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Solving Nonlinear Fractional Differential Equations using a New Decomposition Method

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Abstract
This work introduces a new decomposition method (NDM) for solving nonlinear fractional initial value problems by employing a method proposed by both Elzaki [1] and Adomian decomposition method. The suggested strategy is based on a simple modification of the Adomian decomposition method, in which is combined with the transformation described in [1] for treating the fractional derivatives in the Caputo sense. Some examples to validate this method was introduced, in which the explicit approximate solution is compared to an exact solution or the approximate solution calculated by four-order Runge-Kutta method (RK4).

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Key Word and Phrases
Adomian Decomposition, Fractional Differential Equations, Numerical Solutions.

1. Introduction

Over the last years, fractional differential equations have increased much consideration because of broad utilization in the mathematical modelling of physical problems. These are a generalization of classical integer order ordinary differential equations, are increasingly used to address the needs of problems in fluid mechanics, biology, engineering and other applications [2]-[4]. It is not obvious that an exact solution of these problem types could be calculated. Generally, the numerical solutions can be derived. Various methods have been employed to solve fractional differential equations. As example, Laplace transform method [5], [6], Fourier transforms method [7], Adomain decomposition method [7], [8], [10]- [12], and the new transform method [13]- [15]. The aim of the present paper is to use the NDM, in order to provide explicit approximate solutions for further nonlinear fractional initial value problems. The proposed solutions are highly in agreement with the exact solutions that we could calculate (integer order), or Runge-Kutta (RK4) numerical solutions.

2. Fractional Calculus

We introduce in this section, a general operator of integration and differentiation using the Riemann-Liouville then Caputo fractional definitions.

Definition 2.1
A real function $f(x); x>0$ *is said to be in space* $C_\mu;\ \mu \in \mathrm{P}$ *if there exists a real number* $p>\mu$, *such that* $f(t)=t^p f_1(t)$, *where* $f_1(t)\in C(0,\infty)$, *and it is said to be in the space* C_μ^n *if and only if* $f^n \in C_\mu,\ n\in \mathrm{N}$.

Definition 2.2
The Riemann-Liouville fractional integral operator of order $\alpha>0$, *is defined as follows:*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} f(v)\, dv,\ \alpha > 0, \tag{2.1}$$

Definition 2.3
Consider a function $f \in C_{\mu}$ *and* $\mu \geq -1$. *The fractional derivative of* $f(t)$ *in the Caputo sense is defined as:* $D^{\alpha} f(t) = J^{m-\alpha} D^{m} f(t)$, *for* $m-1 < \alpha \leq m, m \in \mathrm{N}, t > 0,$ *and* $f \in C_{-1}^{m}$

Caputo fractional derivative starting to compute an ordinary derivative then a fractional integral.

3. New Transform

Building on its valuable properties, the proposed transform has as of now demonstrated much efficacy. It is uncovered that it can take care of nonlinear differential problems resulting from some physical issues. Among those are, for example, solving fractional Navier-Stokes equations [16] and proposed an analytic solution of two-dimensional coupled differential Burger's equation [17]. The basic definitions of this transformation is defined as follows, the new transform of the function $f(t)$ is:

$$E[f(t)] = v\int_0^{\infty} f(t) e^{-\frac{t}{v}} dt = T(v), \forall t > 0. \tag{3.1}$$

Theorem 3.1 [13]
Let $T'(v)$ be the transform (3.1) of the derivative of $f(t)$. Then,

$$(i) T'(v) = \frac{T(v)}{v} - v f(0) \quad (ii) T^{(n)}(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0), n \geq 1$$

where $T^{n}(v)$ is the transform (3.1) of the nth derivative of the function $f(t)$.

The transform (3.1) can certainly handle all problems that are ordinarily handled by the well-recognized and extensively used Laplace transform. Indeed, as the next theorem shows the duality between the transform (3.1) and Laplace transform $F(s)$.

Theorem 3.2 [14], [15]

Let : $$f(t) \in A = \left\{ \begin{array}{l} f(t) \mid \exists M, k_1, k_2 > 0 \mid, such\ that: \left| f(t) \right| < Me^{|t|/k_i}, \\ if\ t \in (-1)^j \times [0, \infty) \end{array} \right\}$$

Then with Laplace transform $F(s)$ and the transform (3.1) $T(v)$ of $f(t)$, we obtain;

- $T(v) = vF\left(\frac{1}{v}\right).$
- $F(s) = sT\left(\frac{1}{s}\right).$

Theorem 3.3
If $m-1 < \alpha \leq m,\ m \in N$, then the transform (3.1) *of the fractional derivative* $D^{\alpha} f(t)$ *is,*

$$E\left[D^{\alpha} f(t)\right]=T^{\alpha}(v)=\frac{T(v)}{v^{\alpha}}-\sum_{k=0}^{m-1} f^{(k)}(0) v^{2-\alpha+k}$$

where $T(v)$ is the transform (3.1) of $f(t)$.

Proof. Laplace transform of the fractional derivative is:

$\ell\left[D^{\alpha} f(t)\right]=s^{\alpha} F(s)-\sum_{k=0}^{m-1} f^{(k)}(0) s^{\alpha-k-1}\ t>0,$ and from Theorem 3.2, we have; $\ell\left[D^{\alpha} f(t)\right]=sT^{\alpha}\left(\frac{1}{s}\right)$ *and* $E\left[D^{\alpha} f(t)\right]=vF^{\alpha}\left(\frac{1}{v}\right)$

Then:

$$E\left[D^{\alpha} f(t)\right]=v\left[\frac{1}{v^{\alpha}} F\left(\frac{1}{v}\right)-\sum_{k=0}^{m-1} f^{(k)}(0)\left(\frac{1}{v}\right)^{\alpha-k-1}\right]$$

$$E\left[D^{\alpha} f(t)\right]=\frac{T(v)}{v^{\alpha}}-\sum_{k=0}^{m-1} f^{(k)}(0) v^{2-\alpha+k}$$

4. Algorithm of the Method

The ADM is used to provide approximate answers for nonlinear problems in terms of convergent series with easily computable components. In the current section we employ the NDM to discuss about problems. To express the basic thought, let us study the following fractional differential equation,

$$\begin{aligned} & D^{\alpha} u(t)+a_m u^{(m)}(t)+a_{m-1} u^{(m-1)}(t)+\cdots+a_1 u'(t)+a_0 u(t) \\ & +N(u(t), u'(t))=f(t), \quad t>0,\ m-1<\alpha \leq m, \end{aligned} \tag{4.1}$$

Subject to the initial conditions,

$$u^{(i)}(0)=b_i\ ,\ i=0,1,2,... \tag{4.2}$$

where a_i, b_i are known real constants, N is a nonlinear operator and $f(t)$ is known function. The equation (4.1) is transformed into the following system:

$$\begin{aligned} & D^{\alpha} u(t)+a_m u^{(m)}(t)+a_{m-1} u^{(m-1)}(t)+\cdots+a_1 u'(t)+a_0 u(t) \\ & +N(u(t), u'(t))=f(t)\ t \geq 0 \end{aligned} \tag{4.3}$$

Subject to the initial conditions,

$$u^{(i)}(0)=b_i\ i=0,1,2, \cdots, m-1. \tag{4.4}$$

Taking the transform (3.1) of the equation (4.3), to obtain,

$$\begin{aligned} & E\left[D^{\alpha} u(t)\right]+a_m E\left[u^{(m)}(t)\right]+a_{m-1} E\left[u^{(m-1)}(t)\right]+\cdots+a_1 E\left[u'(t)\right]+ \\ & a_0 E[u(t)]+E[N(u(t), u'(t))]=E[f(t)], \end{aligned}$$

Applying the theorem 3.4, and applying the formula of the transform (3.1), we fix;

$$\frac{1}{v^{\alpha}}E\left[u(t)\right]=\sum_{k=0}^{m-1}f(0)v^{2-\alpha+\_k}+E\left[f(t)\right]-a_mE\left[u^{(m)}(t)\right]-a_{m-1}E\left[u^{(m-1)}(t)\right]$$
$$-\cdots-a_1E\left[u'(t)\right]-a_0\,E\left[u(t)\right]-E\left[N\left(u(t),u'(t)\right)\right],$$

and,

$$E\left[u(t)\right]=\sum_{k=0}^{m-1}f(0)v^{2+k}+v^{\alpha}E\left[f(t)\right]-$$
$$v^{\alpha}\left[a_mE\left[u^{(m)}(t)\right]+a_{m-1}E\left[u^{(m-1)}(t)\right]+\cdots+a_1E\left[u'(t)\right]+a_0E\left[u(t)\right]\right] \quad (4.5)$$
$$-v^{\alpha}E\left[N\left(u(t),u'(t)\right)\right],$$

The new decomposition method represents the solution as an infinite series

$$u(t)=\sum_{r=0}^{\infty}u_r(t). \quad (4.6)$$

and the nonlinear term $N\left(u(t),u'(t)\right)$ decomposes as;

$$N\left(u(t),u'(t)\right)=\sum_{r=0}^{\infty}A_r(t). \quad (4.7)$$

where,

$$A_r(t)=\frac{1}{r!}\frac{d^r}{dp^r}N\left(\sum_{R=0}^{\infty}P^r u_r(t),\sum_{R=0}^{\infty}P^r u'_r(t)\right)_{p=0} \quad (4.8)$$

are Adomian polynomials,

Substituting (4.6), (4.7) and (4.8), into (4.5), to get;

$$E\left[\sum_{r=0}^{\infty}u_r(t)\right]=\sum_{k=0}^{m-1}u^{(k)}v^{2+k}+v^{\alpha}E\left[f(t)\right]-$$
$$v^{\alpha}\left[a_mE\left[\sum_{r=0}^{\infty}u_r^{(m)}(t)\right]+a_{m-1}E\left[\sum_{r=0}^{\infty}u_r^{(m-1)}(t)\right]+\ldots+a_1E\left[\sum_{r=0}^{\infty}u'_r(t)\right]+a_0E\left[\sum_{r=0}^{\infty}u_r(t)\right]\right]-$$
$$v^{\alpha}E\left[\sum_{r=0}^{\infty}A_r(t)\right],$$

The iterations are defined by the recursive relations;

$$E\left[u_{j,0}(t)\right]=\sum_{k=0}^{m-1}u^{(k)}v^{2+k}+v^{\alpha}E\left[f(t)\right]$$
$$E\left[u_r(t)\right]=-v^{\alpha}\left[a_mE\left[u_{r-1}^{(m)}(t)\right]+\cdots+a_1E\left[u'_{r-1}(t)\right]+a_0E\left[u_{r-1}(t)\right]\right]- \quad (4.9)$$
$$v^{\alpha}E\left[A_{r-1}(t)\right],\ r=1,2,3,\cdots$$

5. Numerical Results

In this section we illustrate some examples presented to explain the method.

Example 1.
In this example, a fractional Riccati equation is considered,

$$D^{\alpha}u(t)=1-u^2(t),\ t\geq 0,\ 0<\alpha\leq 1. \quad (5.1)$$

We apply the proposed algorithm. Take the transform (3.1) of the equation (5.1), to get;

$$E\left[D^{\alpha}u(t)\right]=E[1]-E\left[u^{2}(t)\right]\Rightarrow\frac{E[u(t)]}{v^{\alpha}}-u(0)v^{2-\alpha}=v^{2}-E\left[u^{2}(t)\right].$$

Then $E[u(t)]=u(0)v^{2}+v^{\alpha+2}-v^{\alpha}E\left[u^{2}(t)\right]$, or

$$E\left[\sum_{r=0}^{\infty}u_{r}(t)\right]=u(0)v^{2}+v^{\alpha+2}-v^{\alpha}E\left[\sum_{r=0}^{\infty}A_{r}(t)\right]$$

The new decomposition series (4.9) has the form,

$$E\left[u_{0}(t)\right]=c_{j}v^{2}+v^{\alpha+2}$$
$$E\left[u_{i}(t)\right]=-v^{\alpha}E\left[A_{i-1}(t)\right],\ i=1,2,\cdots$$

where:

$$A_{i}(t)=\frac{1}{i!}\frac{d^{i}}{dp^{i}}\left[u_{0}^{2}+2pu_{0}u_{1}+p^{2}(u_{1}^{2}+2u_{0}u_{2})+\cdots\right]_{p=0}.$$

Considering the initial conditions, then we can find;

$$E\left[u_{0}(t)\right]=v^{\alpha+2}\Rightarrow u_{0}(t)=\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$E\left[u_{,1}(t)\right]=-v^{\alpha}E\left[u_{0}^{2}(t)\right]\Rightarrow u_{1}(t)=-\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^{2}(\alpha+1)\Gamma(3\alpha+1)}$$
$$E\left[u_{2}(t)\right]=-v^{\alpha}E\left[2u_{0}(t)u_{,1}(t)\right]$$
$$u_{2}(t)=\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^{3}(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha},\cdots$$

If $\alpha=1$ we obtain:

$$u_{0}(t)=t,u_{1}(t)=-\frac{t^{3}}{3}\text{and}u_{2}(t)=\frac{2t^{5}}{15}$$

This results from the NDM when $\alpha=1$ match the exact solution

$$u(t)=\tanh(x)=t-\frac{t^{3}}{3}+\frac{2t^{5}}{15}+\cdots$$

Thus, the proposed method is a very effective and accurate method that can be utilized to provide analytical results for nonlinear fractional differential equations.

Example 2.
Look at the following fractional nonlinear equation,

$$D^{\alpha}u(t)=0.1-u(t)+0.8u^{2}(t),\ t\geq 0,\ 0<\alpha\leq 1, \tag{5.2}$$

with the initial condition,

$$u(0)=0. \tag{5.3}$$

Take the transform (3.1) of the equation (5.2) and use the initial conditions (5.3), and so we suffer;

$$E[u(t)]=0.1v^{\alpha+2}-v^{\alpha}E[u(t)]+0.8v^{\alpha}E\left[u^{2}(t)\right]$$

According to the relation (4.9), we have the new decomposition series in the form,

$$E[u_0(t)] = 0.1v^{\alpha+2}$$
$$E[u_i(t)] = -v^{\alpha} E[u_{i-1}(t)] + 0.8v^{\alpha} E[A_{i-1}(t)],\ i = 1,2,3,\cdots$$

where :
$$A_i(t) = \frac{1}{i!}\frac{d^i}{dp^i}\left[u_0^2 + 2pu_0u_1 + p^2(u_1^2 + 2u_0u_2) + \cdots\right]_{p=0}$$

Then we have,

$$u_0(t) = \frac{0.1}{\Gamma(\alpha+1)}t^{\alpha},$$
$$u_1(t) = \frac{0.1}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{0.008\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}$$
$$u_2(t) = \frac{0.1}{\Gamma(3\alpha+1)}t^{2\alpha} - \frac{1}{\Gamma(\alpha+1)\Gamma(4\alpha+1)}\left[\frac{0.008\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} + \frac{0.016\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\right]t^{4\alpha}$$
$$+\frac{0.00128\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha},$$

For this example, when $\alpha = 1$, that is easy to calculate the exact solution for this example, but we obtain a solution with a complicate expression. We obtained a good agreement for only two-steps of NDM.

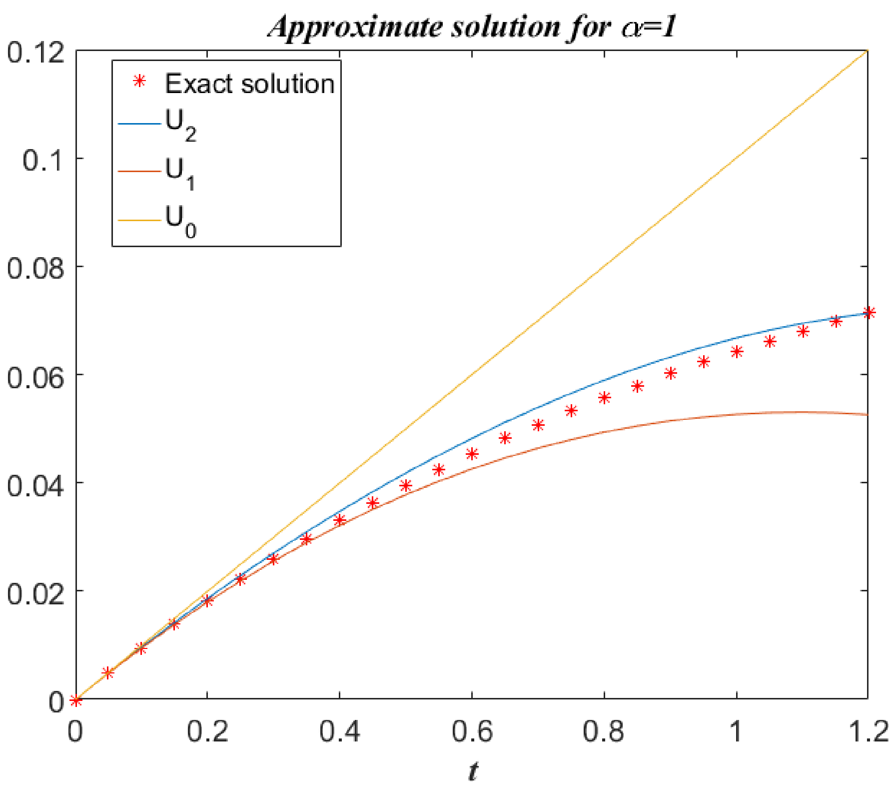

Fig.1 Comparison of the approximation solution U_0, U_1 and U_2 with the exact solution for $\alpha = 1$ (Example 2.).

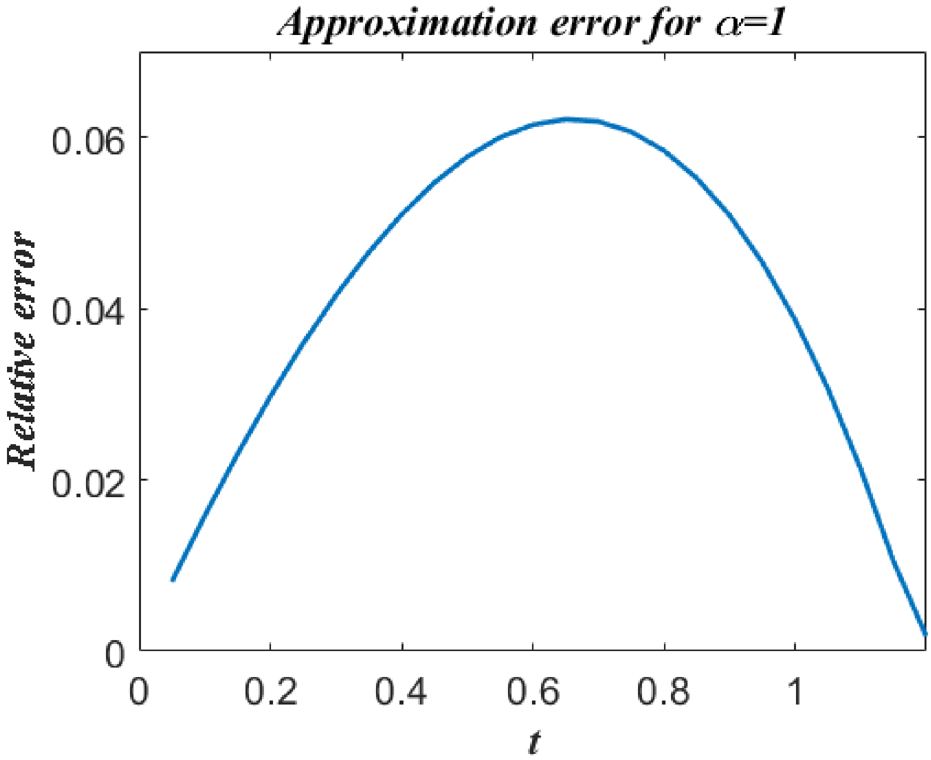

Fig. 2 Relative error of two step NDM when $\alpha = 1$ (Example 2.).

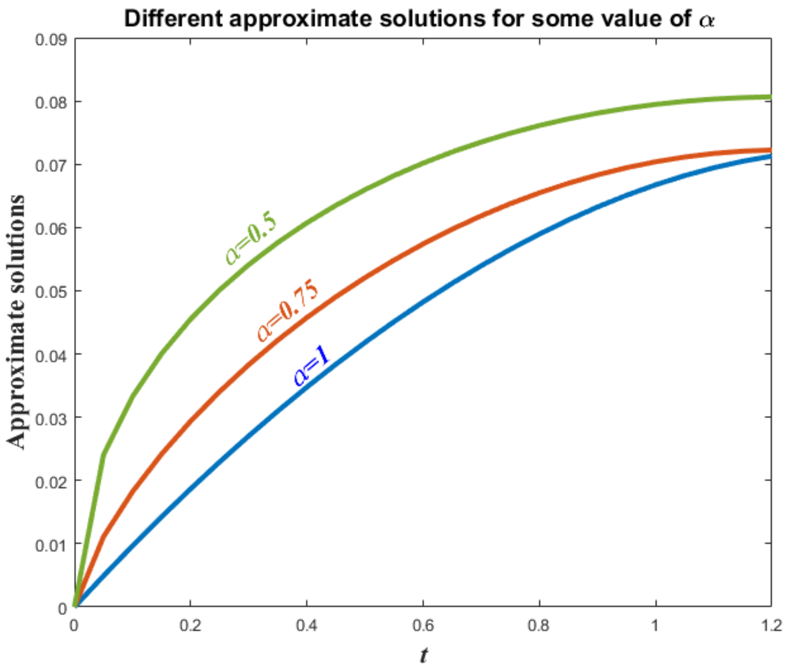

Fig. 3 The different approximate solution for some value of α (Example 2.).

Fig1. shows the comparison between the exact solution for $\alpha = 1$ and some first approximation solution U_n of NDM, with $U_n = \sum_{i=0}^{n} u_i$. In Fig. 2, the relative error of NDM is presented. The different solutions according some values of α are presented the Fig.3.

Example 3.
In this example a Van der Pol equation is considered with a fractional derivative,

$$D^{\alpha}u(t) = 1 + u'(t) - u(t) + u^{2}(t)u'(t)\; t \geq 0 \text{where} 0 < \alpha \leq 2, \tag{5.4}$$

with to the initial conditions,

$$u(0) = 0, u'(0) = 1. \tag{5.5}$$

Take the transform (3.1) of equation (5.4), making use of the initial conditions (5.5). And relation (4.9), we have the new decomposition method in the form,

$$E[u_0(t)] = v^{\alpha+2} + v^{2}u(0) + v^{3}u'(0) - v^{\alpha+1}u(0)$$
$$E[u_i(t)] = v^{\alpha-1}E[u_{i-1}(t)] - v^{\alpha}E[u_{i-1}(t)] - v^{\alpha}E[A_{i-1}(t)], i = 1,2,3,\cdots$$

where:

$$A_i(t) = \frac{1}{i!}\frac{d^i}{dp^i}\left[u_0^2u_0' + p(2u_0u_0'u_1 + u_0^2u_1') + \cdots\right]_{p=0}$$

Then we have,

$$u(t) = t + \frac{2}{\Gamma(\alpha+1)}t^{\alpha} - \frac{1}{\Gamma(\alpha+2)}t^{\alpha+1} - \frac{2}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1}{\Gamma(2\alpha)}t^{2\alpha-1}$$
$$+ \frac{1}{\Gamma(\alpha+1)}t^{2\alpha} - \frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+2)}\left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)}\right]t^{2\alpha+1} + \cdots$$

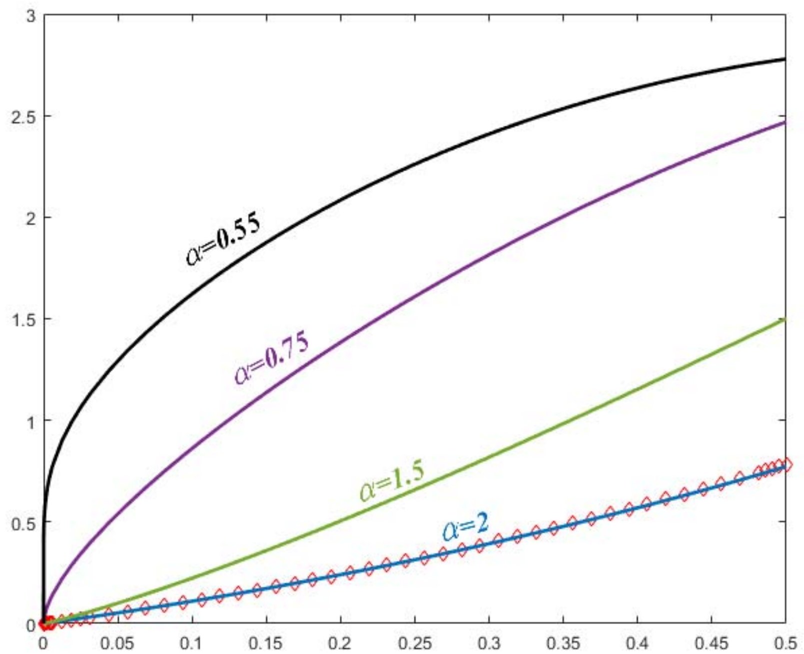

Fig. 4 The different approximate solution for some value of $\alpha = 2,1.5,0.75$ and 0.55 presented in solid lines, adding an approximate solution calculated by RK4 for $\alpha = 2$ in diamond markers (Example 3.).

For $\alpha = 2$, it is found in Fig. 4 that the outcome obtained by using the NDM with that held by the fourth-order Runge-Kutta (RK4) method had accurate. In addition, we also note that solution changes the concavity according to $\alpha < 1$ or $\alpha > 1$.

6. Conclusions

This study is to suggest an efficient algorithm for the solution of nonlinear fractional initial value problems. The Adomian decomposition method has been recognized as a potent technique for solving many nonlinear differential equations. In this work, a combined method that groups together the transform (3.1) and Adomian decomposition are discussed to finding an explicit approximate solution for nonlinear fractional initial value problems. We have noted the simplicity and the performance of EADM particularly if $\alpha = 1$ where the approximate solution is compared to the exact solution or the RK4 one. For a future project, we will interest to extend the NDM for nonlinear fractional PDE of the higher order.

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