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A Singular Integral Transform for the Gibbs-Wilbraham Effect in Inverse Fourier Transforms

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Abstract
This note reports on an asymptotic dispersive singular integral transform representation for the Gibbs-Wilbraham effect in an inverse Fourier transform. A well established fact is that the jump size scale, of 0.089490, is independent of the Gibbs-Wilbraham effect. We demonstrate that this fact is compatible with the consistency of this transform and with skew symmetrization of a part of this effect.

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Singular Integral Equations, Gibbs-Wilbraham Effect, Inverse Fourier Transform.

1. Introduction

It is well known that the Gibbs-Wilbraham effect, see e.g. [1-3], is a significant anomaly of many practical implications for Fourier series or Fourier transform applications. There exist, nonetheless, numerous algorithms for the removal of this effect from Fourier series or expansions in orthogonal polynomials. One of the most well known and widely used methods is spectral reprojection [4], [5]. A more inverse approach to the problem of resolving this effect may be found in [6] and [7]. Regardless of all of that, the design of effective techniques for the resolution of the Gibbs effect remains until now an active area of research [4], [8].

Let x_s be a point in (−L, L) at which *f(x)* has a discontinuity of the first kind. Consider $x_s^- = x_s - \varepsilon$, $x_s^+ = x_s + \varepsilon$, $0 < \varepsilon << 1$, $f_s^- = \lim_{\varepsilon\to 0} f(x_s^-)$, and $f_s^+ = \lim_{\varepsilon\to 0} f(x_s^+)$, to assume, without loss of generality, that $f_s^+ > f_s^-$ and to define the jump discontinuity by :

$$\Delta_s = f_s^+ - f_s^- . \tag{1.1}$$

Moreover, let:

$$f_c(x)=\begin{cases} f(x)\,;\; x \neq x_s \\ undefined\,;\; x = x_s \end{cases} \tag{1.2}$$

A distribution-theoretic proof, see [3], for the Fourier series representation *g(x)* of a 2L-periodic piecewise continuously differentiable function $f(x)\in$ BV(−L,L) stipulates that:

$$g(x)=\begin{cases} f_s^- - \rho\Delta_s\,;\; x = x_s^- \\ \frac{1}{2}(f_s^+ + f_s^-)\,;\; x = x_s \\ f_s^+ + \rho\Delta_s\,;\; x = x_s^+ \\ f_c(x)\,;\; x \neq x_s \text{ or } x_s^\pm \end{cases} \tag{1.3}$$

where: $\rho = u(x_s^+ - x_s) \approx u(0)$, with *u(x)* as the Heaviside unit step function [3]. An uncertain constant, representing the jump size scale satisfying:

$$0 < \rho < 1. \tag{1.4}$$

An uncertainty that had however been known to be $\rho = \frac{1}{\pi} S_i(\pi)$ = 0.089 490, up to 6 decimal places, before the inception of the Dirichlet kernel, in the works of Gibbs and Wilbraham, and by invoking the $S_i(x)$ integral. This well-known result has also been numerically verified in the inverse problem formulation of [3].

This note is an extension of this result to the case of an aperiodic Fourier transformable *f(x)*. Due to the difficulty in computing the spiky Gibbs-Wilbraham effect, [3], we shall aim here at correlating the inverse Fourier transform *g(x)* with a discontinuous *f(x)* that is defined over (-∞,∞). The extension happens to invoke a singular integral transform to model the dispersive behavior of *g(x)*- *f(x)* in the neighborhood of an isolated discontinuity of *f(x)* at x_s. The inverse problem of determining *f(x)* from a known *g(x)* turns out to be a problem of solving a singular integral equation of the second kind. In this respect, it should be noted that singular integral equations are extensively used, see e.g. [9], particularly in the theory of elasticity [10, 11]. Moreover, an inclusion of an asymptotic parameter into this singular integral transform (and associated integral equation) is employed in this work, as in a plane contact problem, [11], or in the sense of [12]. Hence our approach though apparently entirely different from the previously mentioned reprojection and other methods, is meant however to be a new complement to them. A complementation that in no way aims at resolving the Gibbs-Wilbraham effect in their sense. However, the well established fact of the jump size scale, of 0.089490, being independent of the jump size in the Gibbs-Wilbraham effect, is demonstrated to be compatible with the consistency of this transform and with skew-symmetrization of a part of this effect.

The difference $g(x) - f(x)$ at $x = x_s$ is obviously a generic localized anomalous behavior. This fact is verified in Section 4 by an analytical evaluation of the dispersion singular integral transform using Fourier transforms. The main result of this work is reported in Section 3.

2. Construction of the Dispersion Singular Integral Equation

Consider a Fourier transformable function *f(z)* with a dummy variable *z* and let the symbol δ denote Dirac's Delta function. The presence of a jump discontinuity Δ_s in *f(z)* at $z = x_s$ is expected to cause *g(z)* to differ from *f(z)* by two factors. First, a pointwise hereditary sifting at $z = x_s$ which is namely:

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{z} \{f(\tau - \varepsilon) + \tfrac{1}{2}\Delta_s\}\{\,\delta\,[(\tau - x_s) + \varepsilon] - \delta\,[(\tau - x_s) - \varepsilon]\}\, d\tau.$$
$$= \lim_{\varepsilon\to 0} \{f(z - \varepsilon) + \tfrac{1}{2}\Delta_s\}\{\,u\,[(z - x_s) + \varepsilon] - u\,[(z - x_s) - \varepsilon]\}\,.$$
$$= \lim_{\varepsilon\to 0} \{f(z - \varepsilon) + \tfrac{1}{2}\Delta_s\} P_{\varepsilon,s}(z) = p_s(z)\,. \qquad (2.1)$$

Clearly:

$$P_{\varepsilon,s}(z) = \begin{cases} 1\;;\; z \in [x_s^-, x_s^+] \\ 0\;;\; z \notin [x_s^-, x_s^+] \end{cases},$$

and:

$$\lim_{\varepsilon\to 0} P_{\varepsilon,s}(x_s) = p_s(x_s) = 1.$$

Second, an emerging undershoot/overshoot rather uncertain transient $T(z, x_s)$, centered around x_s, that defines the Gibbs-Wilbraham effect spiky transient:

$$H(z, x_s) = H_{\theta,\eta,\alpha,\varepsilon}(z, x_s) = T(z, x_s) - f(z), \tag{2.2}$$

which can phenomenologically be represented as:

$$H(z, x_s) = -\theta\,\delta\,[z-(x_s+\alpha+\varepsilon)] + (\tfrac{1}{2}-\eta)\,\Delta_s\delta\,[z-(x_s+\varepsilon)] \\ +(\tfrac{1}{2}-\eta)\,\Delta_s\delta\,[z-(x_s-\varepsilon)] - \theta\,\delta\,[z-(x_s-\alpha-\varepsilon)], \tag{2.3}$$

where $\theta = \theta(\Delta_s)$ is an uncertain undershoot/overshoot function and α is a free infinitesimal parameter satisfying $0 \le \varepsilon,\alpha << 1$.The unknown parameter η reflects the fact that the rise (varying between 0 and $\frac{1}{2}\Delta_s$) must be averaged over a very short distance ε. Obviously:

$$H(z, x_s) = -\theta\,\delta\,[(z-x_s)-\alpha-\varepsilon)] + (\tfrac{1}{2}-\eta)\,\Delta_s\delta\,[(z-x_s)-\varepsilon)] \\ +(\tfrac{1}{2}-\eta)\,\Delta_s\delta\,[(z-x_s)+\varepsilon)] - \theta\,\delta[(z-x_s)+\alpha+\varepsilon)], \tag{2.4}$$

is a difference kernel, i.e. $H(z, x_s) = H(z - x_s)$ and $H(z - x_s) = H(x_s - z)$, symmetric.

Due to its direct dependence on the uncertain θ and η , the perturbation $H(z - x_s)$ as given by (2.4) is not a universal (in the Green's function sense) rudimentary kernel for the present Gibbs-Wilbraham effect. Moreover $\int_{-\infty}^{\infty} H(z-x_s)\mathrm{d}z = \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty} H(z-x)\mathrm{d}z = (1-2\eta)\,\Delta_s - 2\theta$, but:

$$\int_{-\infty}^{\infty}|H(z-x_s)|\mathrm{d}z = \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty} H(z-x)\mathrm{d}z = (1-2\eta)\,\Delta_s + 2\theta. \tag{2.5}$$

In the language of the theory of elasticity, one can think of this $H(z-x)$ as a fraction of the $g(z) - f(z) - p_s(z)$ perturbation "work" over a unit length of the variable z around a point z, that results from the presence of a discontinuity at the point $z = x$, of the total $g(z) - f(z) - p_s(z)$ perturbation "work" of this discontinuity over the entire z-axis. Conversely, as $H(z-x)$ is a difference kernel, then any interval dz around z of $f(z)$ will contribute to the perturbation $g(x) - f(x) - p_s(x)$ at a discontinuity at $z = x$ by an amount equal to $H(z-x)\,f(z)$. Furthermore, $H(z-x)$ may conventionally be looked at as some kind of a likelihood of a perturbation $\tau(x) = g(x) - f(x) - p_s(x)$ at any point x by an amount equal to $K(z-x)\,f(z)$ from dz around z. Subsequently, adding together all the infinitesimal contributions results in the total anomalous perturbation:

$$\tau(x) = \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty} H(z-x)f(z)\,\mathrm{d}z = q_s(x). \tag{2.6}$$

According to a theorem by Fejer [3], $g(x)$ converges to $f(x)$ at each point x of continuity of $f(x)$.This implies that relation (2.6) rewrites when:

$$Q_s(x) = g(x) - f(x) = \tau(x) - p_s(x),$$

as:

$$Q_s(x) = \lim_{\varepsilon\to 0}\left\{f(x-\varepsilon) + \tfrac{1}{2}\Delta_s\right\}P_{\varepsilon,s}(x) + \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty} H(z-x)f(z)\mathrm{d}z \\ = p_s(x) + q_s(x), \tag{2.7}$$

which is an asymptotic singular integral transform for the Gibbs-Wilbraham effect.

The inversion of this transform, when $g(x)$ is known (to determine $f(x)$), calls for solving the singular integral equation of the second kind:

$$g(x) = f(x) + \lim_{\varepsilon\to 0}\left\{f(x-\varepsilon) + \tfrac{1}{2}\Delta_s\right\}P_{\varepsilon,s}(x) + \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty} H(z-x)f(z)\mathrm{d}z, \tag{2.8}$$

with a symmetric kernel having a bounded norm. Indeed:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^2(z-x_s)f(z)dzdx = \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^2(z-x)f(z)dzdx \\ = \frac{(1-2\eta)^2}{2}\,\Delta_s{}^2 + 2\theta^2. \tag{2.9}$$

Since both the sifting term $p_s(x)$ and the spiky term $q_s(x)$ vanish away from $x = x_s$ and since both the real and imaginary parts of g(x) affect the integral equation, then (2.8) is functionally a dispersion singular integral equation.

In this setting, the 1 that leads $\tau(x)$ in (2.8) is just a constant and not an eigenvalue of the

$H(z-x)$ kernel, which happens to be strongly discontinuous in z. For this integral equation, the spectrum of eigenvalues though real, see e.g. [9], is however either continuous or its discrete eigenvalues may have infinite multiplicity.

3. Main Result

The fact that the right hand side of the norm relation (2.9) should necessarily be a positive quantity, i.e.

$$\frac{(1-2\eta)^2}{2}\,\Delta_s{}^2 + 2\theta^2 \geq 0, \tag{3.1}$$

implies that:

$$\theta^2 \geq -\left(\tfrac{1}{2}-\eta\right)^2\ \Delta_s{}^2\,, \tag{3.2}$$

which indicates an unrealistically imaginary θ.

As a matter of principle, θ in the above inequality is conceived as:

$$\theta = \theta(\Delta_s). \tag{3.3}$$

Incidentally, (3.2) happens to hold true for any free (uncorrelated) θ and η. Moreover, the simplest way for liberating θ and η of Δ_s in (3.2) is by assuming a linear approximation for (3.3) via:

$$\theta = \rho\ \Delta_s, \tag{3.4}$$

where ρ is the jump size scale, representing the associated constant of proportionality. Unfortunately, even with incorporating the (3.4) approximation, relation (3.2) remains unreal (redundant) for estimation of almost all real or imaginary ρ, as (3.2) is equivalent to:

$$\rho \leq -i\left(\tfrac{1}{2}-\eta\right) \text{ and } \rho \geq i\left(\tfrac{1}{2}-\eta\right),$$

where $i=\sqrt{-1}$. Clearly these inequalities can be meaningful for ρ only in their respective upper and lower bounds, i.e.

$$\rho = \mp i\left(\tfrac{1}{2}-\eta\right), \tag{3.5}$$

which are imaginary. Fortunately however, the following Lemma provides for an alternative to (3.5) real estimation of ρ.

Lemma 1 (Consistency)

Assumption of (3.4) in the kernel $H(z-x)$ of (2.4) requires that:

$$\rho \geq \left(\tfrac{1}{2}-\eta\right). \tag{3.6}$$

Proof

Instead of relation (2.9), consider the right hand side of (2.5), which is also necessarily a positive quantity, i.e.

$$(1-2\eta)\,\Delta_s + 2\theta \geq 0.$$

Rearrangement of the terms in this inequality after substitution of (3.4) leads to the required result.

The inequality (3.6) is in fact both a mathematical and a physical consistency condition for the singular integral transform representation of the Gibbs -Wilbraham effect.

Furthermore, consider the Fourier transform pair $f(x)\leftrightarrow\hat{f}(w)$, defined in the symmetrized form by:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{iwx}\ dx, \tag{3.7}$$

and:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{-iwx}\ dw. \tag{3.8}$$

Relation (3.8) holds for any point of continuity of $f(x)$,but is anomalous only at $x = x_s$

The validity of this claim and of the entire previous construction is demonstrated by the result that follows.

Lemma 2
If a Fourier transformable *f(x)*, which is discontinuous at $x = x_s$, is correlated to its inverse Fourier transform *g(x)* by the singular integral equation (2.7),with a kernel:

$$H(z-x) = -\theta\,\delta\,[(z-x)-\alpha-\varepsilon)] + (\tfrac{1}{2}-\eta)\,\Delta_s\delta\,[(z-x)-\varepsilon)]$$
$$+ (\tfrac{1}{2}-\eta)\,\Delta_s\delta\,[(z-x)+\varepsilon)] - \theta\,\delta[(z-x)+\alpha+\varepsilon)], \qquad (3.9)$$

satisfying a fixed jump size scale (3.4) with:

$$\rho = 0.089490, \qquad (3.10)$$

then (2.7) satisfies the consistency condition (3.6) and skew-symmetrization of the spiky part of the Gibbs-Wilbraham effect when:

$$\eta = 0.410\,510. \qquad (3.11)$$

At every $x \neq x_s^{\pm}$, however, $\lim_{\alpha,\varepsilon\to 0} \int_{-\infty}^{\infty} H(z-x)f(z)\mathrm{d}z = 0$ and:

$$g(x) = f(x) + \lim_{\varepsilon\to 0}\{f(x-\varepsilon) + \tfrac{1}{2}\Delta_s\}\,P_{\varepsilon,s}(x) = f(x) + p_s(x). \qquad (3.12)$$

Proof
Consider, without loss of generality, comparison of the step like function:

$$f(x) = f_1 + \Delta_s\, u(x-x_s), \qquad (3.13)$$

in which $\Delta_s = f_s^{+} - f_s^{-} = f_2 - f_1$, with its inverse Fourier transformation *g(x)* via (2.7) to invoke its analytical solution. This can most conveniently be done by application of Fourier transforms.

Obviously since:

$$\int_{-\infty}^{\infty} H(z-x)f(z)\mathrm{d}z \leftrightarrow \widehat{H}(w)\hat{f}(w),$$

then we may write the Fourier transform of (2.7) as $\hat{g}(w)$ in:

$$\hat{g}(w) - \widehat{p_s}(w) = \hat{f}(w) + \lim_{\alpha,\varepsilon\to 0} \sqrt{2\pi}\,\widehat{H}(w)\hat{f}(w). \qquad (3.14)$$

Taking the inverse Fourier transform of $\hat{\tau}(w) = \hat{g}(w) - \hat{f}(w) - \widehat{p_s}(w) = \lim_{\alpha,\varepsilon\to 0} \sqrt{2\pi}\,\widehat{H}(w)\hat{f}(w)$ results with:

$$\tau(x) = \frac{1}{\sqrt{2\pi}} \lim_{\alpha,\varepsilon\to 0} \int_{-\infty}^{\infty} \sqrt{2\pi}\,\widehat{H}(w)\hat{f}(w)e^{-iwx}\,dw. \qquad (3.15)$$

Next we invoke:

$$H(x) = a[\,\delta(x+\alpha) + \delta(x-\alpha)] + b\,[\,\delta(x+\varepsilon) + \delta(x-\varepsilon)],$$
$$a = -\theta,\ \ b = (\tfrac{1}{2}-\eta)\,\Delta_s, \qquad (3.16)$$

to define:

$$\sqrt{2\pi}\,\widehat{H}(w) = a[\,e^{-iw\alpha} + e^{iw\alpha}] + b\,[e^{-iw\varepsilon} + e^{iw\varepsilon}] = 2[a\cos w\alpha + b\cos w\varepsilon] \qquad (3.17)$$

and utilize:

$$\hat{f}(w) = \sqrt{2\pi}\,f_1\,\delta(w) + \sqrt{\frac{\pi}{2}}\,e^{iwx_s}\,\delta(w) + \frac{1}{\sqrt{2\pi}}\,\frac{1}{iw}\,e^{iwx_s}\,\delta(w)\,, \qquad (3.18)$$

in (3.15) to obtain:

$$\tau(x) = a\,\frac{1}{\sqrt{2\pi}} \lim_{\alpha\to 0} \int_{-\infty}^{\infty} e^{-iw\alpha}\hat{f}(w)e^{-iwx}\,dw + a\frac{1}{\sqrt{2\pi}} \lim_{\alpha\to 0} \int_{-\infty}^{\infty} e^{iw\alpha}\hat{f}(w)e^{-iwx}\,dw$$
$$+ b\,\frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-iw\varepsilon}\hat{f}(w)e^{-iwx}\,dw + b\frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{iw\varepsilon}\hat{f}(w)e^{-iwx}\,dw. \qquad (3.19)$$

By the frequency shift property of the Fourier transform, the last relation rewrites as:

$$\tau(x) = a \lim_{\alpha\to 0}[\,f(x+\alpha) + f(x-\alpha)] + b \lim_{\varepsilon\to 0}[\,f(x+\varepsilon) + f(x-\varepsilon)]. \qquad (3.20)$$

Analysis of this function reveals that it is representable as:

$$\tau(x) = \left(\tfrac{1}{2} - \eta\right) \Delta_s \lim_{\varepsilon\to 0} \{ u\,[x-(x_s+\varepsilon)] + u\,[x-(x_s-\varepsilon)]\}$$
$$- \theta \lim_{\alpha\to 0} \{ u\,[x-(x_s+\alpha)] + u\,[x-(x_s-\alpha)]\}, \qquad (3.21)$$

which exhibits a nonsymmetric, with respect to the *x*-axis , undershoot/overshoot spike at $x_s^{\pm}$ that vanishes elsewhere. This fact happens to prove (3.12).

The positive height of this spike, when ε→0, is $\left(\tfrac{1}{2} - \eta\right) \Delta_s$ while the negative height, in the limit when α→ 0, is θ. Skew-symmetrization of this spike requires the satisfaction of:

$$\theta = \left(\tfrac{1}{2} - \eta\right) \Delta_s \,. \qquad (3.22)$$

Further assumption that (3.10) holds in this equation implies (3.11). Also (3.22), (3.10) and (3.11) clearly satisfy the consistency condition (3.6) in its lower bound.

Accordingly, it follows by (2.7) that *f(x)* of (3.11) is representable by the inverse Fourier transformation:

$$g(x) = \left\{ f_1 + \tfrac{1}{2}\Delta_s \right\} \lim_{\varepsilon\to 0} P_{\varepsilon,s}(x) + \begin{cases} f_1 - \rho\Delta_s\,;\; x = x_s^- \\ f_1 + \Delta_s + \rho\Delta_s\,;\; x = x_s^+ \\ f_1 + \Delta_s\, u\,(x - x_s)\,;\; x \neq x_s \;or\; x_s^{\pm} \end{cases}$$
$$= p_s(x) + \{ f(x) + q_s(x) \}, \qquad (3.23)$$

which is the same as:

$$g(x) = \begin{cases} f_1 - \rho\Delta_s\,;\; x = x_s^- \\ f_1 + \tfrac{1}{2}\Delta_s\,;\; x = x_s \\ f_1 + \Delta_s + \rho\Delta_s\,;\; x = x_s^+ \\ f_1 + \Delta_s\, u\,(x - x_s)\,;\; x \neq x_s \;or\; x_s^{\pm} \end{cases} \qquad (3.24)$$

The associated Gibbs-Wilbraham effect is:

$$Q_s(x) = \left\{ f_1 + \tfrac{1}{2}\Delta_s \right\} \lim_{\varepsilon\to 0} P_{\varepsilon,s}(x) + \begin{cases} f_1 - \rho\Delta_s\,;\; x = x_s^- \\ f_1 + \Delta_s + \rho\Delta_s\,;\; x = x_s^+ \\ 0\;;\; x \neq x_s \;or\; x_s^{\pm} \end{cases}$$
$$= p_s(x) + q_s(x)\,. \qquad (3.25)$$

This turns out, as in (1.3), to essentially be:

$$Q_s(x) = p_s(x) + q_s(x) = \begin{cases} f_1 - \rho\Delta_s\;;\; x = x_s^- \\ f_1 + \tfrac{1}{2}\Delta_s\;;\; x = x_s \\ f_1 + \Delta_s + \rho\Delta_s\;;\; x = x_s^+ \\ 0\;;\; x \neq x_s \;or\; x_s^{\pm} \end{cases}.$$

Here the proof completes.

Theorem 1

A Fourier transformable:

$f(x) = f_c(x)\; u(x_s - x) + \Delta_s\; u(x - x_s)$,

emerges from the dispersion singular integral transform (2.7), with a kernel satisfying the conditions of Lemmas 1 and 2, as:

$$g(x) = \lim_{\varepsilon\to 0}\left\{ f_c(x-\varepsilon) + \tfrac{1}{2}\Delta_s\right\} P_{\varepsilon,s}(x) + \begin{cases} f_c(x)\ u(x_s - x) - \rho\Delta_s\ ;\ x = x_s^- \\ f_c(x_s^-)\ u(x_s - x_s^-) + \Delta_s + \rho\Delta_s\ ;\ x = x_s^+ \\ f_c(x)\ u(x_s - x) + \Delta_s\ u(x - x_s)\ ;\ x \neq x_s \text{ or } x_s^\pm \end{cases}$$

Proof

By following the same steps as in the proof of Lemma 2.

Obviously the previous last relation is the same as:

$$g(x) = \begin{cases} f_c(x)\ u(x_s - x) - \rho\Delta_s\ ;\ x = x_s^- \\ f_c(x_s^-) + \tfrac{1}{2}\Delta_s\ ;\ x = x_s \\ f_c(x_s^-)\ u(x_s - x_s^-) + \Delta_s + \rho\Delta_s\ ;\ x = x_s^+ \\ f_c(x)\ u(x_s - x) + \Delta_s\ u(x - x_s)\ ;\ x \neq x_s \text{ or } x_s^\pm \end{cases}$$

with the Gibbs-Wilbraham effect:

$$Q_s(x) = p_s(x) + q_s(x) = \begin{cases} f_c(x)\ u(x_s - x) - \rho\Delta_s\ ;\ x = x_s^- \\ f_c(x_s^-) + \tfrac{1}{2}\Delta_s\ ;\ x = x_s \\ f_c(x_s^-)\ u(x_s - x_s^-) + \Delta_s + \rho\Delta_s\ ;\ x = x_s^+ \\ 0\ ;\ x \neq x_s \text{ or } x_s^\pm \end{cases} .$$

4. Application

In actual fact the singular integral equation (2.7) turns out to be applicable to any Fourier transformable *f(x)* with a single jump discontinuity at $x = x_s$, as to be illustrated by the more general example that follows.

Example 4.1

Study the Gibbs-Wilbraham effect at the discontinuity of the function:

$$f(x) = \begin{cases} e^{-x}\ ;\ x > 0 \\ 0\ ;\ x < 0 \end{cases} . \quad (4.1)$$

This function $f(x) = e^{-x}u(x)$ is obviously discontinuous at the single point $x_s = 0$, with $f_s^- = 0$, $f_s^+ = 1$ and $\Delta_s = 1$. Moreover, $\lim_{\varepsilon\to 0}\left\{ f(0-\varepsilon) + \tfrac{1}{2}\right\} P_{\varepsilon,s}(0) = p_s(0)$. Its Fourier transform is:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x}u(x)\, e^{iwx}\, dx = \frac{1}{\sqrt{2\pi}} \frac{1+iw}{1+w^2} . \quad (4.2)$$

The inverse Fourier transform (real part) of this image is:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1+iw}{1+w^2}\, e^{-ixw}\, dw = \frac{1}{\pi} \int_0^{\infty} \frac{\cos xw}{1+w^2}\, dw + \frac{1}{\pi} \int_0^{\infty} \frac{w \sin xw}{1+w^2}\, dw . \quad (4.3)$$

Now in view of the fact that:

$$\int_0^{\infty} \frac{\cos xw}{1+w^2}\, dw = \frac{\pi}{2}\, e^{-x}\ ;\ x \geq 0,$$

$$\int_0^{\infty} \frac{w \sin xw}{1+w^2}\, dw = \frac{\pi}{2}\, e^{-x}\ ;\ x > 0, \quad (4.4)$$

relation (4.3) rewrites as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+iw}{1+w^2}\quad e^{-ixw}\,dw \;=\; \begin{cases} e^{-x}\;;\; x>0 \\ \frac{1}{2}\;;\; x=0 \\ 0\;;\; x<0 \end{cases}, \tag{4.5}$$

which is obviously different from $g(x) = f(x) + p_s(0) + q_s(0)$. The second line in (4.5) means that $f(0) + p_s(0) = \frac{1}{2}$. Hence we may conclude that inversion of the Fourier transform can in principle reveal, as a by-product, the sifting effect $p_s(x)$, but it definitely fails to reveal the spiking $q_s(x)$ effect.

Dispersion of both the sifting and spiking components of the present Gibbs-Wilbraham effect $Q_s(0) = p_s(0) + q_s(0)$ calls for evaluating the corresponding singular integral transform:

$$Q_s(0) = p_s(0) + \lim_{\alpha,\varepsilon\to 0}\int_{-\infty}^{\infty} H(z-x)e^{-z}u(z)\mathrm{d}z\;, \tag{4.6}$$

with a kernel $H(z-x)$ satisfying the conditions of Lemmas 1 and 2.

Analysis of the first two terms of $H(z-x)$ within (4.6) illustrates that the undershoot in $q_s(0)$ is:

$$\lim_{\alpha,\varepsilon\to 0}\; -\rho\Delta_s\{e^{-(x+\alpha+\varepsilon)}\,u(x+\alpha+\varepsilon)\;-e^{-(x+\varepsilon)}\,u(x+\varepsilon)\}$$

$$= \lim_{\alpha,\varepsilon\to 0}\begin{cases} 0\;;\; x<-\varepsilon-\alpha \\ -\rho\Delta_s\;;\; -\alpha-\varepsilon<x<-\varepsilon \\ \beta(x)\;;\; x>-\varepsilon \end{cases} = \begin{cases} 0\;;\; x<0 \\ -\rho\Delta_s\;;\; x=0^- \\ 0\;;\; x<0 \end{cases}, \tag{4.7}$$

where $\beta(x)$ is some low amplitude decaying positive function.

Similarly the overshoot in $q_s(0)$ is:

$$\lim_{\alpha,\varepsilon\to 0}\;\rho\Delta_s\{e^{-(x-\varepsilon)}\,u(x-\varepsilon)\;-e^{-(x-\alpha-\varepsilon)}\,u(x-\alpha-\varepsilon)\}$$

$$= \lim_{\alpha,\varepsilon\to 0}\begin{cases} 0\;;\; x>\varepsilon+\alpha \\ \rho\Delta_s\;;\; \varepsilon<x<\varepsilon+\alpha \\ \beta(x)\;;\; x<\varepsilon \end{cases} = \begin{cases} 0\;;\; x<0 \\ \rho\Delta_s\;;\; x=0^+ \\ 0\;;\; x<0 \end{cases}. \tag{4.8}$$

Next we add up (4.7) and (4.8) to obtain:

$$q_s(0) \;=\; \begin{cases} -\rho\Delta_s\;;\; x=0^- \\ \rho\Delta_s\;;\; x=0^+ \\ 0\;;\; x\neq 0\; or\; 0^{\mp} \end{cases}. \tag{4.9}$$

Hence solution of the associated singular integral equation reveals the entire anomalous behavior near the $x = 0$ discontinuity of the considered inverse Fourier transform:

$$g(x) = f(x) + p_s(0) + q_s(0) = f(x) + Q_s(0) = \begin{cases} 0\;;\; x<0 \\ -\rho\Delta_s\;;\; x=0^- \\ \frac{1}{2}\;;\; x=0 \\ 1+\rho\Delta_s\;;\; x=0^+ \\ e^{-x}\;;\; x>0 \end{cases}.$$

5. Concluding Remarks

The famous Gibbs-Wilbraham effect is a significant anomaly in the representation of discontinuous functions by orthogonal polynomials at points of their discontinuity. A long history of investigations, that had started between 1848 and 1898, have led to numerous algorithms for removal of this effect. One of the most well known and widely used, is spectral reprojection and related techniques.

The present note reports on a timely entirely different new approach to the understanding of this effect. An approach that correlates the inverse Fourier transform with its discontinuous preimage by a dispersive singular integral equation of the second kind, with a symmetric kernel $H(z-x)$, that satisfies the conditions of Lemmas 1 and 2. It introduces for the first time the techniques of singular integral equations to representation of this classical anomaly.

Finally, it is anticipated that the solvability of this singular integral equation for a variety of discontinuous *f(x)* can hopefully deepen our understanding of the Gibbs-Wilbraham effect.

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