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Numerical Treatments for the Two-Dimensional Mixed Nonlinear Integral Equation in Time and Position

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Abstract
In this paper, the existence of a unique solution of a mixed two-dimensional nonlinear integral equation (**MT-DNIE**), with continuous kernels is discussed and proved using Banach fixed point theorem. A numerical method is used to obtain a system of Hammerstein two-dimensional integral equations (**SHT-DIEs**). Moreover, the Adomian Decomposition Method (**ADM)** and the Homotopy analysis method (**HAM**) are used to obtain, in each method, a nonlinear algebraic system (**NLAS**). The existences of a unique solution of these methods is discussed. The Adomian polynomial formula is used directly to prove the convergence of Adomian series, and the maximum absolute truncation error is obtained. Numerical examples are used to illustrate the preciseness and effectiveness of these proposed methods, and the estimate error, in each case, is computed. Finally, some conclusions are obtained.

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Mixed Two-Dimensional Nonlinear Integral Equation, System of Hammerstein Two-Dimensional Integral Equations, Nonlinear Algebraic System, Adomian Decomposition Method, Homotopy Analysis Method.

1. Introduction

The theory of integral equations has close contact with many different areas of mathematics. Many problems in the field of ordinary and partial differential equations, viscodynamics fluids, contact problems in the theory of elasticity, mixed boundary problems in mathematical physics, biology, chemistry and engineering can be formulated as an integral equation, see [1]-[4]. This suffices to say that there is almost no area of applied sciences and physics where integral equations do not play an important role. Therefore, many different analytic and numeric methods are used to obtain the solution of the linear and nonlinear integral equations, see [5]-[11]. In fact, the numerical methods have a significant progress to obtain the numerical solution of one- dimensional mixed linear and nonlinear integral equations, while the numerical methods for two-dimensional integral equations seem to have been discussed in only a few places, see [12]-[18]. For this, by the current work, a mixed nonlinear two-dimensional nonlinear integral equation will be discussed analytically and numerically.

Consider the following (**MT-DNIE**) of the second kind:

$$\mu\,\phi(x,y;t)=f(x,y;t)+\lambda\int_0^t\int_c^d\int_a^b K(x,u;y,v)F(t,\tau)\gamma\big(u,v;\tau,\phi(u,v;\tau)\big)\,du\,dv\,d\tau. \quad (1.1)$$

Here, $f(x,y;t)$ and $\gamma\big(x,y;t,\phi(x,y;t)\big)$ are given functions in the Banach space $C(J\times[0,T])$, where $J=[a,b]\times[c,d]$, and $t\in[0,T]\,;T<1$, while $\phi(x,y;t)$ is the unknown function. The two kernels, of position $K(x,u;y,v)$ and of time $F(t,\tau)\,;t,\tau\in[0,T]\,,T<1,$ are continuous. The constant μ defines the kind of the integral equation, while λ is a constant, may be complex, that has a physical meaning.

In this paper, the existence of a unique solution of equation (1.1) under certain conditions is discussed and proved by using Banach fixed point theorem. A numerical method is applied on the **MT-DNIE** (1.1) to obtain a **SHT-DIEs**. The **ADM** is used to obtain a **NLAS**. Then Adomian polynomial formula is used directly to prove the convergence of Adomian series, and the maximum absolute truncation error is obtained. The **HAM** is also used to obtain a **NLAS** where, the convergence of the method is discussed. Finally, some numerical results are obtained and the estimate error, in each case, is computed.

2. The Existence of a Unique Solution of MT-DNIE

In this section, Banach fixed point theorem will be used to prove the existence of a unique solution of equation (1.1). For this aim, we rewrite equation (1.1) in the integral operator form:

$$\bar{W}\phi(x,y;t) = \frac{1}{\mu} f(x,y;t) + W\phi(x,y;t) \quad , \tag{2.1}$$

where:

$$W\phi(x,y;t) = \frac{\lambda}{\mu}\int_0^t \int_c^d \int_a^b K(x,u;y,v)F(t,\tau)\gamma\big(u,v;\tau,\phi(u,v;\tau)\big)\, du\, dv\, d\tau. \tag{2.2}$$

Also, we state the following conditions:

(i) The kernel of position $K(x,u;y,v) \in C(J \times J)$ is a continuous function and satisfies: $|K(x,u;y,v)| \le M$, (M is a constant).

(ii) The kernel of time $F(t,\tau) \in C[0,T]$, $0 \le \tau \le t \le T < 1$, is a continuous function and satisfies: $|F(t,\tau)| \le L$, (L is a constant, $\forall\, t,\tau \in [0,T]$).

(iii) The given function $f(x,y;t)$ with its partial derivatives with respect to position and time are continuous functions in the space $C(J \times [0,T])$, and its norm is defined by:
$\|f(x,y;t)\|_{C(J\times[0,T])} = \max_{x,y,t}|f(x,y;t)| \le \mathcal{H}$, ($\mathcal{H}$ is a constant).

(iv) The given function $\gamma\big(x,y;t,\phi(x,y;t)\big)$ satisfies for the constants A_1, A_2 and $A \ge max\{A_1, A_2\}$ the following conditions:

(a) $\max_{x,y,t}\left|\gamma\big(x,y;t,\phi(x,y;t)\big)\right| \le A_1\|\phi(x,y;t)\|_{C(J\times[0,T])}$.

(b) $\left|\gamma\big(x,y;t,\phi_1(x,y;t)\big) - \gamma\big(x,y;t,\phi_2(x,y;t)\big)\right| \le A_2|\phi_1(x,y;t) - \phi_2(x,y;t)|$.

Theorem 1

Equation (1.1) has a unique solution in the space $C(J \times [0,T])$ under the condition:

$$|\mu| > |\lambda|ABMLT. \tag{2.3}$$

The proof of this theorem depends on the following two lemmas.

Lemma 1

The integral operator $\bar{W}$ maps the space $C(J \times [0,T])$ into its self.

Proof

In view of the equations (2.1) and (2.2), we obtain:

$$|\bar{W}\,\phi(x,y;t)| \le \frac{1}{|\mu|}|f(x,y;t)| + \frac{|\lambda|}{|\mu|}\int_0^t\int_c^d\int_a^b |K(x,u;y,v)|\,|F(t,\tau)| \max_{u,v,\tau}\left|\gamma\big(u,v;\tau,\phi(u,v;\tau)\big)\right| du\, dv\, d\tau\,.$$

From the conditions $\big((i) - (iv-a)\big)$, we get:

$$\|\bar{W}\,\phi(x,y;t)\| \le \frac{\mathcal{H}}{|\mu|} + \delta\,\|\phi\|, \left(\delta = \frac{|\lambda|}{|\mu|}\,ABMLT\ ; B = (b-a)(d-c)\right). \tag{2.4}$$

Inequality (2.4) shows that the operator $\bar{W}$ maps the ball S_Ω into its self, where:

$$\Omega = \frac{\mathcal{H}}{(|\mu| - |\lambda|ABMLT)}\,. \tag{2.5}$$

Since, $\Omega > 0$ and $\mathcal{H} > 0$, therefore we have $\delta < 1$. Moreover, the inequality (2.4) involves the boundedness of the operator W, where:

$$\|W\,\phi(x,y;t)\| \leq \delta\,\|\phi(x,y;t)\| \quad . \tag{2.6}$$

Also, the inequalities (2.4) and (2.6) define the boundedness of the operator $\bar{W}$.

Lemma 2

The integral operator $\bar{W}$ is continuous and contraction in the Banach space $C(J \times [0,T])$.

Proof

For the two functions $\phi_1(x,y;t)$ and $\phi_2(x,y;t)$ in the space $C(J \times [0,T])$, the mixed integral equation (1.1), yields:

$$|\bar{W}\,\phi_1(x,y;t) - \bar{W}\,\phi_2(x,y;t)| \leq \frac{|\lambda|}{|\mu|}\left|\int_0^t \int_c^d \int_a^b |K(x,u;y,v)|\,|F(t,\tau)|\max_{u,v,\tau}\left|\gamma\big(u,v;\tau,\phi_1(u,v;\tau)\big) - \gamma\big(u,v;\tau,\phi_2(u,v;\tau)\big)\right|\,du\,dv\,d\tau\right|.$$

Using the conditions (i), (ii) and $(iv-b)$, we get:

$$\|\bar{W}\,\phi_1(x,y;t) - \bar{W}\,\phi_2(x,y;t)\| \leq \delta\|\phi_1(x,y;t) - \phi_2(x,y;t)\| \,. \tag{2.7}$$

From inequality (2.7), we see that $\bar{W}$ is continuous in the space $C(J \times [0,T])$. Moreover, $\bar{W}$ is a contraction operator under the condition $\delta < 1$.

Proof of Theorem 1

Lemmas (1) and (2) show that, the operator $\bar{W}$ of equation (2.1) is contractive in the Banach space $C(J \times [0,T])$. So, from Banach fixed point theorem, $\bar{W}$ has a unique fixed point which of course, the unique solution of equation (1.1).

3. An algebraic System of Hammerstein Two-Dimensional Integral Equations

In this section, a numerical method is applied to the **MT-DNIE** (1.1) to obtain a system of nonlinear integral equations in position.

If we divide the interval $[0,T]$, $0 \leq \tau \leq t \leq T < 1$, as $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_N = T$, where $t = t_k$, $k = 0,1,2,\ldots,N$, the integral term of equation (1.1) becomes:

$$\int_0^{t_k}\int_c^d\int_a^b K(x,u;y,v)\,F(t_k,\tau)\,\gamma\big(u,v;\tau,\phi(u,v;\tau)\big)\,du\,dv\,d\tau$$
$$= \sum_{j=0}^{k} w_j\,F\big(t_k,t_j\big)\int_c^d\int_a^b K(x,u;y,v)\,\gamma\left(u,v;t_j,\phi\big(u,v;t_j\big)\right)du\,dv + O\big(\hbar_k^{p+1}\big)\,,$$
$$(\hbar_k \to 0\,, p > 0)\,. \tag{3.1}$$

where, $\hbar_k = \max_{0\leq j\leq k} h_j$, $h_j = t_{j+1} - t_j$, $w_0 = \frac{1}{2}h_k$, $w_k = \frac{1}{2}h_k$, $w_j = h_j$, $(j \neq 0,k)$. The values of w_j's and p ; $p \approx k$ are depending on the number of derivatives of $F(t,\tau)$ with respect to time , see [19,20] .

Using (3.1) in (1.1), and neglecting $O\big(\hbar_k^{p+1}\big)$, we have:

$$\mu\,\phi_k(x,y) = f_k(x,y) + \lambda\sum_{j=0}^{k}\omega_j\,F_{k,j}\int_c^d\int_a^b K(x,u;y,v)\,\gamma_j\left(u,v,\phi_j(u,v)\right)du\,dv, \tag{3.2}$$

where,

$$\gamma_j\left(x,y,\phi_j(x,y)\right) = \gamma\left(x,y;t_j,\phi\big(x,y;t_j\big)\right)\,,\; F\big(t_i,t_j\big) = F_{i,j}\;,\; f\big(x,y;t_j\big) = f_j(x,y)\,. \tag{3.3}$$

The formula (3.2) represents an algebraic **SHT-DIEs**, and its solution depends on the given functions $f_k(x,y)$, and the kind of the kernel $K(x,u;y,v)$.

Remark 1
Let $C(J)$ be the set of all continuous functions $\{\phi_1(x,y),\ \phi_2(x,y),\ \phi_3(x,y),\dots\}$, where, $\phi_k(x,y)\in C(J)\,,\forall k$. Define on $C(J)$ the norm $\|\phi_k(x,y)\|_{C(J)}=\max_k \max_{x,y\in J}|\phi_k(x,y)|$, then $C(J)$ is a Banach space .

3.1. The Existence of a Unique Solution of the Algebraic SHT-DIEs

In order to guarantee the existence of a unique solution of the algebraic system (3.2) in the Banach space $C(J)$, we write this system in the integral operator form:

$$\bar{U}\phi_k(x,y)=\frac{1}{\mu}f_k(x,y)+U\phi_k(x,y)\,,$$

where:

$$U\phi_k(x,y)=\frac{\lambda}{\mu}\sum_{j=0}^{k}\omega_j\ F_{k,j}\int_c^d\int_a^b K(x,u;y,v)\,\gamma_j\left(u,v,\phi_j(u,v)\right)du\,dv\ .$$

Then, we assume in addition to condition (i) of theorem (1), the following conditions:
(1) $\max_k \max_{(x,y)\in J}|f_k(x,y)|=\|f_k(x,y)\|_{C(J)}\le\mathcal{L}$, ($\mathcal{L}$ is a constant) .
(2) $\sum_{j=0}^{k}\max_k\left|\omega_j F_{k,j}\right|\le\alpha$, (α is a constant) .
(3) The known continuous functions $\gamma_j\left(x,y,\phi_j(x,y)\right)$,$\forall j$ satisfy for the constants ρ_1,ρ_2 and $\rho\ge max\{\rho_1,\rho_2\}$, the following conditions :
(a_1) $\max_k \max_{(x,y)\in J}\left|\gamma_k\big(x,y,\phi_k(x,y)\big)\right|\le\rho_1\|\phi_k(x,y)\|_{C(J)}$.
(a_2) $\left|\gamma_k\left(x,y,\phi_k^{(1)}(x,y)\right)-\gamma_k\left(x,y,\phi_k^{(2)}(x,y)\right)\right|\le\rho_2\left|\phi_k^{(1)}(x,y)-\phi_k^{(2)}(x,y)\right|$.

Theorem 2 **(without proof)**
The algebraic system (3.2) has a unique solution in the Banach space $C(J)$ under the condition :

$$|\mu|>|\lambda|M\rho\alpha. \tag{3.4}$$

4. The Adomian Decomposition Method (ADM)

The **ADM** is considered as a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations and so on. In [15]-[18] the **ADM** was used to solve the integral equation in one-dimensional and two-dimensional.

Here, we will discuss and solve the following system of integral equations by using **ADM**:

$$\phi_k(x,y)=f_k(x,y)+\lambda\sum_{j=0}^{k}\omega_j F_{k,j}\int_c^d\int_a^b K(x,u;y,v)\,\gamma_j\left(u,v,\phi_j(u,v)\right)du\,dv\ . \tag{4.1}$$

Then, assume the functions $\phi_k(x,y)$ in the form:

$$\phi_k(x,y)=\sum_{n=0}^{\infty}\psi_{k,n}(x,y). \tag{4.2}$$

On the other hand, the nonlinear term $\gamma_k\big(u,v,\phi_k(u,v)\big)$ of (4.1) is decomposed into an infinite series of Adomain polynomials:

$$\gamma_k\big(u,v,\phi_k(u,v)\big)=\sum_{n=0}^{\infty}A_{k,n}\,, \tag{4.3}$$

where the traditional formula of $A_{k,n}$ is:

$$A_{k,n}=\frac{1}{n!}\left(\frac{d^n}{d\zeta^n}\gamma_k\big(x,y,\textstyle\sum_{i=0}^{\infty}\zeta^i\psi_{k,i}\big)\right)_{\zeta=0}. \tag{4.4}$$

Another formula of Adomian polynomials, which was deduced in [10], is given by:

$$A_{k,n} = \gamma_k(S_{k,n}) - \sum_{i=0}^{n-1} A_{k,i} \ , \tag{4.5}$$

where, the partial sum is:

$$S_{k,i} = \sum_{i=0}^{n} \psi_{k,i}(x,y) \ . \tag{4.6}$$

After applying the **ADM** on equation (4.1), we have:

$$\psi_{k,0}(x,y) = f_k(x,y) \ ;$$

$$\psi_{k,i}(x,y) = \lambda \sum_{j=0}^{k} \omega_j F_{k,j} \int_c^d \int_a^b K(x,u;y,v)\, A_{j,(i-1)}(u,v)\, du\, dv \ , (\ i \geq 1). \tag{4.7}$$

4. 1. Convergence of the ADM

The next theorem shows the convergence of Adomian series to a unique solution of equation (4.1).

Theorem 3

Under the conditions of theorem (2), the series (4.2) converges uniformly to a unique continuous solution of equation (4.1) in the Banach space $C(J)$.

Proof

Let $\{S_{k,n}\}_{n=1}^{\infty}$ be a sequence of partial sums, where:

$$S_{k,n} = \psi_{k,0} + \psi_{k,1} + \cdots + \psi_{k,n} \ , \ S_{k,0} = \psi_{k,0} \ ; \ \forall 0 \leq k \leq N \ . \tag{4.8}$$

$$\sum_{n=0}^{m} \psi_{k,n} = \sum_{n=0}^{\ell} A_{k,n} \ ; \ \forall 0 \leq k \leq N \ . \tag{4.9}$$

Thus, from equations (4.8) and (4.9), we get:

$$A_{k,0} = \psi_{k,0} = S_{k,0} \ ; \ A_{k,0} + A_{k,1} = \psi_{k,0} + \psi_{k,1} = S_{k,1} \ ; \ k = 0,1,2,\ldots,N \ . \tag{4.10}$$

In general, we have:

$$\sum_{n=0}^{m} A_{k,n} = S_{k,m} \ . \tag{4.11}$$

Let $S_{k,(n+1)}$ and $S_{k,n}$ are two arbitrary partial sums of the sequence $\{S_{k,n}\}_{n=1}^{\infty}$, then from equations (4.7) and (4.11), we get:

$$\left\|S_{k,(n+1)} - S_{k,n}\right\|_{C(J)} = \max_k \max_{x,y\in J} \left|S_{k,(n+1)} - S_{k,n}\right|$$

$$\leq |\lambda| \max_k \max_{x,y\in J} \left|\sum_{j=0}^{k} \left|\omega_j F_{k,j}\right| \int_c^d \int_a^b |K(x,u;y,v)| \left|\gamma_k(S_{k,n}) - \gamma_k(S_{k,(n-1)})\right| du\, dv\right| .$$

Using the conditions (i), (1) and (2), the above inequality becomes:

$$\left\|S_{k,(n+1)} - S_{k,n}\right\|_{C(J)} \leq \alpha \left\|S_{k,n} - S_{k,(n-)}\right\|_{C(J)} , \ (\alpha = |\lambda| M A \mathcal{B}) \ .$$

Applying a successive application, we have:

$$\left\|S_{k,(n+1)} - S_{k,n}\right\|_{C(J)} \leq \alpha \left\|S_{k,n} - S_{k,(n-1)}\right\|_{C(J)} \leq \alpha^2 \left\|S_{k,(n-1)} - S_{k,(n-2)}\right\|_{C(J)} \leq \cdots$$

$$\leq \alpha^n \left\|S_{k,1} - S_{k,0}\right\|_{C(J)} .$$

More generally, if $n > m$, we obtain:

$$\left\|S_{k,n} - S_{k,m}\right\|_{C(J)}$$

$$= \left\|(S_{k,n} - S_{k,(n-1)}) + (S_{k,(n-1)} - S_{k,(n-2)}) + \cdots + (S_{k,(m+1)} - S_{k,m})\right\|_{C(J)}$$

$$\leq \left\|S_{k,n} - S_{k,(n-1)}\right\|_{C(J)} + \left\|S_{k,(n-1)} - S_{k,(n-2)}\right\|_{C(J)} + \cdots + \left\|S_{k,(m+1)} - S_{k,m}\right\|_{C(J)} .$$

$$\le (\alpha^{n-1}+\alpha^{n-2}+\cdots+\alpha^{m})\left\|S_{k,1}-\ S_{k,0}\right\|_{C(J)}=\frac{\alpha^{m}}{1-\alpha}\left\|S_{k,1}-\ S_{k,0}\right\|_{C(J)}.$$

Hence,

$$\left\|S_{k,n}-\ S_{k,m}\right\|_{C(J)}\le\frac{\alpha^{m}}{1-\alpha}\left\|\psi_{k,1}(x,y)\right\|_{C(J)}. \tag{4.12}$$

Provided $0<\alpha<1$, so that $\lim_{n,m\to\infty}\left\|S_{k,n}-\ S_{k,m}\right\|_{C(J)}=0$. It follows that $\{S_{k,n}\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space $C(J)$, hence it converges to a limit which is the solution of equation (4.1) .

Theorem 4

If the conditions of theorem (3) are satisfied, then the maximum absolute truncation error of the series (4.2) to problem (4.1) is estimated to be:

$$\max_{x,y\in J}\left|\phi_k(x,y)-\sum_{i=0}^{m}\psi_{k,i}(x,y)\right|\le\frac{\alpha^{m+1}}{1-\alpha}\max_{x,y\in J}\left|\psi_{k,0}(x,y)\right|.$$

Proof

From the inequality (4.12) of theorem (3), we have:

$$\left\|S_{k,n}-\ S_{k,m}\right\|_{C(J)}\le\frac{\alpha^{m}}{1-\alpha}\max_{x,y\in J}\left|\psi_{k,1}(x,y)\right|.$$

Since $S_{k,n}\to\phi_k(x,y)$ as $k\to\infty$, we obtain:

$$\max_{x,y\in J}\left|\psi_{k,1}(x,y)\right|\le\ \alpha\max_{x,y\in J}\left|\psi_{k,0}(x,y)\right|\ ,\qquad (\alpha=|\ \lambda|MA\mathcal{B}).$$

Finally, the maximum absolute truncation error is:

$$\max_{x,y\in J}\left|\phi_k(x,y)-\sum_{i=0}^{m}\psi_{k,i}(x,y)\right|\le\frac{\alpha^{m+1}}{1-\alpha}\max_{x,y\in J}\left|\psi_{k,0}(x,y)\right|. \tag{4.13}$$

5. The Homotopy Analysis Method (HAM)

The **HAM** is in principle based on Taylor series with respect to an embedding parameter and this method is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, and partial differential equations, see [21]-[24]. More importantly, the **HAM** provides us a simple way to ensure the convergence of solution series.

In this section, the **SHT-DIEs** (4.1) will be solved by using **HAM.** Write the **SHT-DIEs** (4.1) in the form:

$$\widetilde{N}[\phi_k(x,y)]=\ \phi_k(x,y)-\lambda\sum_{j=0}^{k}\omega_jF_{k,j}\int_c^d\int_a^b K(x,u;y,v)\,\gamma_j\left(u,v,\phi_j(u,v)\right)dudv$$
$$-f_k(x,y)=0. \tag{5.1}$$

where, $\widetilde{N}$ is an operator, $\phi_k(x,y)$ are unknown functions and x,y are independent variables. Let $\phi_{k,0}(x,y)$ denote an initial guess of the exact solutions $\phi_k(x,y)$; $\forall\ 0\le k\le N, \ell\neq 0$ an auxiliary parameter, $H(x,y)$ is an auxiliary function and $\tilde{L}$ an auxiliary linear operator with the property $\tilde{L}[f_k(x,y)]=0$ when $f_k(x,y)=0$. Using $r\in[0,1]$ as an embedding parameter, we construct such a homotopy system.

$$(1-r)\,\tilde{L}\left[\psi_k(x,y;r)-\phi_{k,0}(x,y)\right]-r\,\ell\,H(x,y)\widetilde{N}[\psi_k(x,y;r)]$$
$$=\widetilde{H}\left[\psi_k(x,y;r);\phi_{k,0}(x,y),H(x,y),\ell,r\right]. \tag{5.2}$$

It should be emphasized that we have a great freedom to choose the initial guess $\phi_{k,0}(x,y)$, the auxiliary linear operator $\tilde{L}$, the non-zero auxiliary parameter ℓ, and the auxiliary function $H(x,y)$,

$\widetilde{H}$ is the second auxiliary function. Assume the homotopy (5.2) to be zero, i.e., $\widetilde{H}[\psi_k(x,y;r);\phi_{k,0}(x,y),H(x,y),\ell,r]=0$. We have the so–called zero-order deformation equation

$$(1-r)\,\tilde{L}[\psi_k(x,y;r)-\phi_{k,0}(x,y)]=r\,\ell\,H(x,y)\tilde{N}[\psi_k(x,y;r)]\ ;\forall 0\le k\le N. \tag{5.3}$$

When $r=0$, the zero-order deformation (5.3) becomes $\psi_k(x,y;0)=\phi_{k,0}(x,y)$. Also, when $r=1,\ell\neq 0$ and $H(x,y)\neq 0$, the zero–order deformation (5.3) leads to $\psi_k(x,y;1)=\phi_k(x,y)$. By Taylor's theorem, $\psi_k(x,y;r)$ can be represents in a power series form of *r* as follows:

$$\psi_k(x,y;r)=\phi_{k,0}(x,y)+\textstyle\sum_{m=1}^{\infty}\phi_{k,m}(x,y)\,r^m\ , \tag{5.4}$$

where

$$\phi_{k,m}(x,y)=\frac{1}{m!}\left(\frac{\partial^m\psi_k(x,y;r)}{\partial r^m}\right)_{r=0}. \tag{5.5}$$

Here, the nonzero auxiliary parameter ℓ, and the auxiliary function $H(x,y)$ are properly chosen, so that the power series (5.4) of $\psi_k(x,y;r)$ converges at *r* =1. Then, under these assumptions, we have the series solution:

$$\phi_k(x,y)=\psi_k(x,y;1)=\phi_{k,0}(x,y)+\textstyle\sum_{m=1}^{\infty}\phi_{k,m}(x,y)\,. \tag{5.6}$$

Therefore, we can define the vector:

$$\overrightarrow{\phi_{k,n}}(x,y)=\Big(\phi_{k,0}(x,y),\phi_{k,1}(x,y),\dots,\phi_{k,n}(x,y)\Big). \tag{5.7}$$

According to the definition (5.5), the governing equation of $\phi_{k,m}(x,y)$ can be derived from the zero-order deformation equation (5.4). Differentiating (5.4) *m* times with respective to *r*, then dividing the results by $m!$ and setting *r* = 0, we have the so–called m^{th} order deformation equation:

$$\tilde{L}\,[\phi_{k,m}(x,y)-\xi_m\phi_{k,(m-1)}(x,y)]=\ell\,H(x,y)\,R_m[\overrightarrow{\phi_{k,(m-1)}}(x,y)]\ ;$$
$$\phi_{k,m}(0,0)=0. \tag{5.8}$$

where, $\forall\ 0\le k\le N$, we have:

$$R_m[\overrightarrow{\phi_{k,(m-1)}}(x,y)]=\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial r^{m-1}}\,\tilde{N}\big(\textstyle\sum_{i=0}^{\infty}\phi_{k,i}(x,y)\ \ r^i\big)\right)_{r=0},$$
$$\xi_m=\begin{cases}0 & m\le 1\\ 1 & m>1\end{cases}. \tag{5.9}$$

Note that the high-order deformation of equation (5.8) is governing by the linear operator $\tilde{L}$, and the term $R_m[\overrightarrow{\phi_{k,(m-1)}}(x,y)]$ can be expressed simply by (5.9) for any operator $\tilde{N}$.

5.1. The Computational Procedure for the HAM

In this section, we will use the **HAM** to solve the nonlinear system of integral equations of the second kind in two-dimensional (4.1). For this, the corresponding $(m\times k)$ order deformations of equation (5.5) can be written as:

$$\tilde{L}\,[\phi_{k,m}(x,y)-\xi_m\phi_{k,(m-1)}(x,y)]=\ell\,H(x,y)\,R_{k,(m-1)}[\overrightarrow{\phi_{k,(m-1)}}(x,y)]\ ;$$
$$\phi_{k,m}(0,0)=0, \tag{5.10}$$

where,

$$R_{k,m}[\overrightarrow{\phi_{k,(m-1)}}(x,y)]=\phi_{k,m-1}-\frac{(1-\xi_m)}{(m-1)!}\,f_k\,(x,y)$$
$$-\lambda\textstyle\sum_{j=0}^{k}\omega_jF_{k,j}\int_c^d\int_a^b K(x,u;y,v)\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial r^{m-1}}\,\tilde{N}\big(\sum_{m=1}^{\infty}\phi_{j,m}(u,v)\,r^m\big)\right]_{r=0}du\,dv\,. \tag{5.11}$$

So, to obtain a simple iteration formula for $\phi_{k,m}(x,y)$, choose $\tilde{L}\,\Phi = \Phi$ as an auxiliary linear operator, as a zero–order approximation to the desired function $\phi_k(x,y)$, the solution $\phi_{k,0}(x,y) = f_k(x,y)$ is taken, and the nonzero auxiliary parameter ℓ and the auxiliary function $H(x,y)$, can be taken as $\ell = -1, H(x,y) = 1$, then substituting into (5.10) to obtain:

$$\phi_{k,0}(x,y) = f_k(x,y) \quad ,$$

$$\phi_{k,m}(x,y) = \sum_{j=0}^{k} \omega_j F_{k,j} \int_c^d \int_a^b K(x,u;y,v)\, \frac{1}{(m-1)!} \left[\frac{\partial^{m-1}}{\partial r^{m-1}} \widetilde{N}\left(\sum_{m=1}^{\infty} \phi_{j,m}(u,v)\, r^m\right)\right]_{r=0} du\, dv,$$
$$m = 1,2, \dots \; .$$

and the corresponding homotopy series solution is given by:

$$\phi(x,y,t_k) = \phi_k(x,y) = \sum_{m=0}^{\infty} \phi_{k,m}(x,y)\,. \qquad (5.12)$$

It is not difficult to prove the convergence of the method.

6. Numerical Experiments and Discussions

Example (1): Consider the mixed integral equation

$$\phi(x,y;t) = f(x,y;t) + 2\int_0^t \int_0^1 \int_0^1 e^{uv} x\, y\, t\, \tau\; \log^2\left(1 + \phi(u,v;\tau)\right) du\, dv\, d\tau\,. \qquad (6.1)$$

For which the exact solution $\phi(x,y;t) = t\, x^2\, y^2$.

In the following Tables (1, 2, 3), we present the exact and the approximate numerical solutions using **HAM** and **ADM**, and the corresponding errors of equation (6.1), for some points of $0 \le x\,, y \le 1$. These tables show that the errors of the approximate solutions, by **HAM** and **ADM**, have accuracies of $10^{-16}, 10^{-5}, 10^{-2}$ when *T*= 0.001, 0.08 , 0.5 , respectively.

Table (1) *T*=0.001

x	*y*	ϕ **Exact**	ϕ **HAM**	**Error HAM**	ϕ **ADM**	**Error ADM**
0.1	0.1	1.0000E-07	1.0000E-07	5.2940E-23	1.0000E-07	5.2940E-23
	0.4	1.6000E-06	1.6000E-06	8.4703E-22	1.6000E-06	8.4703E-22
	0.7	4.9000E-06	4.9000E-06	8.4703E-22	4.9000E-06	8.4703E-22
	1	1.0000E-05	1.0000E-05	1.6941E-21	1.0000E-05	1.6941E-21
0.4	0.4	2.5600E-05	2.5600E-05	1.3553E-20	2.5600E-05	1.3553E-20
	0.7	7.8400E-05	7.8400E-05	1.3553E-20	7.8400E-05	1.3553E-20
	1	1.6000E-04	1.6000E-04	2.7105E-20	1.6000E-04	2.7105E-20
0.7	0.7	2.4010E-04	2.4010E-04	8.1315E-20	2.4010E-04	8.1315E-20
	1	4.9000E-04	4.9000E-04	0.0000E+00	4.9000E-04	0.0000E+00
1	1	1.0000E-03	1.0000E-03	0.0000E+00	1.0000E-03	0.0000E+00

Table (2) *T*=0.08

x	*y*	ϕ **Exact**	ϕ **HAM**	**Error HAM**	ϕ **ADM**	**Error ADM**
0.1	0.1	8.0000E-06	7.9987E-06	1.2748E-09	7.9987E-06	1.2748E-09
	0.4	1.2800E-04	1.2799E-04	5.0994E-09	1.2799E-04	5.0994E-09
	0.7	3.9200E-04	3.9199E-04	8.9239E-09	3.9199E-04	8.9239E-09
	1	8.0000E-04	7.9999E-04	1.2748E-08	7.9999E-04	1.2748E-08
0.4	0.4	2.0480E-03	2.0480E-03	2.0397E-08	2.0480E-03	2.0397E-08
	0.7	6.2720E-03	6.2720E-03	3.5696E-08	6.2720E-03	3.5696E-08
	1	1.2800E-02	1.2800E-02	5.0990E-08	1.2800E-02	5.0990E-08

0.7	0.7	1.9208E-02	1.9208E-02	6.2470E-08	1.9208E-02	6.2470E-08
	1	3.9200E-02	3.9200E-02	8.9240E-08	3.9200E-02	8.9240E-08
1	1	8.0000E-02	8.0000E-02	1.2746E-07	8.0000E-02	1.2748E-07

Table (3) ***T*=0.5**

x	*y*	ϕ Exact	ϕ HAM	Error HAM	ϕ ADM	Error ADM
0.1	0.1	5.0000E-05	4.0008E-05	9.9919E-06	4.0008E-05	9.9919E-06
	0.4	8.0000E-04	7.6003E-04	3.9968E-05	7.6003E-04	3.9968E-05
	0.7	2.4500E-03	2.3801E-03	6.9943E-05	2.3801E-03	6.9943E-05
	1	5.0000E-03	4.9001E-03	9.9919E-05	4.9001E-03	9.9919E-05
0.4	0.4	1.2800E-02	1.2640E-02	1.5987E-04	1.2640E-02	1.5987E-04
	0.7	3.9200E-02	3.8920E-02	2.7976E-04	3.8920E-02	2.7977E-04
	1	8.0000E-02	7.9601E-02	3.9943E-04	7.9600E-02	3.9968E-04
0.7	0.7	1.2005E-01	1.1956E-01	4.8831E-04	1.1956E-01	4.8960E-04
	1	2.4500E-01	2.4432E-01	6.7644E-04	2.4430E-01	6.9943E-04
1	1	5.0000E-01	4.9940E-01	6.0301E-04	4.9900E-01	9.9919E-04

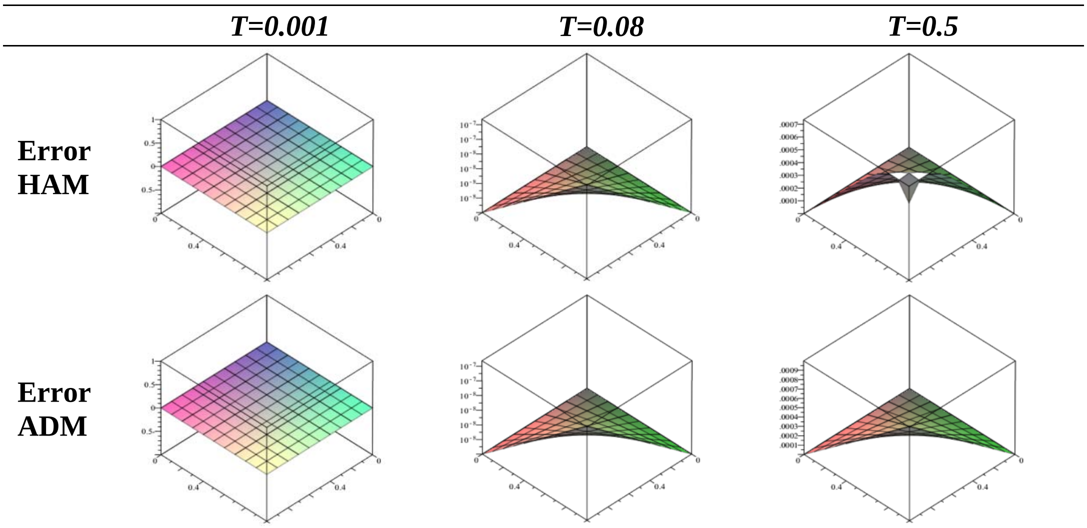

Fig. 1

Example 2: Consider the mixed integral equation

$$\phi(x, y; t) = f(x, y; t) + \int_0^t \int_0^1 \int_0^1 \cos(uv)\, e^{x+y} \tau\, \phi^3(u, v; \tau)\, du\, dv\, d\tau . \quad (6.2)$$

The exact solution is $\phi(x, y; t) = t \sin(xy)$

The exact and the approximate numerical solutions using **HAM** and **ADM**, and the corresponding errors of equation (6.2), for some points of $0 \le x, y \le 1$, with $N = 2$, were detected in the following Tables (4, 5, 6). These tables show that the errors of the approximate solutions, by **HAM** and **ADM**, have accuracies of $10^{-10}, 10^{-9}, 10^{-5}$, when *T*= 0.001, 0.08 , 0.5, respectively.

Table (4) *T*=0.001

x	*y*	ϕ Exact	ϕ HAM	Error HAM	ϕ ADM	Error ADM
0.1	0.1	1.5707E-08	1.5707E-08	1.8207E-18	1.5707E-08	1.8207E-18
	0.4	6.2791E-08	6.2791E-08	1.0687E-17	6.2791E-08	1.0687E-17
	0.7	1.0973E-07	1.0973E-07	8.9547E-18	1.0973E-07	8.9547E-18
	1	1.5643E-07	1.5643E-07	5.9769E-17	1.5643E-07	5.9769E-17
0.4	0.4	2.4869E-07	2.4869E-07	3.5145E-17	2.4869E-07	3.5145E-17
	0.7	4.2578E-07	4.2578E-07	1.3493E-16	4.2578E-07	1.3493E-16
	1	5.8779E-07	5.8779E-07	1.0753E-16	5.8779E-07	1.0753E-16
0.7	0.7	6.9591E-07	6.9591E-07	7.6859E-18	6.9591E-07	7.6859E-18
	1	8.9101E-07	8.9101E-07	1.1163E-16	8.9101E-07	1.1163E-16
1	1	1.0000E-06	1.0000E-06	0.0000E+00	1.0000E-06	0.0000E+00

Table (5) *T*=0.08

x	*y*	ϕ Exact	ϕ HAM	Error HAM	ϕ ADM	Error ADM
0.1	0.1	1.0053E-04	1.0053E-04	4.3477E-15	1.0053E-04	6.9565E-13
	0.4	4.0186E-04	4.0186E-04	1.1239E-13	4.0186E-04	7.8761E-13
	0.7	7.0230E-04	7.0230E-04	1.7310E-14	7.0230E-04	1.1827E-12
	1	1.0012E-03	1.0012E-03	7.4252E-13	1.0012E-03	1.2575E-12
0.4	0.4	1.5916E-03	1.5916E-03	1.4493E-13	1.5916E-03	8.5507E-13
	0.7	2.7250E-03	2.7250E-03	9.8354E-13	2.7250E-03	1.0165E-12
	1	3.7618E-03	3.7618E-03	3.2817E-13	3.7618E-03	1.6718E-12
0.7	0.7	4.4538E-03	4.4538E-03	1.9081E-13	4.4538E-03	2.1908E-12
	1	5.7024E-03	5.7024E-03	1.1944E-12	5.7024E-03	1.8056E-12
1	1	6.4000E-03	6.4000E-03	0.0000E+00	6.4000E-03	4.0000E-12

Table (6) *T*=0.5

x	*y*	ϕ Exact	ϕ HAM	Error HAM	ϕ ADM	Error ADM
0.1	0.1	0.00392683	0.00392529	1.5407E-06	0.0039253	1.5407E-06
	0.4	0.01569763	0.01569555	2.0798E-06	0.0156956	2.0798E-06
	0.7	0.02743358	0.02743077	2.8074E-06	0.0274308	2.8074E-06
	1	0.03910862	0.03910483	3.7895E-06	0.0391048	3.7896E-06
0.4	0.4	0.06217247	0.06216966	2.8070E-06	0.0621697	2.8074E-06
	0.7	0.10644482	0.10644104	3.7845E-06	0.106441	3.7896E-06
	1	0.14694631	0.14694123	5.0878E-06	0.1469412	5.1154E-06
0.7	0.7	0.1739782	0.17397314	5.0610E-06	0.1739731	5.1153E-06
	1	0.22275163	0.22274495	6.6773E-06	0.2227447	6.9050E-06
1	1	0.25	0.24999127	8.7299E-06	0.2499907	9.3209E-06

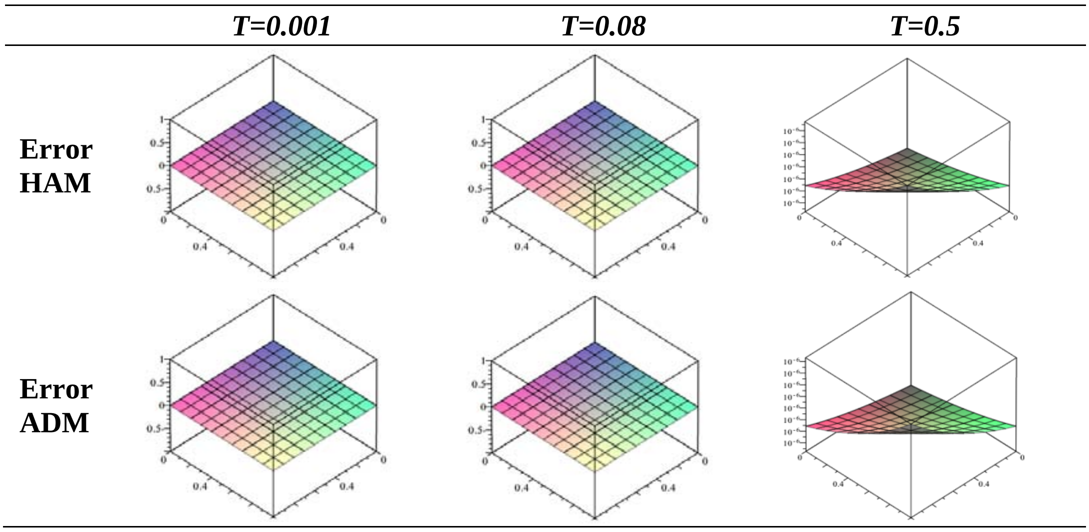

Fig. 2

7. Conclusions

From this paper, we can conclude the following points:

- The existence of a unique solution of a **MT-DNIE** with continuous kernels was discussed and proved.
- A **SHT-DIEs** was obtained from a **MT-DNIE**, by using a numerical method.
- **ADM** and **HAM** transferred the system of Hammerstein two-dimensional integral equations to a nonlinear algebraic system. The existences of a unique solution of these methods was discussed.
- The Adomian polynomial formula was used to prove the convergence of Adomian series, and then the maximum absolute truncation error was obtained.
- Some applications contain numerical results, in some different time, were calculated, and the estimate error, in each case, was computed.
- The **MT-DNIE** with continuous kernels can be transferred to a system of Volterra equations which, also, can be solved by numerical methods.
- In future work, we can consider and solve a **MT-DNIE** with a singular kernel in position.
- From the previous numerical results of Tables (1-6), we deduce the following discussion:
    1. There is a positive relationship between the time and the error, in both numerical methods **ADM** and **HAM**.
    2. Whereas x and y were increasing in the interval [0,1] , the error values of **ADM** and **HAM** are also increasing.
    3. The **HAM** approximate solution is better than that by **ADM.**
    4. The approximate solutions maintained the symmetric property as the exact solution with respect to position, i.e. $\phi(x, y; t) = \phi(y, x; t)$.

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