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Stability Analysis of a Predator-Prey Model

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Abstract
In this paper, a predator-prey model with Holling type II response function is proposed and analyzed. The model is characterized by a couple of system of first order non-linear differential equations. The equilibrium points are computed, boundedness and criteria for stability of the system are obtained.

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Prey-Predator, Stability, Limit Cycle, Functional Response, Hopf Bifurcation.

1. Introduction

There is an extensive literature concerned with the dynamical relationship between predator and prey due to its universal existence and importance. Mathematical modeling provides an effective tool in the study of contemporary population ecology [1]-[3]. In population dynamics, the functional response of predator to prey density refers to the change in the density of prey attacked per unit time per predator as the prey density changes [4].

Although the prey-predator theory has seen much progress, many long standing mathematical and ecological problems remain open [5]-[11].

Consider the prey-predator system:

$$\begin{cases} \frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \frac{\alpha xy}{1+nx} & \equiv \quad F(x,y) \\ \frac{dy}{dt} = y\left(-s+\frac{\alpha\delta x}{1+nx}\right) & \equiv \quad G(x,y) \end{cases} \tag{1.1}$$

where x, y denote prey and predator population respectively at any time t, and r, k, s, α, n, δ are all positive constants. Here, r represents the intrinsic growth rate and k the carrying capacity of the prey, s is the death rate of the predator; $\frac{\alpha}{n}$ is the maximum number of prey that can be eaten by each predator in unit time; $\frac{1}{n}$ is the density of prey necessary to achieve one half that rate; δ is the conversion factor denoting the number of newly born predators for each captured prey. The term $\frac{\alpha x}{(1+nx)}$ denotes the functional response of the predator.

2. Basic Results

2.1 Boundedness of the System

Theorem 2.1.1
All the solutions of system (1.1) are bounded.

Proof:
Define the function $z = x + (1/\delta)y$. Then :

$$\frac{dz}{dt}=\frac{dx}{dt}+\frac{1}{\delta}\frac{dy}{dt}=rx\left(1-\frac{x}{k}\right)-\frac{\alpha xy}{1+nx}-\frac{s}{\delta}y+\frac{\alpha xy}{1+nx}$$

For $u>0$, we have:

$$\frac{dz}{dt}+uz\leq\frac{k}{4r}(u+r)^2-\frac{1}{\delta}(s-u)$$

Now if we choose $u<s$; then $\frac{k}{4r}(u+r)^2-\frac{1}{\delta}(s-u)$ is bounded for all $(x,y)\in R_+^2$. Thus, we choose $\eta>0$, such that $\frac{dz}{dt}+uz<\eta$.

Applying the theory of differential inequality [12] we obtain:

$$0<z(x,y)<\frac{\eta}{u}(1-e^{-ut})+z(x(0),y(0))e^{-ut}$$

which, up on letting $t\to\infty$, yields $0<z<\left(\frac{\eta}{u}\right)$..

So, we have that all the solutions of system (1) that start R_+^2 are confined to the origin A, where $A=\{(x,y)\in R_+^2: z=\frac{\eta}{u}+\varepsilon\}$, for any $\varepsilon>0$.

2.2 Equilibria

The positive equilibria of model (1.1) can be obtained by solving the following equations:

$$r-\frac{rx}{k}-\frac{\alpha y}{1+nx}=0$$

$$-s+\frac{\alpha\delta x}{1+nx}=0$$

All the equilibria of model (1.1) are:

$$E_0(0,0),\ E_1(k,0), and\ E_2(x^*,y^*)$$

where: $x^*=\frac{s}{(\delta\alpha-sn)}$, and $x^*=\frac{r\delta}{k}\left(\frac{k(\delta\alpha-sn)-s}{(\delta\alpha-sn)^2}\right)$

For the existence of positive equilibrium both $\delta\alpha-sn>0$ and $\frac{s}{(\delta\alpha-sn)}<1$ must hold.

2.3 Stability Analysis

In this section we will consider the stability properties of the equilibria of (1.1). Stability of equilibrium points is investigated by finding the Jacobean matrices for each equilibrium points.

Now:

$$J = \begin{pmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{pmatrix} = \begin{pmatrix} r - \frac{2rx}{k} - \frac{\alpha y(1+nx) - \alpha nxy}{(1+nx)^2} & \frac{\alpha x}{1+nx} \\ \frac{\alpha\delta(1+nx) - \alpha n\delta x}{(1+nx)^2} & -s + \frac{\alpha\delta x}{1+nx} \end{pmatrix}$$

From this we have, $J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -s \end{pmatrix}$, $J(E_1) = \begin{pmatrix} -r & \frac{-\alpha kr}{r+rkn} \\ 0 & -s + \frac{\delta\alpha rk}{r+rkn} \end{pmatrix}$, and $J(E_2) = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$

where:

$$A = r - \frac{2r}{k}\frac{s}{(\delta\alpha - sn)} - \frac{r}{k\delta\alpha}(k(\delta\alpha - sn) - s),$$

$$B = -\frac{s}{\delta}, \text{ and } C = \frac{r}{k\alpha}(k(\delta\alpha - sn) - s)$$

The eigenvalues of this system are roots of the equation $(r-\lambda)(-\lambda - s) = 0$. Therefore, $E_0(0,0)$ is unstable (saddle). $E_1(k,0)$ is locally asymptotically stable when $\frac{s}{k(\delta\alpha - ns)} > 1$ because, Jacobean matrix of $E_1(k,0)$, has negative value if $\frac{\sigma}{k(\delta\alpha - ns)} > 1$ and unstable (saddle) when $\frac{s}{k(\delta\alpha - ns)} \leq 1$.

Remark: When both $E_0(0,0)$, and $E_1(k,0)$ are saddle, the system is *persistent* [14].

Proposition 2.3.1
Suppose $\det(E_2) = -BC$ and $tr(E_2) = A$, then E_2 is locally asymptotically stable if $-BC > 0$ and $A < 0$.

Proof:
Now **A** will be negative if $\frac{s}{k(\delta\alpha - sn)} + \frac{\delta\alpha}{kn(\delta\alpha - sn)} > 1$. From this it is clear that if $\frac{s}{k(\delta\alpha - sn)} + \frac{\delta\alpha}{kn(\delta\alpha - sn)} > 1$, then E_2 is locally asymptotically stable.

Remark
If $\frac{s}{k(\delta\alpha - sn)} + \frac{\delta\alpha}{kn(\delta\alpha - sn)} = 1$, then the system enters into Hopf type small amplitude periodic solutions (limit cycles) near E_2.

2.4 Existance of Limit Cycle

In two dimensions, it is well known for prey-predator systems that the existence and stability of a limit cycle is related to the existence and stability of a positive equilibrium. We assume that a positive equilibrium exists, for otherwise the predator population tends to extinction [13].

If the equilibrium is asymptotically stable, there may exist limit cycles, the innermost of which must be unstable from the inside and the outermost of which must be stable from the outside. Besides, if the limit cycles do not exist in this case, the equilibrium is globally asymptotically stable. Also, if the positive equilibrium exists and is unstable, there must occur at least one limit cycle.

By the present subsection, we shall prove that system (1.1) has unique stable limit cycle, when E_2 becomes locally unstable.

Let us consider system (1.1) in the form:

$$\frac{dx}{dt} = xg(x) - yp(x), \quad x(0) > 0 \tag{2.1}$$

$$\frac{dy}{dt} = y(-s + q(x)), \quad y(0) > 0$$

where: $g(x) = r\left(1 - \frac{x}{k}\right)$, $p(x) = \frac{\alpha x}{1+nx}$, $q(x) = \frac{\delta \alpha x}{1+nx}$. We will prove the following theorem regard uniqueness of limit cycle of this system.

Lemma 2.4.1

Suppose in system (2.1), $\frac{d}{dx}\left(\frac{xg'(x)+g(x)-xg(x)\frac{p'(x)}{p(x)}}{-s+q(x)}\right) \leq 0$ in $0 \leq x < x^*$ *and* $x^* < x \leq k$. The system (2) has exactly one limit cycle which is globally asymptotically stable with respect to the set $\{(x,y): x > 0, y > 0\} \backslash \{E_2(x^*, y^*)\}$

Theorem 2.4.1

If $\frac{s}{k(\delta\alpha - sn)} + \frac{\delta\alpha}{kn(\delta\alpha - sn)} \leq 1$, then system (1) has exactly one limit cycle which is globally asymptotically stable with respect to the set $\{(x,y): x > 0, y > 0\} \backslash \{E_2(x^*, y^*)\}$.

Proof:

This will be equivalent to proving:

$$\frac{d}{dx}\left(\frac{x\left(1-\frac{x}{k}\right) + r\left(1-\frac{x}{k}\right) - r\left(1-\frac{x}{k}\right)\frac{1}{1+nx}}{-s + \frac{\delta\alpha x}{1+nx}}\right) \leq 0$$

Equivalently, $\frac{d}{dx}\left(\frac{x(2x+\frac{1}{n}-k)}{x-\lambda}\right) \geq 0$,

where: $\lambda = \frac{s}{\delta\alpha - ns}$. It is equivalent to proving $(x-\lambda)^2 + \lambda\left(\frac{k-\frac{1}{n}}{2}\right) - \lambda^2 \geq 0$ or $\frac{k-\frac{1}{n}}{2} \geq \lambda$

That is if : $\frac{s}{k(\delta\alpha - sn)} + \frac{\delta\alpha}{kn(\delta\alpha - sn)} \leq 1$.

The equality holds if and only if $\frac{s}{k(\delta\alpha - sn)} + \frac{\delta\alpha}{kn(\delta\alpha - sn)} = 1$.

This completes the proof.

3. Conclusions

By the current paper we considered a prey-predator system assuming that the predator response is of Holling type II. We gave conditions for existence and stability of the equilibria and persistent criteria for the system. Besides, we proved that exactly one stable limit cycle occurs in this system when the positive equilibrium is unstable. This proof also enables us to conclude that local asymptotic stability of the positive equilibrium implies its global asymptotic stability.

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