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Proper Generalized Decomposition Method for Solving Singularity Volterra-Fredholm Integral Equations with Non-smooth Input Function

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Abstract
A conditioned and robust method based on product integration method can be used to calculate integrals with singularities. We write the integrand function as a product of the smooth function and a singular kernel function. We transform the Volterra Fredholm integral equations to a system of Fredholm integral equations of the second kind. A special transformation used to solve the system of Fredholm equation for both smooth and non-smooth input functions. The system of algebraic equation will be derived and solved. Numerical examples are included to demonstrate the validity and applicability of the proposed technique with both singular Kernels.

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Volterra-Fredholm Integral Equations, Logarithmic Kernels, Integral equation, Collocation Matrix Method, Legendre and Chebychev Polynomials.

1. Introduction

We consider the Volterra-Fredholm integral equation of the second kind with Logarithm Kernel:

$$\Phi(x,t)-\int_0^t\int_{-1}^1 \ln|x-y|\Phi(y,\tau)dyd\tau = f(x,t),(x,t)\in[-1,1]\times[0,T] \quad (1.1)$$

where $0\leq T$ and f is a given function. The elements $K(x,y)=\ln|x-y|$ which is the Logarithm Kernel or the Abel kernel defined as $K(x,y)=|x-y|^{-\delta}$, where $\delta\in]0,1[$. The function $f(x,t)$ can be smooth or non-smooth input function. The two cases are slightly different. The main objective of this paper is to derive a new strategy to solve the Volterra-Fredholm integral equation with smooth and non-smooth given function $f(x,t)$. The method will be presented throughout the paper and discussed for both introduced Kernel above. For solving Volterra-Fredholm integral equations (VFIE), many methods with enough accuracy and efficiency have been used before by many researches [4] - [12]. Maleknejad and Fadaei Yami [8] solved the system of Volterra-Fredholm integral equations by Adomian decomposition method. Kauthen in [7], used continuous time collocation method for Volterra-Fredholm integral equations. Legendre wavelets also were applied for solving Volterra-Fredholm integral equations [13]. In [14], Yalsinbas developed numerical solution of nonlinear Volterra-Fredholm integral equations by using Taylor polynomials. In this paper, we use numerical technique based on Trapezoidal rule, to reduce the Volterra-Fredholm integral Equations to a linear system of Fredholm Integral equations which will be solved using Legendre, Chebyshev collocation method (this technique is presented for the Abel Kernel in [1], [2]) and new technique based on smooth transformation. The paper is organized as follows. In Sections 2, we recall some definitions of the Legendre and Chebychev collocation Method. In section 3, a system of Fredholm integral equations of the second kind is obtained from the Volterra-Fredholm Integral equation. In section 4, we derive a smooth transformation to cancel the singularities for smooth and non smooth input function. In the remainder of the paper, we give a practical example to certify the validity of the proposed technique and we conclude.

2. Preliminary

We present a short summary of Legendre and Chebychev methods to approximation an integral with smooth integrants.

2.1 Approximation of Integral using Legendre Collocation Method

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common set of orthogonal polynomials is the Legendre polynomials. We choice $x_k, k = 0,\dots,n$ (k integer) the zeros of the Legendre polynomial of degree equal $n+1$, P_{n+1}. We determine a suitable interpolating elements $\phi_j(x), j = 0,1,\dots,n$, such that:

$$\Phi_n(x) = \sum_{j=0}^{n} \phi_j(x)\Phi(x_j) \tag{2.1}$$

is the unique interpolating polynomial of degree n, which interpolates Φ at the points $x_i, i = 0,1,\dots,n$. The elements $\phi_j(x), j = 0,1,\dots,n$ are called the basic functions associated with the Legendre interpolation polynomial and they satisfy $\phi_j(x_i) = \delta_{ij}$. Then we get an approximation of the exactly integral, let say:

$$I_n(\Phi) = \int_{-1}^{1} K(x,y)\Phi_n(y)dy \tag{2.2}$$

This type of approximation must be chosen so that the integral (2.2) can be evaluated (either explicitly or by an efficient numerical technique). The functions $P_0(x), P_1(x),\dots,P_n(x)$ will be called interpolating elements. In this dissertation, the interpolating function Φ_n will be assumed to be the interpolating polynomial:

$$\Phi_n(x) = \sum_{j=0}^{n} \beta_j P_j(x) \tag{2.3}$$

where P_j are Legendre polynomials of degree j, n is the number of Legendre polynomials, and β_j are unknown parameters, to be determined. The coefficients β_j are obtained by multiplying both sides of (2.4) by $P_m, m \le n$ (as weight functions), and integrating the resulting equation with respect to x over the interval $[-1,1]$ to obtain:

$$\int_{-1}^{1} P_m(x)\Phi_n(x)dx = \sum_{j=0}^{n} \beta_j \int_{-1}^{1} P_m(x)P_j(x)dx = \beta_m \frac{2}{2m+1}$$

Therefore,

$$\beta_m = \frac{2m+1}{2}\int_{-1}^{1} P_m(x)\Phi_n(x)dx \tag{2.4}$$

Here the integrand $P_m\Phi_n$ is a polynomial of degree $n+m \le 2n$ then its integration in (2.4) can exactly be obtained from just $n+1$ point Gauss-Legendre method, by using the following formula:

$$\beta_m = \frac{2m+1}{2}\sum_{j=0}^{n} w_j P_m(x_j)\Phi(x_j) \tag{2.5}$$

where $w_j, j = 0,\ldots,n$ are the $(n+1)$-point Gauss-Legendre weights. The $n+1$ grid points (x_i) of Gauss Legendre integration in (2.6) giving us the exact integral of an integrand polynomial of degree $n+m \leq 2n$ can be obtained as the zeros of the $n+1$-th-degree Legendre polynomial. Then, given the $n+1$ grid point x_i, we can get the corresponding weight w_i of the i point Gauss Legendre integration formula by solving the system of linear equations. Now, the interpolating polynomial Φ_n can be written as:

$$\Phi_n(x) = \sum_{m=0}^{n}\left(\frac{2m+1}{2}\sum_{j=0}^{n} w_j P_m(x_j)\Phi(x_j)\right)P_m(x)$$
$$= \sum_{j=0}^{n}\left(w_j \sum_{m=0}^{n}\frac{2m+1}{2}P_m(x_j)P_m(x)\right)\Phi(x_j) \tag{2.6}$$

Using (2.2) and (2.7) we get :

$$\phi_j(x) = w_j \sum_{m=0}^{n}\frac{2m+1}{2}P_m(x_j)P_m(x), j = 0,\ldots,n \tag{2.7}$$

2.2 Approximation of Integral using Chebychev Collocation Method

Like Legendre Methods, here we will use the Chebyshev polynomials T_n of the first kind. The polynomial T_{n+1} has $n+1$ zeros in the interval $[-1;1]$, which are located at the points:

$$x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right), k = 0,\ldots,n \tag{2.8}$$

The Chebyshev polynomials of the first kind of degree n, T_n, satisfy discrete orthogonality relationships on the grid of the $(n+1)$ zeros of T_{n+1} (which are referred to as the Chebyshev nodes):

$$\sum_{k=0}^{N} T_i(x_k)T_j(x_k) = \begin{cases} 0 & : i \neq j \\ N+1 & : i = j = 0 \\ \frac{N+1}{2} & : i = j \neq 0 \end{cases} \tag{2.9}$$

For an arbitrary interval $[a,b]$, we can find a mapping that transform $[a,b]$ into $[-1,+1]$:

$$y_k = \frac{b-a}{2}x_k + \frac{a+b}{2} = \frac{b-a}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right) + \frac{a+b}{2}, k = 0,\ldots,n$$

and the Chebyshev nodes defined by (2.8) are actually zeros of this Chebyshev polynomial. Based on the discrete orthogonality relationships of the Chebyshev polynomials, various methods of solving linear and nonlinear ordinary differential equations, see [3] for more details. The solution of linear ordinary differential systems, with polynomial coefficients, can be approximated by a finite polynomial or a finite Chebyshev series. The computation can be performed so that the solution satisfies exactly a perturbed differential system, the perturbations being computed multiples of one or more Chebyshev polynomials and integral differential equations, see [15] were devised at about the same time and were found to have considerable advantage over finitedifferences methods. Since then, these methods have become standard [16]. They rely on expanding out the unknown function in a large series of Chebyshev polynomials, truncating this series, substituting the approximation in the actual equation, and determining equations for the coefficients. In our approach we follow closely the procedures like Legendre Method. Let us say, that similar procedures can be applied for a second grid given by the extremas of T_n as nodes. It is important to stress that our goal is not to approximate a function f on the interval $[-1;1]$, but rather to approximate the values of the function f corresponding to a given discrete set of points like those given in equation (2.8). Here, let $(T_0, T_1, T_2, \ldots, T_n)$ the interpolating elements. Then equation (2.3) becomes:

$$\Phi_n(x) = {\sum_{j=0}^{n}}' \beta_j T_j(x) \tag{2.10}$$

where the prime indicates that the first term is to be halved (which is convenient for obtaining a simple formula for all the coefficients β_j). The function Φ_n interpolates Φ at the $n+1$ Chebyshev nodes, we have at these nodes $\Phi(x_k) = \Phi_n(x_k)$. Hence, using the discrete orthogonality relation (2.9) we get:

$$\beta_j = \frac{2}{n+1} \sum_{k=0}^{n} \Phi(x_k) T_j(x_k),\ j = 0,1,\ldots,n \tag{2.11}$$

$$\Phi_n(x) = {\sum_{j=0}^{n}}' \beta_j T_j(x) = {\sum_{j=0}^{n}}' \frac{2}{n+1} \sum_{k=0}^{n} \Phi(x_k) T_j(x_k) T_j(x)$$

$$= \sum_{k=0}^{n} \frac{2}{n+1} \left({\sum_{j=0}^{n}}' T_j(x_k) T_j(x)\right) \Phi(x_k) \tag{2.12}$$

Using (2.1) and (2.10) we get:

$$\phi_k(x) = \frac{2}{n+1} {\sum_{j=0}^{n}}' T_j(x_k) T_j(x) \tag{2.13}$$

3. Derivation of System of Fredholm Equations

We consider the Volterra-Fredholm integral equation of the second kind with Logarithm Kernel (1.1). First, if $t = 0$ the Volterra-Fredholm integral equations is reduced to: $\Phi(x,0) = f(x,0)$. For $t \neq 0$, we apply Trapezoidal Method to solve the Volterra integral equations according to the variable τ. For a given t, we divide the interval of integration $(0;t)$ into m equal subintervals $\delta\tau = t_m / m$, where $t_m = t$.

Let $\tau_0 = 0, t_0 = \tau_0, t_m = \tau_m = t, \tau_j = j\delta\tau, t_j = \tau_j$. By using the trapezoid rule,

$$\int_0^t \int_{-1}^{1} \ln|x-y| \Phi(y,\tau) dy d\tau = \delta\tau {\sum_{j=0}^{m}}'' \int_{-1}^{1} \ln|x-y| \Phi(y,\tau_j) dy$$

where the double prime indicates that the first and last term to be halved, where:

$$\delta\tau = \frac{\tau_j - 0}{j} = \frac{t-0}{m}, \tau_j \boxtimes t,\ j \geq 1, t = t_m = \tau_m$$

In all our approximations, the error assumed negligible, this help us to get a system of Fredholm Integral equations. Now, for $0 \leq r \leq m$, the Volterra Fredholm integral equations become a system of Fredholm integral equations:

$$\Phi(x,t_r) - \delta\tau \sum_{j=0}^{r}{}'' \int_{-1}^{1} \ln|x-y|\Phi(y,\tau_j)dy = f(x,t_r), 1 \le r[$$

and $\Phi(x,0) = f(x,0)$. We get the system:

$$\Phi(x,0) = f(x,0)$$

$$\Phi(x,t_1) - \frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi(y,t_1)dy = f(x,t_1) + \frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi(y,0)dy$$

$$\Phi(x,t_2) - \frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi(y,t_2)dy = f(x,t_2) + \left\{ \begin{array}{r} \frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi(y,0)dy \\ + \\ \delta\tau\int_{-1}^{1} \ln|x-y|\Phi(y,t_1)dy \end{array} \right.$$

$$\vdots$$

$$\Phi(x,t_m) - \frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi(y,t_m)dy = f(x,t_m) + \delta\tau \sum_{j=0}^{m-1}{}' \int_{-1}^{1} \ln|x-y|\Phi(y,t_j)dy,$$

where the prime indicates that the first term to be halved. Denote:

$$f(x,t_\ell) = f^{\ell}(x), \Phi(y,\tau_\ell) = \Phi^{\ell}(y), \ell = 0,\ldots,m$$

Putting:

$$F^m(x) = f^m(x) + \sum_{j=0}^{m-1}{}' \int_{-1}^{1} \ln|x-y|\Phi^j(y)dy,$$

An astitute computation gives:

$$F^m(x) = f^m(x) + 2\sum_{j=1}^{m-1}(-1)^{j+m}(f^j(x) - \Phi^j(x))$$

$$+(-1)^{m+1}\frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi^0(y)dy$$

Now, our problem becomes:

$$\Phi^{\ell}(x) - \frac{\delta\tau}{2}\int_{-1}^{1} \ln|x-y|\Phi^{\ell}(y)dy = F^{\ell}(x), \ell = 1,\ldots,m \tag{3.1}$$

$$\Phi(x,0) = f(x,0)$$

Eqn (3.1) represents a system of Fredholm integral equations of the second kind. In the next section, we will present the well known techniques of Legendre and Chebychev collocation methods to solve this system and later we present a smooth transformation to solve the system of Fredholm integral equation with singular kernel.

4. Smoothing Transformation for Non-smooth Input Function

This section deals with a new transformation that can be introduced in the case where the input function $f(x,t)$ has a finite singularities. For both type of kernel we will propose an appropriate transformation which lead to better convergence.

4.1 Somooting Transformation: Case of Logarithmic Kernel

Monegato and Scuderi [17] introduce a simple smoothing change of variable to solve one-dimensional linear weakly singular integral equations on bounded intervals, with input functions which may be smooth or not. In both cases either the input function is smooth or non-smooth, they define the smoothing transformation κ using piecewise Hermite interpolation polynomial H_m, so we will call this transformation as the Hermite transformation. The Fredholm integral equation of the second kind with Logarithmic Kernel will be solved by using a smooth transformation. In our case we will present the Hermite smoothing transformation which reduce a second kind Fredholm integral equation with a weakly singular kernel, for both smooth and non-smooth input functions, to an equivalent equation with smoother solution. We choose a nonlinear transformation: $\kappa:[-1,1]\to[-1,1]$ is a sufficiently smooth monotone function having as fixed points $x_0=-1<x_1<x_2<\ldots<x_{n+1}=1$ and vanishing derivative at these points. An example of this mapping function is the Hermite interpolation polynomial and its is define in each subinterval by the conditions:

$$j\in\{k,k+1\}, H_n(x_j)=x_j, H_n^{(i)}(x_j)=0, i=0,1,\ldots,\alpha_j-1, \alpha_j\geq 2$$

The integers $\alpha_k, k=0,\ldots,n$, are chosen accordingly to the smoothing effect that ought to produce at the points $x_k, k=0,\ldots,n$. Notice that the smoothness of κ itself does not depend on the choice of α_0 and α_n. A predecious choice is, $\forall k=0,\ldots,n$:

$$H_k(t)=x_k+(x_{k+1}-x_k)^{2-\alpha_k-\alpha_{k+1}}\frac{(\alpha_k+\alpha_{k+1}-1)!}{(\alpha_k-1)!(\alpha_{k+1}-1)!}\int_{x_k}^{t}(z-x_k)^{\alpha_k-1}(x_k-z)^{\alpha_{k+1}-1}dz.$$

Now, using the Hermite interpolation polynomial to solve the Fredholm integral equation (in our case is nothing but the system of integral equation). Fixing $\ell\in\{1,\ldots,m\}$ and putting $x=\kappa(t)=H_n(t)$ in the following system of Fredholm integral equation:

$$\Phi^{\ell}(H_m(t))-\frac{\delta\tau}{2}\int_{-1}^{1}\ln|H_m(t)-y|\Phi^{\ell}(y)dy=F^{\ell}(H_m(t)) \tag{4.1}$$

A simple change of variable $y=H_m(s)$, then (2.16) becomes:

$$\Phi^{\ell}(H_m(t))-\frac{\delta\tau}{2}\int_{-1}^{1}\ln|H_m(t)-H_m(s)|\Phi^{\ell}(H_m(s))H_m^{'}(s)ds=F^{\ell}(H_m(t)) \tag{4.2}$$

Multiplying both sides of (4.2) by $H_m^{'}(t)$ and setting $\Phi^{\ell}(H_m(t))H_m^{'}(t)=\eta^{\ell}(t)$ and $F^{\ell}(H_m(t))H_m^{'}(t)=\zeta^{\ell}(t)$ we obtain:

$$\eta^{\ell}(t)-\frac{\delta\tau}{2}\int_{-1}^{1}\ln|H_m(t)-H_m(s)|H_m^{'}(t)\eta^{\ell}(s)ds=\zeta^{\ell}(t)$$

Using:

$$\ln|H_m(t)-H_m(s)|=\ln\left|\frac{H_m(t)-H_m(s)}{t-s}\right|+\ln|t-s|$$

For simplicity we define:

$$\theta(t,s)=\begin{cases}\ln\left|\frac{H_m(t)-H_m(s)}{t-s}\right|H_m^{'}(t),\ if\ \ t\neq s\\ \ln\left|H_m^{'}(t)\right|H_m^{'}(t),\ if\ \ t=s\end{cases}$$

Putting $t=x_i, i=1,\ldots,n$

$$\eta^{\ell}(x_i)-\frac{\delta\tau}{2}\int_{-1}^{1}\left(\theta(x_i,s)+\ln|x_i-s|H_m^{'}(x_i)\right)\eta^{\ell}(s)ds=\zeta^{\ell}(x_i) \tag{4.3}$$

The function $H_m^{'}(x_i)\eta^{\ell}(s)$ and $\theta(x_i,s)\eta^{\ell}(s)$ are continuous as functions of s, Now we will approximate the function $\chi_1^{\ell}(s)=H_m^{'}(x_i)\eta^{\ell}(s)$ and $\chi_2^{\ell}(s)=\theta(x_i,s)\eta^{\ell}(s)$ by the n^{th} degree interpolating polynomials:

$$\chi_1^{\ell}(s)=\sum_{j=0}^{n}H_m^{'}(x_i)\eta^{\ell}(x_j)\phi_j(s),\chi_2^{\ell}(s)=\sum_{j=0}^{n}\theta(x_i,x_j)\eta^{\ell}(x_j)\phi_j(s) \tag{4.4}$$

which interpolates $\chi_1^{\ell}(s)$ and $\chi_2^{\ell}(s)$ at $x_i, i=1,\ldots,n$ and $\phi_j(s)$ is given by (2.8) (if we use Legendre polynomial) and (2.14) (for Chebyshev polynomial). Substituting (4.4) into (4.3) we get: $\forall i=1,\ldots,n$

$$\eta^{\ell}(x_i)-\frac{\delta\tau}{2}\sum_{j=0}^{n}\left(\chi_1^{\ell}(x_j)\int_{-1}^{1}\phi_j(s)ds+\chi_2^{\ell}(x_j)\int_{-1}^{1}\phi_j(s)\ln|x_i-s|ds\right)\eta^{\ell}(x_j)=\zeta^{\ell}(x_i), \tag{4.5}$$

Eqn (4.5) can be written as the $(n+1)\times(n+1)$ linear system:

$$(Id-\frac{\delta\tau}{2}A)\eta^{\ell}=\zeta^{\ell}$$

where:

$$\eta^{\ell}=\left(\eta^{\ell}(x_i)\right)_{i=1,\ldots,n},\quad \zeta^{\ell}=\left(\zeta^{\ell}(x_i)\right)_{i=1,\ldots,n}$$

$$A=\left(A_{i,j}=\left(\chi_1^{\ell}(x_j)\int_{-1}^{1}\phi_j(s)ds+\chi_2^{\ell}(x_j)\int_{-1}^{1}\phi_j(s)\ln|x_i-s|ds\right)\right)_{i,j=1,\ldots,n}$$

• Substituting (2.7) into equation (4.5) (case of Legendre polynomial) we obtain:

$$a^1_{i,j} = w_j \sum_{k=0}^{n} \frac{2k+1}{2} P_k(x_j) u_k(x_i)$$

where $u_k(x_i), i, k = 0, \dots, n$ are defined as:

$$u_k(x_i) = \int_{-1}^{1} \ln|x_i - y| P_k(y) dy$$

The constants $u_k(x_i), i, k = 0, \dots, n$, can be evaluated from the recurrence relation:

$$(k+3)u_{k+2}(x_i) = (2k+3)x_i u_{k+1}(x_i) - k u_k(x_i), k1$$
$$u_0(x_i) = (1+x_i)\ln|1+x_i| + (1-x_i)\ln|1-x_i| - 2$$
$$u_1(x_i) = \frac{1}{2}(1-x_i^2)\ln\left|\frac{1+x_i}{1-x_i}\right| - 2$$
$$u_2(x_i) = x_i u_1(x_i) + \frac{2}{3}$$

• Using Chebychev polynomial we have:

$$a^1_{ij} = \frac{2}{n+1} \sum_{j=0}^{n}{}' v_k(x_i) T_k(x_j)$$

where $v_k(x_i), i, k = 0, \dots, n$ are defined as:

$$v_k(x_i) = \int_{-1}^{1} \ln|x_i - y| T_k(y) dy$$

The constants $v_k(x_i), i, k = 0, \dots, n$, can be evaluated from the recurrence relation:

$$(1+\frac{1}{m+1})v_{m+1}(x_i) - 2x_i v_m(x_i) + (1-\frac{1}{m-1})v_{m-1}(x_i)$$
$$= \frac{2}{1-m^2}((1-x_i)\ln|1-x_i| - (-1)^m(1+x_i)\ln|1+x_i|) - 6\frac{(1-(-1)^m)}{(m^2-1)(m^2-4)}$$

with starting values:

$$v_0(x_i) = (1+x_i)\ln|1+x_i| + (1-x_i)\ln|1-x_i| - 2$$
$$v_1(x_i) = x_i(u_0(x_i)+1) + \frac{1}{2}((1-x_i)^2\ln|1-x_i| - (1+x_i)^2\ln|1+x_i|)$$

$$v_2(x_i) = 4x_i v_1(x_i) - (2x_i^2+1)v_0(x_i) + \frac{2}{3}((1-x_i)^3 + (1+x_i)^3\ln|1+x_i|) - \frac{4}{9}(1+3x_i^2)$$
$$v_3(x_i) = 2x_i(3+2x_i^2)v_0(x_i) - 3(4x_i^2+1)v_1(x_i) - 6x_i v_2(x_i)$$
$$+ (1-x_i)^4\ln(1-x_i) + (1+x_i)^4\ln(1+x_i) + 2x_i(1+x_i^2)$$

The second coefficient $\int_{-1}^{1}\phi_j(s)ds$ can be computed as follows:

1. Using Legendre collocation method: substituting (2.8) into $\int_{-1}^{1}\phi_j(s)ds$ and a simple computation gives:

$$\int_{-1}^{1} P_m(x)dx = \begin{cases} 2 & if \ \ m = 0 \\ 0 & if \ \ m \ge 1 \end{cases}$$

Now we get:

$$\int_{-1}^{1}\phi_j(s)ds = w_j \sum_{m=0}^{n} \frac{2m+1}{2} P_m(x_j) \int_{-1}^{1} P_m(x)dx = w_j$$

2. Using Chebychev collocation method: substituting (2.14) into $\int_{-1}^{1}\phi_j(s)ds$ and simple computation gives:

$$\int_{-1}^{1} T_m(x)dx = \begin{cases} \dfrac{2}{1-m^2} & if \ \ m \ even \\ 0 & if \ \ m \ odd \end{cases}$$

In this case we have:

$$\begin{aligned} \int_{-1}^{1}\phi_j(s)ds &= \frac{2}{n+1} {\sum_{j=0}^{n}}' T_j(x_k) \int_{-1}^{1} T_j(s)ds \\ &= \frac{2}{n+1} {\sum_{j=0}^{[n/2]}}' T_{2j}(x_k) \frac{2}{1-4j^2} \\ &= \frac{2}{n+1} {\sum_{j=0}^{[n/2]}}' \cos\left(\frac{j\pi(2k+1)}{n+1}\right) \frac{2}{1-4j^2} \end{aligned}$$

where $[n/2]$ is is the greatest integer less than or equal to $n/2$.

4.2 Smoothing Transformation: Case of Abel Kernel

A modified transformation can be proposed in the case of non-smooth input function $f(x,t)$. Suppose that $F(x)$ obtained in the system of Fredholm integral equation has finite singularities at a finite number of points of $(-1,1)$. For instance, let us say the jump points are:

$$-1 < x_1 < x_2 < \ldots < x_N < 1$$

We define further the function:

$$\eta(t) = \frac{3}{10}t^3 + \left(\frac{1}{2} + \frac{t}{5}\right), -1 \le t \le 1$$

where q is a positive integer. We define the transformation:

$$h(t) = \frac{\eta(t)^5 - \eta(-t)^5}{\eta(t)^5 + \eta(-t)^5}, -1 \le t \le 1 \qquad (4.6)$$

We define our transformation by:

$$h_k(t) = \frac{x_k + x_{k+1}}{2} + \frac{x_{k+1} - x_k}{2} h(\frac{x_{k+1} + x_k}{x_{k+1} - x_k} + \frac{2}{x_{k+1} - x_k} t), t \in [x_k, x_{k+1}], k = 0, \ldots, M \quad (4.7)$$

The transformation (4.7) ($h_k(x)$) defined on $[x_k, x_{k+1}]$ proposed satisfies the following special conditions:

- x_k, x_{k+1} are invariant under h_k , i.e $h_k(x_k) = x_k, h_k(x_{k+1}) = x_{k+1}$
- $x_k, x_{k+1}, \frac{x_k + x_{k+1}}{2}$ are zero of h'_k , i.e $h'_k(x_k) = h'_k(x_{k+1}) = h_k(\frac{x_k + x_{k+1}}{2}) = 0$

This transformation defined on the interval $[-1,1]$ as:

$$h(t) = h_k(t), t \in [x_k, x_{k+1}], k = 0, \ldots, M$$

where M is the number of singularities of the input function F .

Fixing $\ell \in \{1, \ldots, m\}$ and putting $x = h_k(t)$ in the following system of Fredholm integral equation:

$$\Phi^{\ell}(h_m(t)) - \frac{\delta\tau}{2} \int_{-1}^{1} | h_m(t) - y |^{-\gamma} \Phi^{\ell}(y) dy = F^{\ell}(h_m(t)) \quad (4.8)$$

A simple change of variable $y = h_m(s)$, then (4.8) becomes:

$$\Phi^{\ell}(h_m(t)) - \frac{\delta\tau}{2} \int_{-1}^{1} | h_m(t) - h_m(s) |^{-\gamma} \Phi^{\ell}(h_m(s)) h'_m(s) ds = F^{\ell}(h_m(t)) \quad (4.9)$$

Multiplying both sides of (4.9) by $h'_m(t)$ and setting $\Phi^{\ell}(h_m(t)) h'_m(t) = \eta^{\ell}(t)$ and $F^{\ell}(h_m(t)) h'_m(t) = \zeta^{\ell}(t)$ we obtain:

$$\eta^{\ell}(t) - \frac{\delta\tau}{2} \int_{-1}^{1} | h_m(t) - h_m(s) |^{-\gamma} h'_m(t) \eta^{\ell}(s) ds = \zeta^{\ell}(t)$$

For simplicity we define:

$$\theta(t,s) = \begin{cases} \left| \frac{h_m(t) - h_m(s)}{t - s} \right|^{-\gamma} h'_m(t), \text{ if } t \neq s \\ \left| h'_m(t) \right|^{-\gamma} h'_m(t), \text{ if } t = s \end{cases}$$

Putting $t = x_i, i = 1, \ldots, n$:

$$\eta^{\ell}(x_i) - \frac{\delta\tau}{2} \int_{-1}^{1} \left(\theta(x_i, s) | x_i - s |^{-\gamma} h'_m(x_i) \right) \eta^{\ell}(s) ds = \zeta^{\ell}(x_i) \quad (4.10)$$

The function $h'_m(x_i)\eta^{\ell}(s)$ and $\theta(x_i, s)\eta^{\ell}(s)$ are continuous as functions of s . Now, we will approximate the function $\chi_1^{\ell}(s) = h'_m(x_i)\eta^{\ell}(s)$ and $\chi_2^{\ell}(s) = \theta(x_i, s)\eta^{\ell}(s)$ by the n^{th} degree interpolating polynomials:

$$\chi_1^\ell(s) = \sum_{j=0}^{n} h_m^{'}(x_i)\eta^\ell(x_j)\phi_j(s), \chi_2^\ell(s) = \sum_{j=0}^{n} \theta(x_i, x_j)\eta^\ell(x_j)\phi_j(s) \qquad (4.11)$$

which interpolates $\chi_1^\ell(s)$ and $\chi_2^\ell(s)$ at $x_i, i = 1,\ldots,n$ and $\phi_j(s)$ is given by (2.8) (if we use Legendre polynomial) and (2.14) (for Chebyshev polynomial). Substituting (4.11) into (4.10) we get:

$$\eta^\ell(x_i) - \frac{\delta\tau}{2}\sum_{j=0}^{n}\chi_2^\ell(x_j)\int_{-1}^{1}\phi_j(s)\,|\,x_i - s\,|^{-\gamma}\,ds\,\eta^\ell(x_j) = \zeta^\ell(x_i), i = 1,\ldots,n \qquad (4.12)$$

Eqs (4.12) can be written as the $(n+1)\times(n+1)$ linear system:

$$(Id - \frac{\delta\tau}{2}A)\eta^\ell = \zeta^\ell$$

where:

$$\eta^\ell = \left(\eta^\ell(x_i)\right)_{i=1,\ldots,n}, \quad \zeta^\ell = \left(\zeta^\ell(x_i)\right)_{i=1,\ldots,n}$$

$$A = \left(A_{i,j} = \left(\chi_2^\ell(x_j)\int_{-1}^{1}\phi_j(s)\,|\,x_i - s\,|^{-\gamma}\,ds \right)\right)_{i,j=1,\ldots,n}$$

The coefficient $\int_{-1}^{1}\phi_j(s)\,|\,x_i - s\,|^{-\gamma}\,ds$ used in the matrix A can be computed as follows:

- Legendre collocation method: Substituting (2.8) into equation (4.12), this lead to compute:

$$a_{i,j}^1 = w_j\sum_{k=0}^{n}\frac{2k+1}{2}P_k(x_j)u_k(x_i)$$

where $u_k(x_i), i,k = 0,\ldots,n$ are defined by:

$$u_k(x_i) = \int_{-1}^{1}|\,x_i - y\,|^{1-\gamma}\,P_k(y)dy$$

The constants $u_k(x_i), i,k = 0,\ldots,n$, can be evaluated from the recurrence relation:

$$(k+2-\gamma)u_{k+1}(x_i) = (2k+1)x_i u_k(x_i) - (k-1+\gamma)u_{k-1}(x_i), k1$$

$$u_0(x_i) = \frac{1}{1-\gamma}\left((1-x_i)^{1-\gamma} + (1+x_i)^{1-\gamma}\right)$$

$$u_1(x_i) = x_i u_0(x_i) + \frac{1}{2-\gamma}((1-x_i)^{2-\gamma} - (1+x_i)^{2-\gamma})$$

- Chebyshev collocation method: If we substitute (2.13) in (4.12), we obtain:

$$a_{i,j}^1 = \frac{2d_j}{n}\sum_{k=0}^{n}d_k T_k(x_j)u_k(x_i)$$

where $u_k(x_i), i,k = 0,\ldots,n$ are defined as:

$$u_k(x_i) = \int_{-1}^{1} | x_i - y |^{-\delta} T_k(y)dy$$

The constants $u_k(x_i), i, k = 0, \dots, n$, can be evaluated from the recurrence relation:

$$(1+\frac{1-\gamma}{k+1})u_{k+1}(x_i) - 2x_i u_k(x_i) + (1-\frac{1-\gamma}{k-1})u_{k-1}(x_i) =$$
$$\frac{2}{1-k^2}((1-x_i)^{1-\gamma} - (-1)^k(1+x_i)^{1-\gamma}), k \ge 2$$

with the starting values the follwing values :

$$u_0(x_i) = \frac{1}{1-\gamma}\left((1-x_i)^{1-\gamma} + (1+x_i)^{1-\gamma}\right)$$
$$u_1(x_i) = x_i u_0(x_i) + \frac{1}{2-\gamma}((1-x_i)^{2-\gamma} - (1+x_i)^{2-\gamma})$$
$$u_2(x_i) = 4x_i u_1(x_i) - (2x_i^2+1)u_0(x_i) + \frac{2}{3-\gamma}((1-x_i)^{3-\gamma} - (1+x_i)^{3-\gamma})$$

5. Numerical Implementation

In this section, to achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we will give some computational results. The computations, associated with the example, are performed by MATLAB 7. Practically, the error function en at step n is computed as $\Phi^N - \Phi_n$ where un is the result with n used in the approximation and N is large enough so that Φ^N is much closer to the (discrete) solution η^∞ than the numerical tolerance $\varepsilon = 10^{-14}$.

5.1 First Example

We consider the solution with $n = 128$, and we compute the error between the solution for different value at different value of n and the η^{150}. We consider in this first example the case of Logarithm Kernel which is singular at the diagonal ($x = y$). Our transformation proposed let us use both Chebychev and Legendre collocation in order to have a more accurate results. In our computation, we consider $f(x,t) = | x + t |$. The method of smoothing the kernel used to solve Volterra-Fredholm integral equation with:

- Smoothing transformation used with Logarithm kernel and Legendre approximation with $\alpha_0 = \alpha_1 = 2$, to solve the Volterra Fredholm integral equation. Fig. 1 presents the log_{10} of error computed for the η function introduced. Let us recall that Φ^{150} is used as reference solution in this example. We call this method ”LASK”. The error is presented for three different value of t.
- Smoothing transformation used with Logarithm kernel and Chebychev approximation, with $\alpha_0 = \alpha_1 = 2$, we get Fig.2 . It’s clear that the error is deceasing function for different fixed value of t. We call this method ”CASK”.
- Table 1 shows the $\| \Phi - \Phi_n \|_\infty$ with respect the variation of n (number of term used in our approximation) for different value of t.

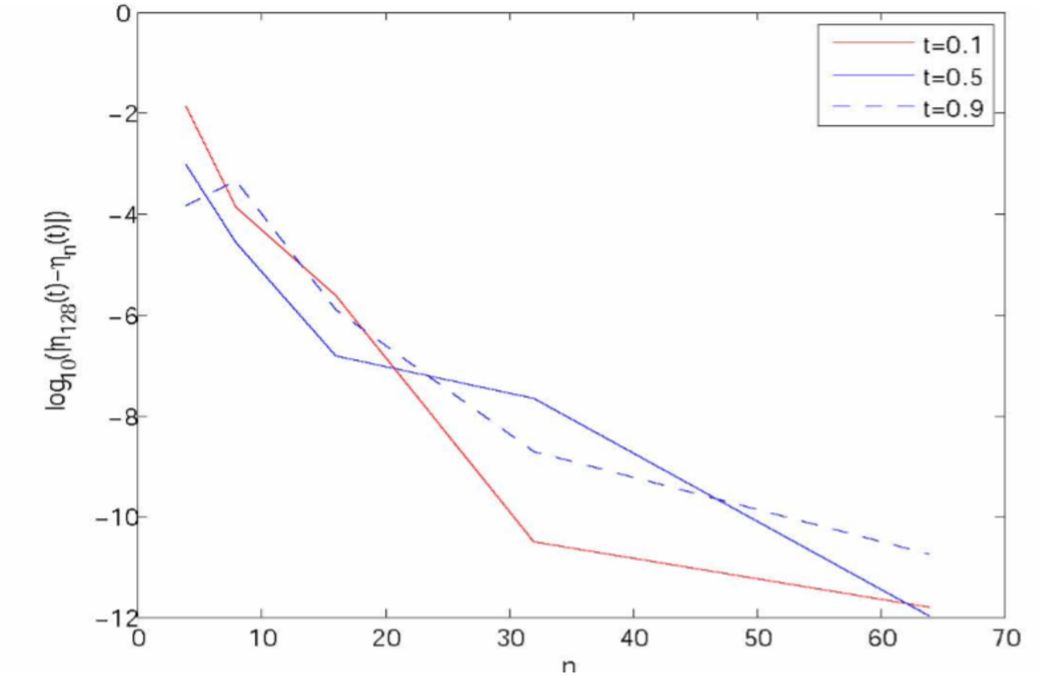

Fig. 1 LASK at three different time

Table 1 Abel Kernel with legendre expansion. Infinity norm of the error for different value of n, with $f(x,t) = |x+t|$

T	*n=8*	*n=15*	*n=20*	*n=50*
0.1	3.7554e-5	5.0241e-6	3.3377e-9	1.4721e-11
0.2	4.4443e-5	5.7561e-6	1.3291e-8	1.0759e-11
0.3	4.1594e-5	5.3988e-6	8.8816e-10	2.2396e-11
0.4	3.3997e-5	1.1361e-6	1.4303e-8	1.1753e-11
0.5	2.0721e-6	7.4281e-6	1.4958e-8	8.4115e-12
0.6	7.0072e-5	1.1446e-6	5.5378e-9	1.5124e-11
0.7	5.9575e-5	4.4120e-6	9.1627e-9	6.0130e-12
0.8	5.1439e-6	1.9976e-6	1.3127e-8	7.0615e-12
0.9	7.6804e-6	1.2737e-6	5.1083e-9	5.6299e-12

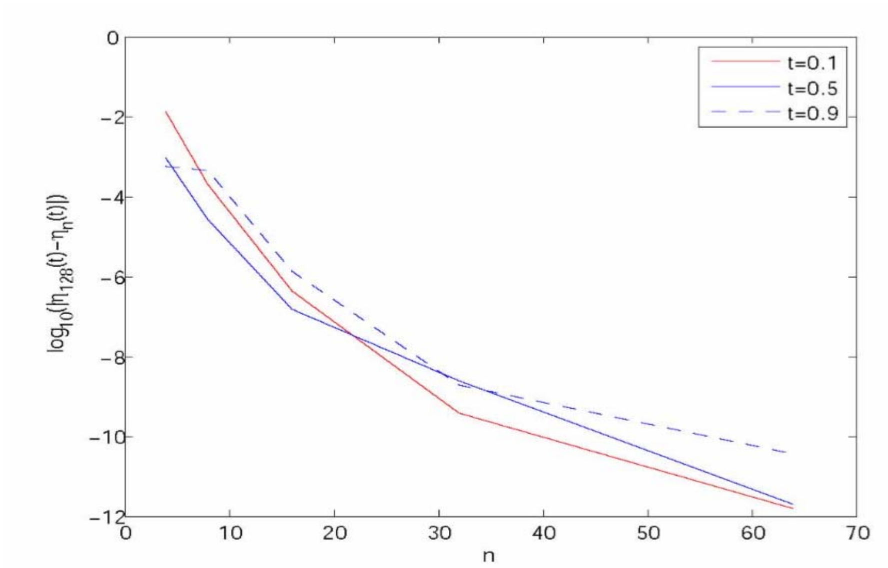

Fig. 2 CASK at three different time

5.2 Second Example

In the case of Abel kernel, we consider $f(x,t) = tx/\sqrt{2-x}$. The reference solution is considered at $n = 320$. Tables 2 and 3 present the infinity norm of error $\|\Phi - \Phi_n\|_\infty$ for different value of n and different value of t for both Legendre and Chebychev approximation used with smoothing transformation. We use a singular function in this example.

• Here, we denote by $\|\Phi - \Phi_n\|_{\infty,AbSL}$ the error between the reference solution and the solution computed using Legendre Polynomial and smoothing transformation.

• $\|\Phi - \Phi_n\|_{\infty,AbSC}$ denotes the error between the reference solution and the solution computed using Chebychev and smoothing transformation.

We present further two tables to show the error versus the parameter.

Table 2 Abel Kernel with Legendre expansion. Infinity norm of the error for different value of n

T	*n=5*	*n=7*	*n=15*	*n=30*	*n=60*
0.1	5.3633e-02	1.4798e-03	3.0538e-07	5.1949e-12	2.7756e-16
0.2	9.1459e-02	1.7370e-03	1.1436e-07	1.1271e-11	3.8858e-15
0.3	1.1501e-01	4.6156e-04	2.4473e-07	1.8913e-11	3.9958e-15
0.4	8.1724e-02	1.3090e-03	2.3363e-07	2.7467e-11	8.3157e-15
0.5	2.6437e-02	1.8735e-03	7.0656e-08	3.1924e-11	2.3304e-15
0.6	4.7376e-02	3.4607e-04	2.5040e-07	1.6363e-11	6.4383e-15
0.7	1.1135e-01	1.8060e-03	1.9136e-07	3.7064e-11	4.3309e-16
0.8	1.1918e-01	1.2394e-03	8.9153e-08	2.3764e-11	1.4494e-14
0.9	1.1592e-02	2.2443e-03	5.5983e-08	2.2166e-11	2.4417e-14

Table 3 Abel Kernel with Chebychev expansion. Infinity norm of the error for different value of n

T	$n=5$	$n=7$	$n=15$	$n=30$	$n=60$
0.1	4.3622e-02	1.5698e-03	1.5751e-10	2.5555e-12	4.5991e-14
0.2	8.2462e-02	2.6370e-03	3.9498e-10	6.4197e-12	1.0081e-13
0.3	2.1721e-01	4.7156e-04	8.1604e-10	1.2664e-11	1.6098e-13
0.4	7.1782e-02	1.4290e-03	1.4873e-9	2.0295e-11	1.1535e-13
0.5	3.6031e-02	1.8735e-03	2.3945e-9	1.9885e-11	1.6065e-13
0.6	5.7351e-02	1.4607e-04	2.9040e-9	8.3604e-12	1.2618e-13
0.7	6.1151e-01	2.8060e-03	1.0222e-9	1.6510e-11	2.0528e-13
0.8	2.1211e-01	2.2394e-03	3.6632e-9	1.0956e-11	1.6201e-13
0.9	2.12911e-02	3.2443e-03	4.4391e-9	2.4230e-11	3.2971e-13

6. Conclusions

In this paper, two methods have been proposed using smoothing transformation for two different kernels. Both, smoothing transformation used to cancel the singularities in the integral equation and therefore the Legendre and Chebychev method are presented and used to solve the modified integral equation obtained after smoothing transformation. The linear system obtained is well conditioned and solved to get an accurate solution which was compared to the analytical solution. Our methods presented can be extended to other Volterra or Fredholm integral equation.

References

1. *Chniti C.*, 'Numerical Approximations of redholm Integral equations with Abel Kernel using Legendre and Chebychev Polynomials', *J. Math. Comput. Sci.*, ***3****(2013), 655-667.*
2. *Chniti C.*, 'On The Numerical Solution of Volterra-Fredholm Integral Equations With Abel Kernel Using Legendre Polynomials', *IJAST.*, ***1*** *(2013), 404-412.*
3. *Fox L.*, 'Chebyshev Methods for Ordinary Differential Equations', *Comput Journal,* ***4****, (1962), 318-331*
4. *Ezzati R.*, Mokhtari F., 'Numerical solution of Fredholm integral equations of the second kind by using fuzzy transforms', *Int. Journal of Physical Scienc.*, ***7*** *(2012) ,1578-1583.*
5. *Ezzati R., Najafalizadeh S.*, 'Numerical solution of nonlinear Voltra-Fredholm integral equation by using Chebyshev polynomials', *Mathematical Sciences Quarterly Journal.*, ***5*** *(2011), 14-22.*
6. *Guoqiang H., Liqing Z.*, 'Asymptotic expantion for the trapezoidal Nystrom method of linear Voltra-Fredholm integral equations', *J. Comput. Math. Appl.*, ***51*** *(1994), 339-348.*
7. *Kauthen P.G..*, 'Continuous time collocation methods for Voltra-Fredholm integral equations', *Numer. Math.*, ***56*** *(1989), 409-424.*
8. *Maleknejad K., Fadaei Yami M.R..*, 'A computational method for system of Volterra- Fredholm integral equations', *Appl. Math. Comput.*, ***188*** *(2006), 589-595.*
9. *Maleknejad K., Hadizadeh M.*, 'A new computational method for Volterra-Fredholm integral equation', *J. Comput. Math. Appl.*, ***37*** *(1999), 37-48.*
10. *Maleknejad K., Mahmodi Y.*, 'Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations', *Appl. Math. Comput.*, ***145*** *(2003), 641-653.*
11. *Maleknejad K., Sohrabi S. , Rostami Y.*, 'Numerical solution of nonlinear Voltra integral equation of the second kind by using Chebyshev polynomials', *Appl. Math. Comput.*, ***188*** *(2007), 123-128.*
12. *Maleknejad K., Tavassoli M., Mahmoudi Y.*, 'Numerical solution of linear Fredholm and Volterra integral equations of the second kind by using Legendre wavelet' *Kybern Int. J. Syst. Math.*, ***32*** *(2003), 1530-1539.*
13. *Yousefi S., Razzaghi S.M.*, 'Legendre wavelets method for the nonlinear Volterra Fredholm integral equations', *Math, Comput. Simul.*, ***70*** *(2005) 1-8.*
14. *Yalsinbas S.*, 'Taylor polynomial solution of nonlinear Volterra-Fredholm integral equations', *Appl. Math. Comput.*, ***127*** *(2002), 195-206.*

15. *Elliott D., '*Chebyshev series method for the numerical solution of Fredholm integral equations'*, Comput. J., **6** (1963/1964), 102-111.*
16. *Press W. H., Teukolsky S. A., Vetterling W. T., Flannery B. P., 'Numerical Recipes in FORTRAN :The art of scientific computing',* Cambridge University Press, New York*, 1992.*
17. *Monegato G., Scuderi L.,* 'High order methods for weakly singular integral equations with non-smooth input functions', *Mathematics of Computation., **67** (1998), 1493-1515.*