $\frac{R}{2}$, $\Delta$ as $\frac{\Delta}{4}$ and angles remains same we get:

$$NM^2=\frac{R^2}{2\Delta}\left(\sin 2A.\mathrm{DM}^2+\sin 2B.\mathrm{EM}^2+\sin 2C.\mathrm{FM}^2-\frac{\Delta}{2}\right)$$

By using Apollonius theorem we can replace the values of DM, EM and FM by:

$$DM^2=\frac{1}{2}BM^2+\frac{1}{2}CM^2-\frac{1}{4}a^2,$$

$$EM^2=\frac{1}{2}AM^2+\frac{1}{2}CM^2-\frac{1}{4}b^2$$

and $FM^2 = \frac{1}{2}AM^2 + \frac{1}{2}BM^2 - \frac{1}{4}c^2$

Then:

$$NM^2 = \frac{R^2}{2\Delta}\left[\left(\frac{\sin 2\mathrm{B} + \sin 2\mathrm{C}}{2}\right)AM^2 + \left(\frac{\sin 2A + \sin 2\mathrm{C}}{2}\right)BM^2 + \left(\frac{\sin 2A + \sin 2B}{2}\right)CM^2 - \frac{1}{4}\left(\sum_{a,b,c} a^2 \sin 2A\right) - \frac{\Delta}{2}\right]$$

Using lemma-**2** it can be further simplified as:

$$NM^2 = \frac{R^2}{2\Delta}\left[\sum\left\{\left(\frac{\sin 2\mathrm{B} + \sin 2\mathrm{C}}{2}\right)AM^2\right\} - \frac{2\Delta}{4R^2}\left(a^2 + b^2 + c^2 - 6R^2\right) - \frac{\Delta}{2}\right]$$

Further simplification gives the required conclusion:

$$4NM^2 = \frac{R^2}{\Delta}\left[\left(\sin 2\mathrm{B} + \sin 2\mathrm{C}\right)AM^2 + \left(\sin 2A + \sin 2\mathrm{C}\right)BM^2 + \left(\sin 2A + \sin 2B\right)CM^2\right] - \left(a^2 + b^2 + c^2 - 5R^2\right)$$

***Remark***

The above relation can also be proved using the idea of Euler's line and its related ratios.

***Theorem -2***

Nine point circle of triangle ABC touches its incircle internally, that is if N and I are the centers of nine point circle and incircle respectively whose radii are $\frac{R}{2}$ and $r$ then: $\boldsymbol{NI = \left|\frac{R}{2} - r\right|}$

***Proof***

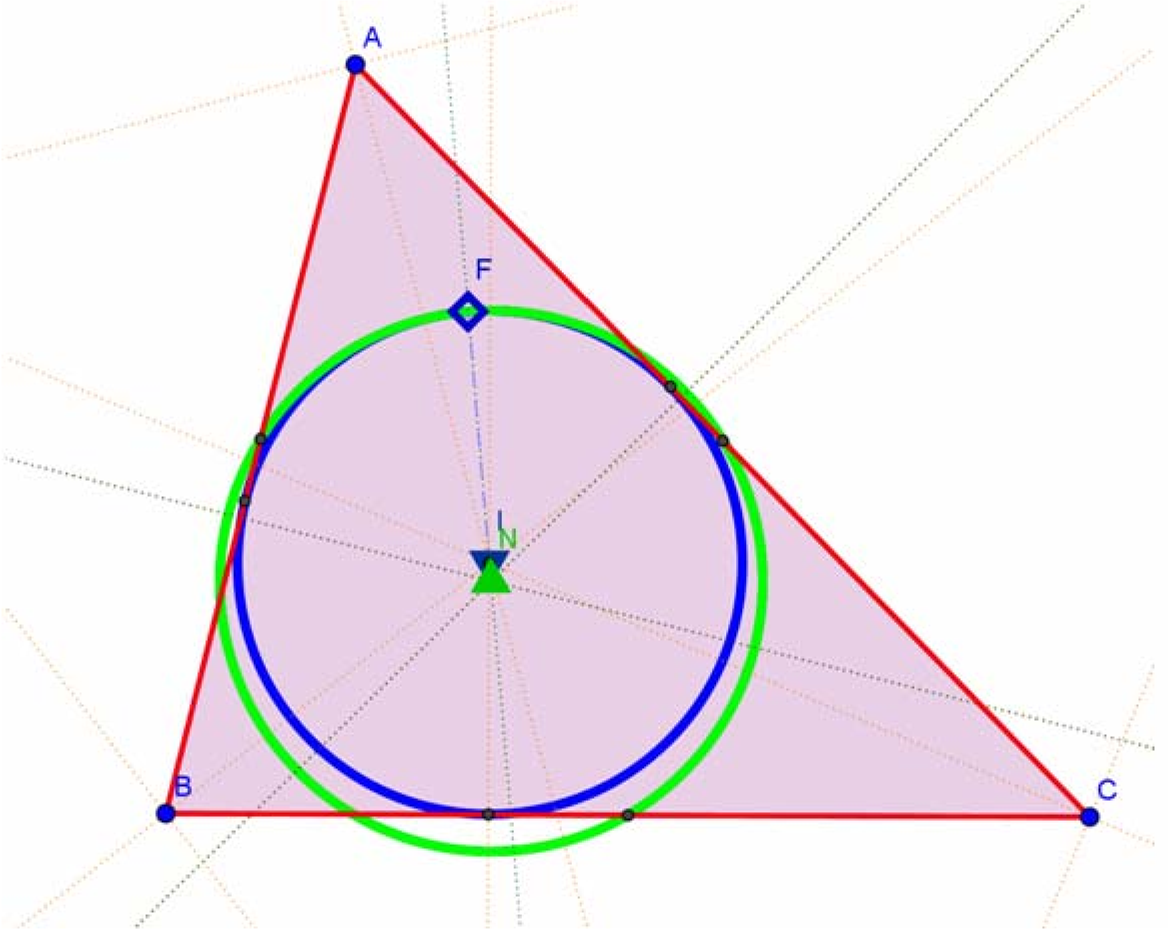


**Fig. 1** Nine Point Circle (N) touches Incircle at Inner Feuerbach's point (F).

We have by using Theorem-**1**,

$$4NM^2 = \frac{R^2}{\Delta}\left[(\sin 2B + \sin 2C)AM^2 + (\sin 2A + \sin 2C)BM^2 + (\sin 2A + \sin 2B)CM^2\right] - \left(a^2 + b^2 + c^2 - 5R^2\right)$$

……....………(Ω)

Since (Ω**)** is true for any M let us fix M as I (incenter) we get:

$$4NI^2 = \frac{R^2}{\Delta}\left[(\sin 2B + \sin 2C)AI^2 + (\sin 2A + \sin 2C)BI^2 + (\sin 2A + \sin 2B)CI^2\right] - \left(a^2 + b^2 + c^2 - 5R^2\right)$$

…………………(€)

Now by replacing AI, BI and CI using (d), (€) can be rewritten as:

$$4NI^2 = \frac{R^2}{\Delta}\left[\sum(\sin 2B + \sin 2C)\left(\frac{r^2}{\sin^2\frac{A}{2}}\right)\right] - \left(a^2 + b^2 + c^2 - 5R^2\right)$$

It can be further simplified as:

$$4NI^2 = \frac{R^2}{\Delta}\left[4r^2\sum\cot\frac{A}{2}\cos(B-C)\right]-\left(a^2+b^2+c^2-5R^2\right)$$

Using lemma-4 (4.1) , lemma-1 (1.2) and Further simplification gives:

$NI^2 = (\frac{R}{2}-r)^2$ (since $R \geq 2r$ )

It further gives: $NI = \left|\frac{R}{2}-r\right|$

Hence proved.

***Theorem -3***

Nine point circle of triangle ABC touches their excircles externally, that is if N and $I_1$, $I_2$, $I_3$ are the centers of nine point circle and excircles respectively whose radii are $\frac{R}{2}$ and $r_1$, $r_2$, $r_3$ then: $NI_1 = \frac{R}{2}+r_1$, $NI_2 = \frac{R}{2}+r_2$ and $NI_3 = \frac{R}{2}+r_3$

***Proof***

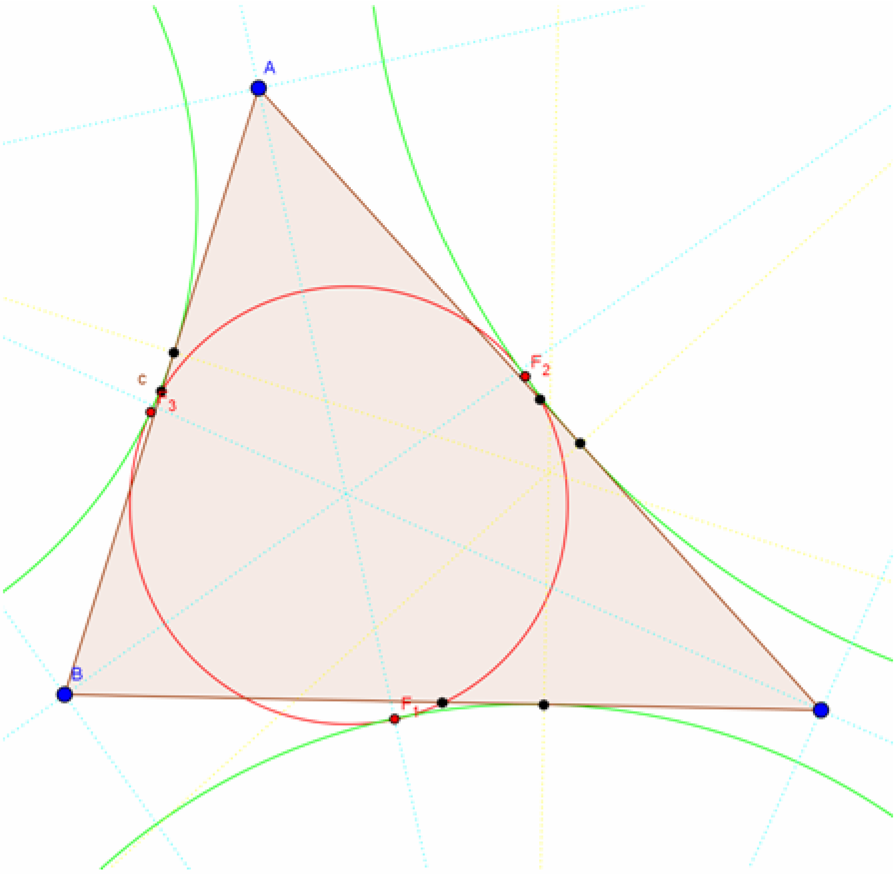


**Fig. 2** Nine Point Circle (N) touches 3 Excircles at 3 outer Feuerbach's points ($F_1$,$F_2$,$F_3$).

We have by using Theorem-1,

$$4NM^2=\frac{R^2}{\Delta}\left[\left(\sin 2\mathrm{B}+\sin 2\mathrm{C}\right)AM^2+\left(\sin 2A+\sin 2\mathrm{C}\right)BM^2+\left(\sin 2A+\sin 2B\right)CM^2\right]-\left(a^2+b^2+c^2-5R^2\right)$$

………………(Ω)

Since (Ω) is true for any M let us fix M as $I_1$ (excenter opposite to vertex A of triangle ABC )

We get:

$$4NI_1^{\,2}=\frac{R^2}{\Delta}\left[\left(\sin 2\mathrm{B}+\sin 2\mathrm{C}\right)AI_1^{\,2}+\left(\sin 2A+\sin 2\mathrm{C}\right)BI_1^{\,2}+\left(\sin 2A+\sin 2B\right)CI_1^{\,2}\right]-\left(a^2+b^2+c^2-5R^2\right)$$

………………(π)

Now by replacing $AI_1$, $BI_1$ and $CI_1$ using (e), (π) can be rewritten as:

$$4NI_1^{\,2}=\frac{R^2}{\Delta}\left[\left(2\sin\frac{A}{2}\cos\frac{A}{2}\cos(B-C)\frac{r_1^{\,2}}{\sin^2\frac{A}{2}}\right)+\left(2\sin\frac{B}{2}\cos\frac{B}{2}\cos(\mathrm{C}-A)\frac{r_1^{\,2}}{\cos^2\frac{B}{2}}\right)+\left(2\sin\frac{C}{2}\cos\frac{C}{2}\cos(\mathrm{A}-B)\frac{r_1^{\,2}}{\cos^2\frac{C}{2}}\right)\right]-\left(a^2+b^2+c^2-5R^2\right)$$

It can be further rewritten as:

$$4NI_1^{\,2}=\frac{r_1^{\,2}R^2}{\Delta}\left[\cot\frac{A}{2}\cos\left(B-C\right)+\tan\frac{B}{2}\cos\left(C-A\right)+\tan\frac{C}{2}\cos\left(A-B\right)\right]-\left(a^2+b^2+c^2-5R^2\right)$$

Now using lemma- 4(4.2) , lemma-1(1.2) and Further simplification gives $I_1N^2=(\frac{R}{2}+r_1)^2$

It further gives $NI_1=\frac{R}{2}+r_1$

Similarly we can prove $NI_2=\frac{R}{2}+r_2$ and $NI_3=\frac{R}{2}+r_3$

***Theorem – 4 ( Feuerbach, 1822 )***

In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and externally tangent to the three excircles.

***Proof***

Theorem -2 and Theorem-3 completes the proof of Feuerbach’s Theorem.

For historical details of this theorem see [5] , [6], [8] and [12].

## 5. Conclusions

By the current paper in Theorem-1 we proved a metric relation on nine point center which we used as a tool in proving the Feuerbach's theorem, using this metric relation we can prove all the fundamental properties of nine point center and we can investigate for the new properties of nine point center and also using the metric relation we can find the distance between the nine point center and all other triangle centers.

Not only these, this metric relation is also useful in proving the famous theorems related to nine point circle like lester circle theorem, etc.

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***REMARKS***

**1.** The relation mentioned in Theorem-1 can be also proved by using the idea of Euler's line and its related ratios.

**2.** If N is the nine point center of triangle ABC whose circumcenter is S then: $AN^2 + BN^2 + CN^2 + SN^2 = 3R^2$

***PROOF***

We know that (using [1])

If N is the nine point center of the triangleABC then:

$$AN = \frac{1}{2}\sqrt{R^2 + c^2 + b^2 - a^2} = \frac{1}{2}\sqrt{R^2 + 2bc\cos A}$$

$$BN = \frac{1}{2}\sqrt{R^2 + c^2 + a^2 - b^2} = \frac{1}{2}\sqrt{R^2 + 2ac\cos\mathrm{B}}$$

$$CN = \frac{1}{2}\sqrt{R^2 + a^2 + b^2 - c^2} = \frac{1}{2}\sqrt{R^2 + 2ab\cos\mathrm{C}}$$

So: $4AN^2 + 4BN^2 + 4CN^2 = 3R^2 + a^2 + b^2 + c^2$ ………(α)

Now by using Theorem-1 we have:

$$4NM^2 = \frac{R^2}{\Delta}\left[\left(\sin 2B + \sin 2C\right)AM^2 + \left(\sin 2A + \sin 2C\right)BM^2 + \left(\sin 2A + \sin 2B\right)CM^2\right] - \left(a^2 + b^2 + c^2 - 5R^2\right)$$

Since it is true for any M let us fix M as S(circumcenter), we have:

$$4NS^2 = \frac{R^2}{\Delta}\left[\left(\sin 2B + \sin 2C\right)AS^2 + \left(\sin 2A + \sin 2C\right)BS^2 + \left(\sin 2A + \sin 2B\right)CS^2\right] - \left(a^2 + b^2 + c^2 - 5R^2\right)$$

Further simplification by replacing AS=BS=CS=R and using (c), (α) gives required conclusion.