Universal Journal of Computational Analysis **3 (2015), 32-37**
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Reproducing Kernel Method for Singularly Perturbed 2D Elliptic Partial Differential Equations

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Abstract
The current article is using a reproducing kernel method for solving a class of singularly perturbed 2D elliptic partial differential equations which are often used to describe steady-state mass and energy transport in solids. By this method, with the reproducing kernel $H(D) = W_2^3[0,1] \otimes W_2^3[0,1]$, the two numerical experiments analyses are conducted to show that the n-term approximations of the present solutions have a good agreement with the exact solutions, which verify the precision and efficiency of the method in this work. The reproducing kernel method has the characteristics of simple and high precision, and provides a new numerical method for solving the singularly perturbed problems.

Key Word and Phrases
Perturbed, Reproducing Kernel, Partial Differential Equations, Analytical Solution.

1. Introduction

Singularly perturbed partial differential equations appeared in many branches of applied mathematics, which has been widely used in many fields such as physics and engineering applications, solid state physics, plasma physics, fluid mechanics, kinematics mathematical biology and chemistry, etc. Among them, the perturbed elliptic partial differential equation model structure (such as construction, beam and machine) is the basis of atomic physics equation. This kind of equation solution illustrates linear phenomenon occurs in many systems such as biological, engineering, aerospace industry. Over the recent years, the numerical solution of linear partial differential equation has been made significant progress through various methods, but it is still a mathematician problem has been the main focus on. This article is based on the reproducing kernel theory to study this class of partial differential equation.

In 1986, Cui M.G.[1] proposed the reproducing kernel Hilbert space method for the first time. He gave a reproducing kernel space $W_2^1[a,b]$ which is a reproducing kernel Hilbert space, export the limited expression of reproducing kernel, and pioneer the research in the field of reproducing kernel. The current years, the reproducing kernel method to solve the partial differential equation is no longer strange. Du J. and Cui M.G.[2] using reproducing kernel method for solving the semilinear heat partial differential equation of heat conduction with integral boundary conditions and giving the exact solution in form of series; Wang Y.L. and Su L.J.[3] solved the singularly perturbed parabolic partial differential equations, they divided the domain into three subdomains and applied the reproducing kernel space method on each subdomain; Wu B.Y. and Li X.Y.[4] gave a new algorithm for a class of linear nonlocal boundary value problems based on the reproducing kernel method; Wang Y.L.[5]-[7] used reproducing kernel space method for solving a series of singular perturbation delay problems. References [2]-[7] provide a theoretical basis in this research.

By the present paper, we consider the singularly perturbed 2D elliptic partial differential equation as follows:

$$\begin{cases} \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial^2 u(x,t)}{\partial t^2} + a(x,t)\dfrac{\partial u(x,t)}{\partial x} + b(x,t)\dfrac{\partial u(x,t)}{\partial t} + c(x,t)u(x,t) = f(x,t), \\ u(x,0)=0, u(x,1)=0, u(0,t)=0, u(1,t)=0 \end{cases} \quad (1.1)$$

where ε is a positive small parameter. the functions, $u(x,t)$ is an unknown function to be determined, $(x,t)\in[0,1]\times[0,1]$, $a(x,t), b(x,t), c(x,t), f(x,t)$ are sufficiently smooth. Besides, let us assume that (1.1) has a unique solution which belongs to $W_2^3[0,1]\times W_2^3[0,1]$, which is a reproducing kernel space defined in the second section.

2. Reproducing Kernel Method

2.1. Space $W_2^1[0,1]=\{u(x)\,|\,u \text{ is one-variable absolutely continuous function }, u'\in L^2[0,1]\}$

An inner product is defined:

$$\langle u(x),v(x)\rangle_{W_2^1} = u(0)v(0)+\int_0^1 u'(x)v'(x)dx,\quad u(x),v(x)\in W_2^1[0,1] \quad (2.1)$$

The reproducing kernel is:

$$R_x(y)=\begin{cases}1+x, & y>x\\ 1+y, & x>y\end{cases} \quad (2.2)$$

2.2. Space $W_2^3[0,1]=\{u(x)\,|\,u,u',u'' \text{ are one-variable absolutely continuous functions}, u(0)= u(1)=0.\ u''\in L^2[0,1]\}$

An inner product is defined:

$$< u(x),v(x) >_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0)+\int_0^1 u'''(x)v'''(x)dx, u(x),v(x)\in W_2^3[0,1] \quad (2.3)$$

The reproducing kernel is:

$$K_x(y)=\begin{cases}K(x,y), & y>x\\ K(y,x), & x>y\end{cases} \quad (2.4)$$

$$K(x,y)=-\frac{(1+115x+40x^2)(1+115y+40y^2)}{18720}+\frac{1}{120}x(x^4+120y-5x^3y+30xy^2+10x^2y^2),$$

$$K(y,x)=-\frac{(1+115x+40x^2)(1+115y+40y^2)}{18720}+\frac{1}{120}y(y^4+10x^2y(3+y)-5x(-24+y^3)).$$

3. Exact and Approximate Solution

3.1. Analytical Solution

In order to solve (1.1), let:

$$(Lu)(x,t)=F(x,t) \quad (3.1)$$

where L : $H(D)\to H_1(D)$ is a bounded linear operator[3], $D=[0,1]\times[0,1]$, L^{-1} is existent, $H(D)=W_2^3[0,1]\otimes W_2^3[0,1]$ and $H_1(D)=W_2^1[0,1]\otimes W_2^1[0,1]$ is the reproducing kernel of

$K_{(\xi,\eta)}(x,t)$ and $\overline{K}_{(\xi,\eta)}(x,t)$. So, the solution of (3.1) is the solution of (1.2).

Let:

$$\varphi_i(x,t) = \overline{K}_{(x_i,t_i)}(x,t), \quad \psi_i(x,t) = L^*\varphi_i(x,t), \quad B\phi = b, \tag{3.2}$$

where $b = [\psi_1(x,t), \psi_2(x,t), ...]^T$, $\phi = [\zeta_1, \zeta_2, ...]^T$, $B = (L\psi_i(x,t)|_{(x,t)=(x_j,t_j)})_{i,j=1,2,...}$, L^* is the adjoint operator. If B^{-1} exists, then $(L\zeta_j(x,t)|_{(x,t)=(x_i,t_i)})_{i,j=1,2,...}$ is an identity matrix. If $\{x_i,t_i\}_{i=1}^{\infty}$ is dense on D, $\psi_i\{x_i,t_i\}_{i=1}^{\infty}$ is a complete function system in $H(D)$, then an analytical solution of (3.1) is:

$$u(x,t) = \sum_{j=1}^{\infty} F(x_j,t_j)\zeta_j(x,t) \tag{3.3}$$

3.2. Approximate Solution

If (1.1) is nonlinear, the analytical solution can be written by:

$$u(x,t) = \sum_{j=1}^{\infty} F(x_j,t_j,u(x_j,t_j))\zeta_j(x,t) \tag{3.4}$$

The approximate solution to (1.1) can be obtained using the following method.

We can give initial function $u_1(x,t) \in H(D)$, by using the form (3.4), then an iterative sequence is constructed by:

$$u_n(x,t) = \sum_{j=1}^{\infty} F(x_j,t_j,u_{n-1}(x_j,t_j))\zeta_j(x,t), \ n = 2, ... \tag{3.5}$$

The convergence of $u_n(x,t)$ has been proved by [3].

4. Numerical Experiment

Experiment 1. Considering the heat transfer in a rectangle of length *L* and height *H*. The governing equation for the dimensionless temperature distribution [8] can be written as: [9] p. 166, [10]

$$\begin{cases} \varepsilon^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial^2 u(x,t)}{\partial t^2} = 0, & x \in [0,1], \ t \in [0,1] \\ u(x,0) = 0, u(x,1) = \sinh(\varepsilon\pi)\sin(\pi x), u(0,t) = 0, u(1,t) = 0 \end{cases} \tag{4.1}$$

The exact solution [10] is $u_\varepsilon(x,t) = \sinh(\varepsilon\pi \cdot t)\sin(\pi x)$, where $\varepsilon = H/L$ denotes the aspect ratio. By Mathmatica7.0, the absolute errors of $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ are tabulated in Table 1 and Fig.1.

Table 1. Absolutes Errors of $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$

(x,y)	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
(0.1,0.1)	8.56700E—05	7.02109E—08	7.00680E—11
(0.2,0.2)	3.17871E—04	2.61703E—07	2.61184E—10
(0.3,0.3)	5.93865E—04	4.92400E—07	4.91462E—10
(0.4,0.4)	8.17019E—04	6.82799E—07	6.81556E—10
(0.5,0.5)	9.00715E—04	7.59013E—07	7.57700E—10
(0.6,0.6)	8.14811E—04	6.92273E—07	6.91136E—10
(0.7,0.7)	5.91083E—04	5.06034E—07	5.05243E—10
(0.8,0.8)	3.13920E—04	2.70551E—07	2.70148E—10
(0.9,0.9)	8.63240E—05	7.47023E—08	7.45941E—11

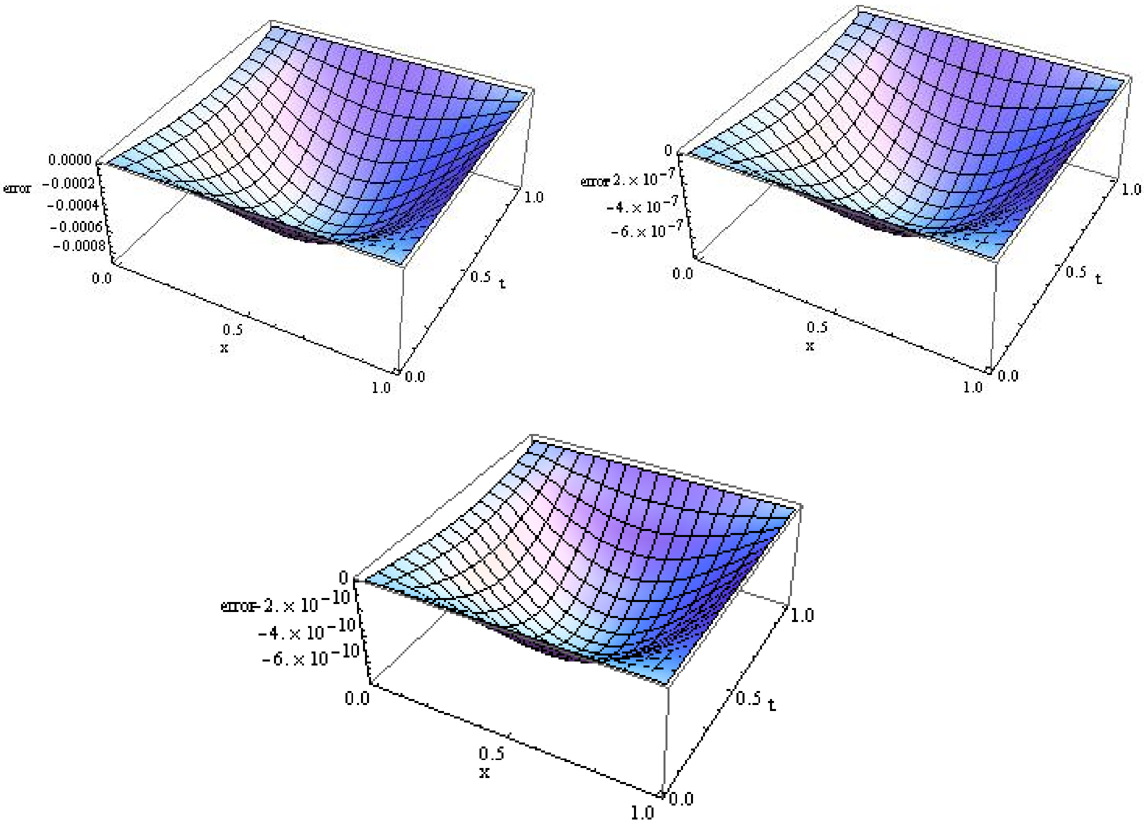

Fig. 1 Errors of $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ in Experiment 1

Experiment 2. Consider the singularly perturbed 2D elliptic partial differential equation:

$$\begin{cases} \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial^2 u(x,t)}{\partial t^2} = F(x,t), \qquad x \in [0,1], t \in [0,1] \\ u(x,0) = \sin(\pi x), u(x,1) = e^{-\varepsilon \pi^2} \sin(\pi x), u(0,t) = 0, u(1,t) = 0 \end{cases} \tag{4.2}$$

where $F(x,t)=\varepsilon\pi^2 e^{-\varepsilon\pi^2 t}\sin\pi x(\varepsilon\pi^2-1)$. The exact solution is $u_\varepsilon(x,t)=e^{-\varepsilon\pi^2 t}\sin(\pi x)$. By Mathmatica7.0, the absolute errors of $\varepsilon=10^{-1},10^{-2},10^{-3}$ are tabulated in Table 2 and Fig.2.

Table 2. Absolutes Errors of $\varepsilon=10^{-1},10^{-2},10^{-3}$

(x,y)	$\varepsilon=10^{-1}$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-3}$
(0.1,0.1)	2.47913E—04	6.75168E—07	3.73995E—09
(0.2,0.2)	2.88218E—04	1.51498E—07	3.37415E—09
(0.3,0.3)	2.19615E—04	7.61038E—07	1.07009E—10
(0.4,0.4)	3.35814E—04	2.01352E—06	5.09732E—09
(0.5,0.5)	5.71202E—04	2.44583E—06	8.29154E—09
(0.6,0.6)	6.95810E—04	2.37097E—06	8.70669E—09
(0.7,0.7)	5.73567E—04	1.51458E—06	6.08297E—09
(0.8,0.8)	2.87201E—04	6.81333E—07	2.75895E—09
(0.9,0.9)	5.43020E—05	9.64790E—08	4.23511E—10

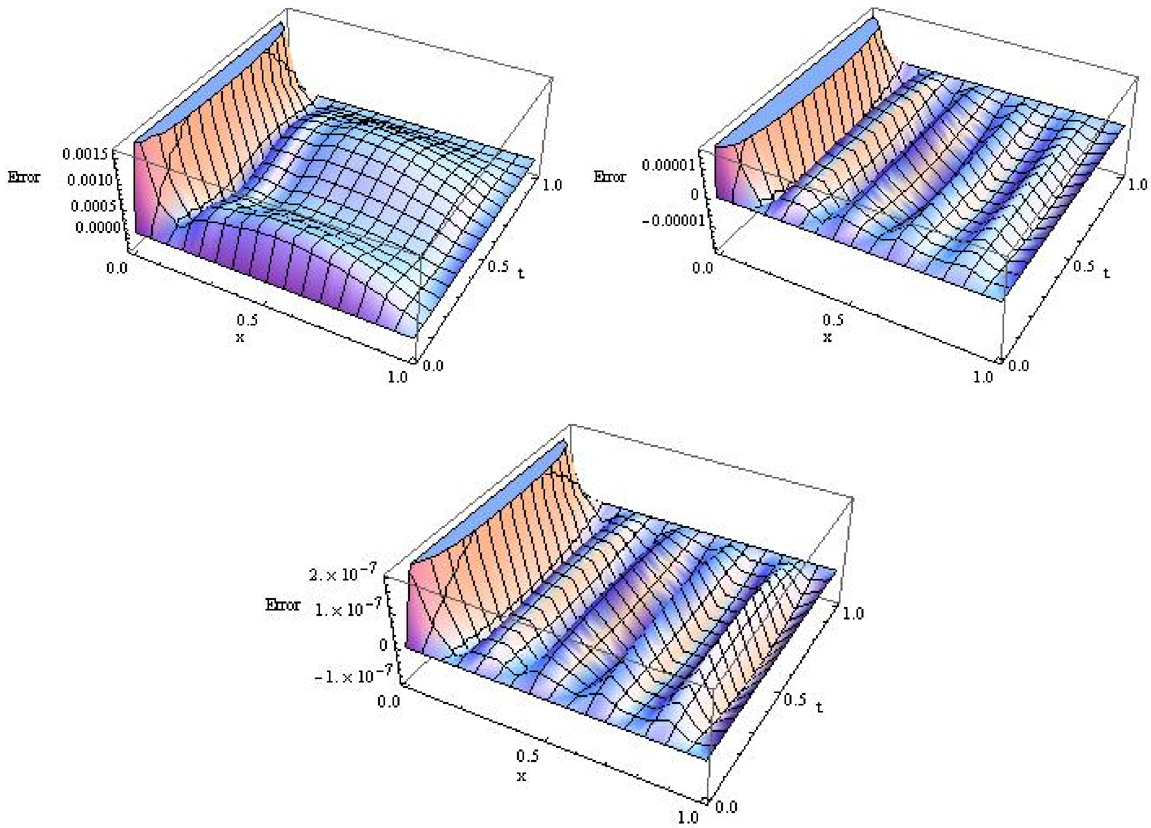

Fig. 2 Errors of $\varepsilon=10^{-1},10^{-2},10^{-3}$ in Experiment 2

5. Conclusions

By the present paper, the reproducing kernel method was employed successfully for solving a class of 2D elliptic partial differential equations with singularly perturbed. In traditional singularly perturbed problem, the smaller ε is, the fast change solutions are. But by this method, it is avoided such problem. So, from Table 1-2 and Fig.1-2 of experiment 1-2, we can find that the absolutes errors of $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ are more and more small. The two numerical errors show that the reproducing kernel method is very simple and high precision. Consequently, the work of this research provide a theoretical basis for further researching such as singularly perturbed delay parabolic partial differential equations and fractional partial differential equations.

Acknowledgements

The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. This paper is supported by the Natural Science Foundation of China (No.11361037), the Natural Science Foundation of Inner Mongolia (No.2013MS0109), Project Application Technology Research and Development Foundation of Inner Mongolia (No.20120312) and Autonomous Region PhD Research Innovation Project of Inner Mongolia (No.B20141012808).

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