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Degenerate Kernel Method for Three Dimension Nonlinear Integral Equations of the Second Kind

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Abstract
The present paper is concerned to solve three dimension nonlinear integral equations (3DNIE) of the second kind with continuous kernel in the space $L_2^3(\Omega)$, where Ω is the domain of this problem. An existence of a unique solution for a 3DNIE of the second kind is considered. A degenerate kernel method is applied to obtain a nonlinear algebraic system (NAS) where the existence and uniqueness solution of this system is guaranteed. Numerical results are calculated and the estimated error in each example is computed using Maple 18.

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Key Word and Phrases
Three - dimension Nonlinear Integral Equations (3DNIE), Degenerate Kernel Method, Nonlinear Algebraic System (NAS).

1. Introduction

Over the past fifty years, substantial progress in developing analytically and numerically solutions of IE in linear and nonlinear cases with different kinds is considered. Several numerical methods for approximate solution of linear and NIE are represented in Golberg [1], [2], Linz [3], Baker [4], Delves and Mohamed [5] and Atkinson [6]. The interested reader should also consult the books by Tricomi [7], Hochstadt [8] and Green [9] for analytical solution methods. Besides, Kaneko and Xu [10] used degenerate kernel method to obtain the solution of Hammerstein equation. Abdou *et. al* discussed the existence and uniqueness of the solution for a two dimensional NIE and solved it by using degenerate kernel method [11], [12]. The existence of NIE, of type Hammerstein-Volterra of the second kind, is proved by using the technique of Schauder fixed point theorem. Guoqiang *et. al*, in [13], obtained numerically the solution of two-dimensional nonlinear Volterra integral equation by collocation and iterated collocation methods. In [14], Guoqiang and Jiong analyzed the existence of asymptotic error expansion of the Nystrom solution for two-dimensional nonlinear Fredholm integral equation of the second kind.

Consider the 3DNIE:

$$\mu\varphi(x,,y,z) - \lambda \iiint_{\Omega} k(x,u;y,v;z,w)\psi\big(u,v,w,\varphi(u,v,w)\big)dudvdw = f(x,y,z). \quad (1.1)$$

under the static condition:

$$\iiint_{\Omega} \varphi(x,,y,z)dxdydz = p < \infty. \quad (1.2)$$

Here, k and f are known functions considered, in general, in the space L_2^3 and called the kernel and free term of the integral equation, respectively. Also, ψ is known continuous function, while φ is unknown. The constant λ, may be complex, has a physical meaning, while the constant μ defined the kind of NIE. The integral equation (1.1) can be written in the integral operator form:

$$W\varphi = \lambda \iiint_{\Omega} k(x,u;y,v;z,w)\psi\big(u,v,w,\varphi(u,v,w)\big)dudvdw. \quad (1.3)$$

So, we have:

$$\mu\overline{W}\varphi = f + W\varphi. \quad (1.4)$$

In order to guarantee a unique solution, we assume the following conditions:

(i) The kernel *k* satisfies the equality:

$$\left[\iiint_{\Omega}\ \iiint_{\Omega} |k(x,u;y,v;z,w)|^2\, d\Omega d\Omega\right]^{1/2} = C, \quad C\ is\ a\ constant.$$

(ii) The given function *f*(*x*,*y*,*z*) and its partial derivatives with respect to *x*, *y*, *z* belong to L_2^3 space and its norm is as following:

$$\|f(x,y,z)\| = \left[\iiint_{\Omega} |f(x,y,z)|^2 d\Omega\right]^{1/2} = H, \quad H\ is\ a\ constant.$$

(iii) For the two unknown functions φ_1 and φ_2, the known function:

$\psi\big(x,y,z,\varphi(x,y,z)\big)$ satisfies Lipschitz condition, which is:

$$\left\|\psi\big(x,y,z,\varphi_1(x,y,z)\big) - \psi\big(x,y,z,\varphi_2(x,y,z)\big)\right\| \le E\|\varphi_1(x,y,z) - \varphi_2(x,y,z)\|, \ \ E\ is\ a\ constant.$$

Theorem of a unique solution can be easily proved after discussing the bounded and continuity of the operator (1.3). Moreover, it must be a contraction mapping, see [11].

In section two, a degenerate kernel method in the 3DNIE is used to obtain a NAS. In section three, several examples are solved to explain this method.

2. Degenerate Kernel Method

Suppose that the approximate kernel $k_{n,m,l}(x,u;y,v;z,w)$ takes the form:

$$k_{n,m,l} = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} \alpha_i(x)\zeta_i(u)\beta_j(y)\eta_j(v)\gamma_k(z)\xi_k(w), \tag{2.1}$$

where:

$$\left|k_{n,m,l} - k\right| \to 0 \quad \text{as} \quad n\to\infty, m\to\infty, l\to\infty. \tag{2.2}$$

From cond. (i) and (2.2), we have the following condition:

$$\left[\iiint_{\Omega}\ \iiint_{\Omega} \left|k_{n,m,l}\right|^2 d\Omega d\Omega\right]^{1/2} \le C. \tag{2.3}$$

Therefore, the integral equation (1.1) yields:

$$\mu\varphi_{n,m,l}(x,,y,z) - \lambda\iiint_{\Omega} k_{n,m,l}(x,u;y,v;z,w)\psi\left(u,v,w,\varphi_{n,m,l}(u,v,w)\right) dudvdw = f(x,y,z) + R_{n,m,l}. \tag{2.4}$$

where $R_{n,m,l}$ is the error function of $O(n^{-r_1}m^{-r_2}l^{-r_3})$; r_1, r_2 and r_3 are constants.

Definition 2.1 The degenerate kernel method is said to be convergent of order $r_1 + r_2 + r_3$ in the domainL_2^3, if and only if for large *n*, *m*, *l*, there exist a constant D > 0 independent of *n*, *m*, *l* such that:

$$\left\|\varphi(x,y,z) - \varphi_{n,m,l}(x,y,z)\right\| \le D\, n^{-r_1}m^{-r_2}l^{-r_3}. \tag{2.5}$$

The reader can establish the uniqueness of (2.4) after using cond.(ii), (2.3) and (2.5).

By replacing (2.1) in (2.4), after ignoring the error, we have:

$$\mu\varphi_{n,m,l}(x,,y,z)-\lambda\sum_{i,j,k}\alpha_i(x)\beta_j(y)\gamma_k(z)\iiint_\Omega \zeta_i(u)\eta_j(v)\xi_k(w)\psi\left(u,v,w,\varphi_{n,m,l}(u,v,w)\right)dudvdw = f(x,y,z). \tag{2.6}$$

Assume, the unknown constants:

$$A_{ijk}=\iiint_\Omega \zeta_i(u)\eta_j(v)\xi_k(w)\psi\left(u,v,w,\varphi_{n,m,l}(u,v,w)\right)dudvdw. \tag{2.7}$$

Hence, the formula (2.4) becomes:

$$\varphi_{n,m,l}(x,y,z)=\frac{\lambda}{\mu}\sum_{i,j,k}^{n,m,l}\alpha_i(x)\beta_j(y)\gamma_k(z)A_{ijk}+\frac{1}{\mu}f(x,y,z),\quad (\mu\neq 0). \tag{2.8}$$

In order to determine A_{ijk}, we substitute (2.8) into (2.7), to get the following NAS:

$$A_{ijk}=\iiint_\Omega \zeta_i(u)\eta_j(v)\xi_k(w)\psi\left(u,v,w,\frac{\lambda}{\mu}\sum_{o,p,q}^{n,m,l}\alpha_o(u)\beta_p(v)\gamma_q(w)A_{opq}+\frac{1}{\mu}f(x,y,z)\right)dudvdw. \tag{2.9}$$

The formula (2.9) represents a system of nonlinear algebraic equations.

Theorem 2.1

Under the above assumptions, the sequence solution $\varphi_{n,m,l}$ of (2.4) converges to the unique solution $\varphi(x,y,z)$ of (1.1), see[11].

Theorem 2.2

The NAS of (2.9) has a unique solution A_{ijk} in a Banach space l_2^3 under the following conditions, see [11].

$(i)\left[\sum_{i,j,k}\iiint_\Omega\left|\alpha_i(x)\beta_j(y)\gamma_k(z)\right|^2dxdydz\right]^{\frac{1}{2}}.\left[\sum_{i,j,k}\iiint_\Omega\left|\zeta_i(x)\eta_j(y)\xi_k(z)\right|^2dxdydz\right]^{\frac{1}{2}}\leq C^*$

$(ii)\left[\sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^l\iiint\left|\psi\left(x,y,z,\varphi_{n,m,l}(x,y,z,A_{ijk})\right)\right|^2dxdydz\right]^{\frac{1}{2}}\leq H^*$

(iii) For $\{\bar{A},\bar{B}\}\in l_2^3$ in which its norm is $\|\bar{A}\|_{l_2^3}=\left[\sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^l\left|A_{ijk}\right|^2\right]^{\frac{1}{2}}$, we have

$\left[\sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^l\iiint\left|\psi\left(x,y,z,\varphi_{n,m,l}(x,y,z,A_{ijk})\right)-\psi\left(x,y,z,\varphi_{n,m,l}(x,y,z,B_{ijk})\right)\right|^2dxdydz\right]^{\frac{1}{2}}\leq$ $E^*\|A-B\|$, where C^*, H^* and E^* are constants.

3. Numerical Examples

Example 1

Consider the integral equation:

$$\varphi(x,,y,z)-0.01\int_0^1\int_0^1\int_0^1 x^2u(y+v^2)wz\,\varphi^2(u,v,w)dudvdw=f(x,y,z). \tag{3.1}$$

where: $f(x,y,z)=x^2y^2z-0.0000595x^2z-0.000083x^2$. (Exact solution: x^2y^2z)

(3.1) represents a 3DNIE with degenerate kernel, where:

$$\alpha_1(x) = x^2, \quad \zeta_1(u) = u, \quad \beta_1(y) = y, \quad \beta_2(y) = 1,$$

$$\eta_1(v) = 1, \ \eta_2(v) = v^2, \ \gamma_1(z) = z, \ \xi_2(w) = w. \tag{3.2}$$

Applying (2.9), we obtain:

$$A_{111} = 0.00833, \quad A_{121} = 0.00595. \tag{3.3}$$

By replacing (3.3) in (2.8), we get the numerical solution of the nonlinear case:

$$\varphi_N(x, y, z) = x^2y^2z + 10^{-14}x^2z + 2(10^{-14})x^2yz. \tag{3.4}$$

Here, the error is given by:

$$R(x, y, z) = \|\varphi - \varphi_N\| = 10^{-14}x^2z + 2(10^{-14})x^2yz, \tag{3.5}$$

which is only the result of the rounding-off errors.

Example 2

Consider the integral equation:

$$\varphi(x,, y, z) - 0.01 \int_0^1 \int_0^1 \int_0^1 e^{zw} \sin(xu) \cos(yv)\, \varphi^2(u, v, w) dudvdw = f(x, y, z). \tag{3.6}$$

where: $f(x, y, z) = xyz - \frac{0.01}{x^3y^3z^3}(2x^2 \cos x + 2 \cos x + 2x \sin x)(y^2 \sin y - 2 \sin y + 2y \cos y)(-2 + 2e^z - 2ze^z + z^2e^z)$. (Exact solution: *xyz*)

Here:

$$k(x, u; y, v; z, w) = e^{zw} \sin(xu) \cos(yv). \tag{3.7}$$

In addition, we take two approximations for $k(x,u;y,v;z,w)$

1- Assume the first approximation $k_1(x, u; y, v; z, w) = xu(1 + wz)$, for this:

$$\alpha_1(x) = x^2, \quad \zeta_1(u) = u, \quad \beta_1(y) = 1, \quad \eta_1(v) = 1,$$

$$\gamma_1(z) = 1, \ \gamma_2(z) = z, \ \xi_1(w) = 1, \ \xi_2(w) = w. \tag{3.8}$$

Applying (2.9), we get:

$$A_{111} = 0.027784, \quad A_{112} = 0.020837. \tag{3.9}$$

Replacing (3.9) in (2.8), and taking $z = 1$, we calculate φ_{N1} and the corresponding error, R_{N1}, then we deduce that max $R_{N1} = 12 \times 10^{-5}$.

2- Assume the second approximation $k_2(x, u; y, v; z, w) = xu\left(1 - \frac{y^2v^2}{2}\right)(1 + wz)$, for this:

$$\alpha(x) = x, \ \zeta(u) = u, \ \beta(y) = \begin{pmatrix} 1 \\ -y^2 \end{pmatrix}, \ \eta(v) = \begin{pmatrix} 1 \\ \frac{v^2}{2} \end{pmatrix}, \ \gamma(z) = \begin{pmatrix} 1 \\ z \end{pmatrix}, \ \xi(w) = \begin{pmatrix} 1 \\ w \end{pmatrix}. \tag{3.10}$$

Applying (2.9), we get:

$$A_{111} = 0.0278, \quad A_{121} = -0.0083, \quad A_{112} = 0.0208, \quad A_{122} = -0.0063. \tag{3.11}$$

Then calculating the corresponding error the corresponding error, R_{N2}, then we deduce that max $R_{N2}= 3 \times 10^{-5}$, see Table1.

Table 1

		1st *approximation*		**2nd *approximation***	
x	*y*	φ_{N1}	R_{N1}	φ_{N2}	R_{N2}
0	0.0	0.0	0.0	0.0	0.0
	0.25	0.0	0.0	0.0	0.0
	0.5	0.0	0.0	0.0	0.0
	0.75	0.0	0.0	0.0	0.0
	1.0	0.0	0.0	0.0	0.0
0.25	0.0	1.9e-5	1.9e-5	1.9e-5	1.9e-5
	0.25	0.06248	1.7e-5	0.06248	1.9e-5
	0.5	0.12499	9.3e-6	0.12498	1.8e-5
	0.75	0.18750	3.2e-6	0.18748	1.7e-5
	1.0	0.25002	2.0e-5	0.24998	1.6e-5
0.5	0.0	3.3e-5	3.3e-5	3.3e-5	3.3e-5
	0.25	0.12497	2.8e-5	0.12496	3.3e-5
	0.5	0.24998	1.3e-5	0.24996	3.1e-5
	0.75	0.37501	1.1e-5	0.37497	2.9e-5
	1.0	0.50004	4.4e-5	0.49997	2.8e-5
0.75	0.0	3.6e-5	3.6e-5	3.6e-5	3.6e-5
	0.25	0.18747	2.8e-5	0.18746	3.6e-5
	0.5	0.37499	6.6e-5	0.37496	3.4e-5
	0.75	0.56252	2.9e-5	0.56246	3.2e-5
	1.0	0.75007	7.6e-5	0.74996	3.2e-5
1.0	0.0	2.2e-5	2.2e-5	2.2e-5	2.2e-5
	0.25	0.24998	1.2e-5	0.24997	2.2e-5
	0.5	0.50001	1.5e-5	0.49997	2.1e-5
	0.75	0.75006	6.0e-5	0.74997	2.1e-5
	1.0	1.00012	1.2e-4	0.99997	2.4e-5

4. Conclusions

1- The error decreases when we increase *n*, *m* and *k*.

2- For the continuous kernel, the degenerate kernel method is considered as the best method to obtain the solution of the 3DNIE.

3- In our experiment computational, we note that the error decreases when $0 < z < 1$ and take maximum value at the endpoint.

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