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Determination of the Coefficient in the Advection Diffusion Equation using Collocation and Radial Basis Functions

H. Azari[1] and F. Parzlivand[2]
[1]Department of Mathematics Science, Shahid Beheshti University, Tehran, Iran
h_azari@sbu.ac.ir
[2]Department of Mathematics, Alzahra University, Vanak, Tehran, Iran
fparzlivand@yahoo.com

Abstract
In this paper, the collocation with radial basis functions method is used for finding an unknown parameter $p(t)$ in the inverse parabolic partial differential equation. The radial basis functions (RBF) method is an efficient mesh free technique for the numerical solution of partial differential equations. The main advantage of numerical methods which use radial basis functions over traditional techniques is the meshless property of these methods. In a meshless method, a set of scattered nodes are used instead of meshing the domain of the problem. Some experimental numerical results using the newly proposed numerical procedure are discussed.

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Key Word and Phrases
Inverse Problem, Radial Basis Functions, Collocation.

1. Introduction

Inverse problems in partial differential equations can be used to model many real problems in engineering and other physical sciences (cf. [18, 20, 6, 17, 26, 23, 22, 12, 25, 24, 9, 8, 38, 11, 2, 3, 36, 1, 33, 21] for examples). Studying these problems has a great deal of importance both theoretically and practically. By the present paper, we shall investigate of parabolic inverse problems where unknown coefficient is assumed to be time–dependent only. Research on parabolic inverse problems has been receiving considerable attention recently. The reader can find a lot of references in recent conference proceedings [18] and [20]. They arise for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion, and control theory [12, 10, 15, 13, 7, 19].

Consider the inverse problem of finding the function $u(x,t)$ and the unknown positive coefficient $p(t)$ in the parabolic initial–boundary value problem

$$\begin{aligned} u_t &= u_{xx} + p(t)u_x + f(x,t), \qquad \text{in} \quad Q_T, \\ u(x,0) &= u_0(x), \qquad 0 \le x \le 1, \\ u_x(0,t) &= g_1(t), \qquad 0 \le t \le T, \\ u_x(1,t) &= g_2(t), \qquad 0 \le t \le T, \end{aligned} \tag{1.1}$$

along with an extra condition

$$u_x(x^*,t) = h(t), \qquad 0 \le t \le T, \tag{1.2}$$

where x^* or 1, $Q_T = \{(x,t) : 0 < x < 1, 0 < t < T\}$, $T > 0$, and $u_0, g_2 > 0, g_2 < 0$, are known function. The existence and uniqueness of this inverse problem are discussed in [14, 10, 15]. Also some other numerical and theoretical discussions about this problem are found in [30] and [31].

From (1.1) and (1.2) we have

$$h'(t) = u_{xx}(0,t) + p(t)u_x(0,t) + f(0,t),$$

and it follows that:

$$p(t)=\frac{h'(t)-u_{xx}(0,t)-f(0,t)}{g_1(t)}$$

the inverse problem (1.1)–(1.2) is equivalent to the following non–local parabolic problem:

$$\begin{aligned} &u_t=u_{xx}+\frac{h'(t)-u_{xx}(0,t)-f(0,t)}{g_1}u_x+f(x,t), && \text{in} \quad Q_T,\\ &u(x,0)=u_0(x), && 0\le x\le 1,\\ &u_x(0,t)=g_1(t)>0, && 0\le t\le T,\\ &u_x(1,t)=g_2(t)<0, && 0\le t\le T, \end{aligned} \qquad (1.3)$$

where $h'(t)>0,\ u_{xx}(0,t)<0$ and $u_0(x)>0$.

2. Radial Basis Functions

The numerical solution of partial differential equations by RBF methods is based on a scattered data interpolation problem which we review in this section.

Let $x_0, x_1,..., x_N \in \Omega\subset R^d$ be a given set of scattered data. A radial basis function $x_j\in R^d$. So that the radial basis function ϕ_i is radially symmetric about the center x_j.

In addition consider r be the Euclidean distance between a fixed point $x_j\in R^d$ and $x\in R^d$ i.e. $\|x-x_j\|_2$. A radial function interpolation problem may be described as, given data $f_i=f(x_i)$, $i=0,1,\cdots,N$ and $x=(x_1,x_2,\cdots,x_d)$, the interpolation RBF approximation is:

$$S_f(x)=\sum_{i=0}^{N}(\lambda_i\phi_i(x)), \qquad (2.1)$$

where λ_i are chosen so that $S_f(x_i)=f_i$, above equation can be written without the additional polynomial Ψ. In that case, ϕ must be unconditionally positive definite to guarantee the solvability of the resulting system (e.g., Gaussian or inverse multiquadrics). In this case we have

$$S_f(x)=\sum_{i=0}^{N}\lambda_i\phi_i(x). \qquad (2.2)$$

Thus the interpolation conditions provide the linear system:

$$\mathbf{A}\lambda=f, \qquad (2.3)$$

where $A_{ij}=\phi_i(x_j), \lambda=[\lambda_0,\lambda_1,\cdots,\lambda_N]^T$ and $f=[f_0,f_1,\cdots,f_N]^T$. for the RBFs that we have considered in table 1. The interpolation matrix can be shown to be invertible for distinct interpolation points for GA, IMQ, IQ, MQ by C. Micchelli [29] and H. Vandeven [39].

Although the matrix $\mathbf{A}$ is nonsingular in the above cases, usually it is very ill-conditioned i.e. the condition number of $\mathbf{A}$:

$$\kappa_s\mathbf{A}=\|\mathbf{A}\|_s\|\mathbf{A}^{-1}\|_s, \qquad s=1,2,\infty, \qquad (2.4)$$

is a very large number. Therefore a small perturbation in initial data may produce a large amount of perturbation in the solution. Thus we have to use more precision arithmetic than the standard floating point arithmetic in our computation. For a fixed number of interpolation points the condition number of $\mathbf{A}$ depends on the shape parameter *c*, support of the RBFs and minimum separation distance of interpolation points. Also the condition number grows with *N* for fixed values of shape parameter c. In practice, the shape parameter c must be adjusted with the number of interpolating points in order to produce an interpolation matrix which is well conditioned enough to be inverted in finite precision arithmetic by S.A. Sarra [35].

Despite research done by many scientists to develop algorithms for selecting the values of c which produce the most accurate interpolation (e.g. see [16, 34]), the optimal choice of shape parameter is still an open question. Generally for a fixed number of collocation points N, smaller values of c produce better approximations, but the matrix $\mathbf{A}$ will be more ill-conditioned. Spectral accuracy is obtained in interpolating smooth data using global, infinitely differentiable radial basis functions (e.g. see [27, 28, 5, 4, 32, 37]).

Table 1 Some well-known functions that generate RBFs

Name or Radial Basis Function	Definition
Multiquadric(MQ)	$\phi(r)=\sqrt{c^2+r^2}$
Inverse Quadratic(IQ)	$\phi(r)=\dfrac{1}{c^2+r^2}$
Inverse Multiquadric(IMQ)	$\phi(r)=\dfrac{1}{\sqrt{c^2+r^2}}$
Gaussian(GA)	$\phi(r)=\exp(-cr^2)$

3. Non-local Parabolic Problem and Radial Basis Function

In this section the radial basis functions method is used for solving the problem (1.3).

Consequently, let $\Omega=\{(x_i,\mathrm{t}_i),0\le x_i\le 1,0\le t_i\le T,\quad i=1\cdots N\}$ be a set of scattered nodes, then the solution of the problem (1.3) is considered as follows:

$$\tilde{u}=\sum_{i=0}^{N}\lambda_i\phi_i(x,t), \tag{3.1}$$

where $\phi_i(x,t)=\phi(\|(x,t)-(x_i,t_i)\|_2)$ and λ_i are unknown constants that must be found for $i=0\cdots N$.

The collocation technique is used for finding unknown λ_i , $i=0\cdots N$, let

$\Omega=\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4$ where:

$$\begin{aligned}
\Omega_1&=\{(x_i,t_i),\quad 0\le x_i\le 1,\quad t_i=0,\quad i=0,\cdots,N\},\\
\Omega_2&=\{(x_i,t_i),\quad x_i=0,\quad 0<t_i\le T,\quad i=0,\cdots,N\},\\
\Omega_3&=\{(x_i,t_i),\quad x_i=1,\quad 0<t_i\le T,\quad i=0,\cdots,N\},\\
\Omega_4&=\{(x_i,t_i),\quad 0<x_i<1,\quad 0<t_i\le T,\quad i=0,\cdots,N\}.
\end{aligned} \tag{3.2}$$

Also we assume $\Omega_i\neq 0$ for $1\le i\le 4$. Now (1.3) approximated using (3.1). Thus we have:

$$\begin{aligned}
&\sum_{i=0}^{N}\phi_i(x_k,t_k)\lambda_i=f(x_k), && (x_k,t_k)\in\Omega_1,\\
&\sum_{i=0}^{N}\frac{\partial}{\partial x}\phi_i(x_k,t_k)\lambda_i=g_1(t_k), && (x_k,t_k)\in\Omega_2,\\
&\sum_{i=0}^{N}\frac{\partial}{\partial x}\phi_i(x_k,t_k)\lambda_i=g_2(t_k), && (x_k,t_k)\in\Omega_3,
\end{aligned}$$

$$\sum_{i=0}^{N}\Big(\frac{\partial}{\partial t}\phi_i(x_k,t_k)-\frac{\partial^2}{\partial x^2}\phi_i(x_k,t_k)-\frac{h'(t_k)}{g_1(t_k)}\frac{\partial}{\partial x}\phi_i(x_k,t_k)$$
$$+f(0,t_k)\frac{\partial}{\partial x}\phi_i(x_k,t_k)\Big)\lambda_i+\sum_{i=0}^{N}\frac{\partial}{\partial x^2}\phi_i(0,t_k)\lambda_i\sum_{i=0}^{N}\frac{\partial}{\partial x}\phi_i(x_k,t_k)\lambda_i \quad (3.3)$$
$$=f(x_k,t_k),\qquad (x_k,t_k)\in\Omega_4.$$

Thus, we have a nonlinear system of equations. Solving this nonlinear system the approximate solution of the transformed problem (1.3) is obtained.

Furthermore, to find the approximate value of $p(t)$ results:

$$\tilde{p}(t)=\frac{h'(t)-\tilde{u}_{xx}(0,t)-f(0,t)}{g_1(t)}. \quad (3.4)$$

4. Numerical Experiment

To show the efficiency of the new method on the inverse parabolic partial differential equation, two examples are given. These tests are chosen such that their analytical solutions are known. But the method developed in this research can be applied to more complicated problems.

Example 4.1

Let us solve the inverse problem (1.1)-(1.2) with the following conditions

$$u_0(x)=2+(0.5-x)x,$$
$$g_1(t)=0.5,$$
$$g_2(t)=-1.5,$$
$$h(t)=2\exp(4t)-2t,$$
$$f(x,t)=4\exp(4t)(1.5+2x),$$

and $x^*=0$, for which the exact solution is

$$u(x,t)=-(x-0.5)x+2\exp(4t)-2t,$$
$$p(t)=4\exp(4t).$$

In Figure 1 the error function $u-\tilde{u}$ is plotted using GA-RBF with $c=0.04, \delta=90$ (the number of floating point arithmetic) and the set of collocation points $x_i=(i-1)0.125$ and $t_i=(i-1)0.125$ for $N=81$. Also the corresponding error function $p-\tilde{p}$ is plotted in Figure 2.

Two types of errors were measured, the max error:

$$E^{\infty}=\max_{0\le i\le N}|u(x_i,t_i)-\tilde{u}(x_i,t_i)|,$$

where u is the exact value and $\tilde{u}$ is the RBF approximation, and the RMS error:

$$E^2=\sqrt{\frac{1}{N}\sum_{0\le i\le N}|u(x_i,t_i)-\tilde{u}(x_i,t_i)|^2}.$$

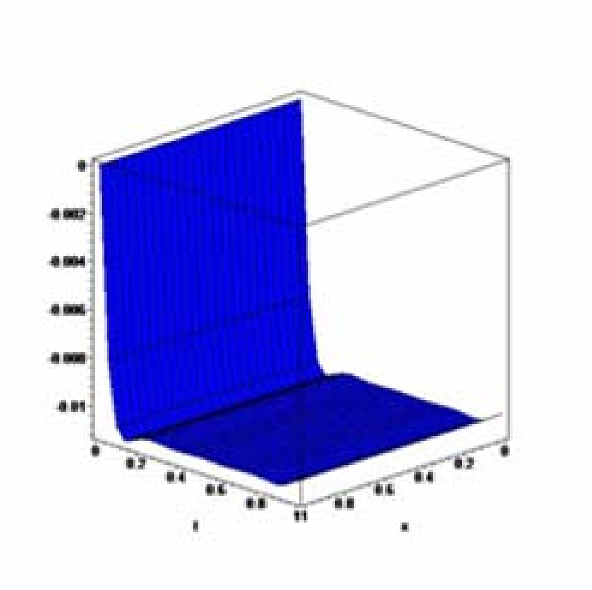

Fig. 1 Plot of function $u - \tilde{u}$ for $N = 81$, $c = 0.04$ and $\delta = 90$.

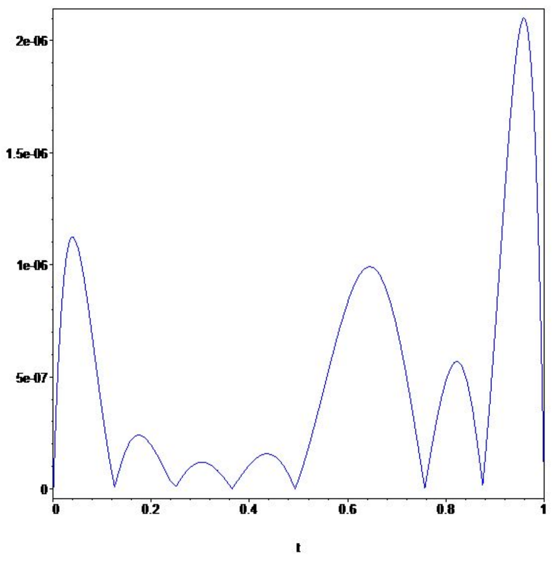

Fig. 2 Plot of function $p - \tilde{p}$ for $N = 81$, $c = 0.04$ and $\delta = 90$.

Table 2 List of value of RBF approximation for $t = 0.5$

x	GA-RBFs error
0	1×10^{-2}
0.125	1×10^{-2}
0.25	1×10^{-2}
0.375	1×10^{-2}
0.5	1×10^{-2}
0.625	1×10^{-2}
0.75	1×10^{-2}
0.875	1×10^{-2}
1	1×10^{-2}

Table 3 List of value of RBF approximation for $x = 0.5$

T	GA-RBFs error
0	0.36×10^{-55}
0.125	1×10^{-2}
0.25	1×10^{-2}
0.375	1×10^{-2}
0.5	1×10^{-2}
0.625	1×10^{-2}
0.75	1×10^{-2}
0.875	1×10^{-2}
1	1×10^{-2}

In Table 2 are listed some value of RBFs error for $t = 0.5$, $N = 81$ and $c = 0.04$, in Table 3 are listed some value of RBFs error for $x = 0.5$, $N = 81$ and $c = 0.04$, and in Table 4 are listed some value of parameter *c*, E^2 and E^∞ for *u*, *p*.

Table 4 Some values of shape parameter *c*, E^2, and E^∞ for *u, p*

c	$E^2(u)$	$E^2(p)$	$E^\infty(u)$	$E^\infty(p)$
0.04	9×10^{-2}	3×10^{-7}	1×10^{-2}	2×10^{-6}
0.06	1×10^{-2}	1×10^{-6}	1×10^{-2}	1×10^{-5}
0.08	1×10^{-2}	4×10^{-6}	1×10^{-2}	3×10^{-5}
0.1	1×10^{-1}	1×10^{-5}	1×10^{-2}	6×10^{-5}

Example 4.2

Consider problem (1.1)-(1.2):

$$u_0(x) = (0.5 - x)x,$$
$$g_1(t) = 0.5,$$
$$g_2(t) = -1.5,$$
$$h(t) = 2t^2 - 2t,$$
$$f(x,t) = 4t(0.5 + 2x),$$

and $x^* = 0$, for which the exact solution is

$$u(x,t) = -(x - 0.5)x + 2t^2 - 2t,$$
$$p(t) = 4t.$$

So, in Figure 3 the error function $u - \tilde{u}$ is plotted using GA-RBF with $c = 0.02$, $\delta = 90$ (the number of floating point arithmetics) and the set of collocation points $x_i = (i-1)0.2$ and $t_i = (i-1)0.2$ for $N = 36$. Also the corresponding error function $p - \tilde{p}$ is plotted in Figure 4.

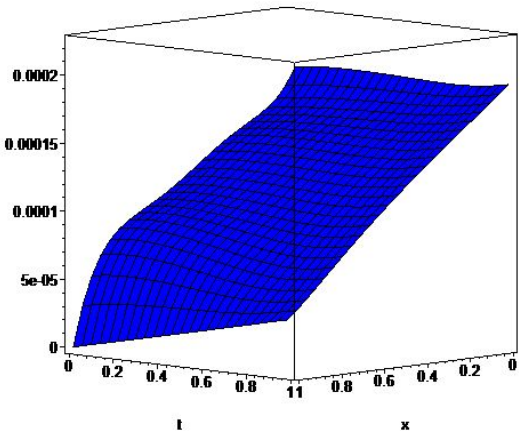

Fig. 3 Plot of function $u - \tilde{u}$ for *N*=36 and *c*=0.02 and $\delta = 90$.

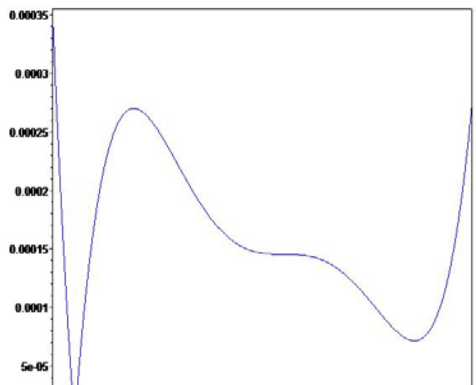

Fig. 4 Plot of function $p - \tilde{p}$ for *N*=36 and *c*=0.02 and $\delta = 90$.

Table 5 List of value of RBF approximation for t=0.5.

x	GA-RBFs Error
0	1×10^{-4}
0.2	1×10^{-4}
0.4	11×10^{-5}
0.6	12×10^{-5}
0.8	13×10^{-5}
1	13×10^{-5}

Table 6 List of value of RBF approximation for $x = 0.5$.

t	GA-RBFs Error
0	0.45×10^{-7}
0.2	0.63×10^{-4}
0.4	0.99×10^{-4}
0.6	0.13×10^{-3}
0.8	0.17×10^{-3}
1	0.20×10^{-3}

Table 7 Some values of shape parameter c, E^2, and E^∞ for u, p

c	$E^2(u)$	$E^2(p)$	$E^\infty(u)$	$E^\infty(p)$
0.02	1×10^{-2}	9×10^{-5}	2×10^{-4}	3×10^{-4}
0.05	8×10^{-4}	5×10^{-4}	1×10^{-3}	2×10^{-3}
0.08	2×10^{-3}	1×10^{-3}	3×10^{-3}	5×10^{-3}
0.1	3×10^{-2}	2×10^{-3}	5×10^{-3}	8×10^{-3}
0.4	5×10^{-2}	3×10^{-2}	8×10^{-2}	1×10^{-2}

5. Conclusions

Radial basis functions are used to solve an inverse parabolic equation. The meshless property of the RBFs method is the most important advantage of this scheme over the traditional mesh dependent techniques such as finite difference methods, finite element methods, and boundary element methods. The mesh free nature of the new technique allows us to solve the problems with non-regular geometry. A comparison with some well-known finite difference methods for numerical solution of the inverse parabolic problem shows that the present method is more accurate. In conclusion we mention that the RBFs technique can be extended to similar two and three dimensional inverse parabolic problems subject to temperature overspecification.

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