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Analytic Solution to the System of Time Fractional Partial Differential Equations via Laplace Transform

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Abstract
By the current article, the author solved certain system of time fractional heat equations using integral transform. The transform method is a powerful tool for solving partial fractional differential equations and evaluation of integrals. The result reveals that the transform method is very convenient and effective.

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Caputo Fractional Derivative, Non-homogeneous Time Fractional Heat Equation, Laplace Transforms, Fourier Transforms.

1. Introduction

Over the recent years, it has turned out that many phenomena in fluid mechanics, physics, biology, engineering and other areas of the sciences can be successfully modelled by the use of fractional derivatives. That is because of the fact that, a realistic modelling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. Fractional differential equations arise in the unification of diffusion and wave propagation phenomenon. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, [1], [2], [3], [12], the Fourier transform method [11], the iteration method [18] and operational method [11]. However, most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients.

More detailed information about some of these results can be found in a survey paper by Kilbas and Trujillo [10], Atanackovic and Stankovic [4], [5] and Stankovic [20], used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model for visco - elastic rod. Oldham and Spanier [13] and [14], respectively, by reducing a boundary value problem involving Fick's second law in electro-analytic chemistry to a formulation based on the partial Riemann – Liouville fractional with half derivative. Oldham and Spanier [14] gave other application of such equations for diffusion problems. Wyss [21] and Schneider [19] considered the time fractional diffusion and wave equations and obtained the solution in terms of Fox functions.

Definition 1.1

The left Caputo fractional derivatives of order $\alpha > 0$ $(n-1 < \alpha \le n\ ,\ n \in N)$ is defined by:

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx . $$

Definition 1.2

The Laplace transform of function $f(t)$ is as:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt := F(s).$$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)\,ds$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

Definition 1.3

The two-parameter function of the Mittag-Leffler type is defined by the series expansion:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

The simplest Wright function is given by the series:

$$W(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n + \beta)},$$

for $\alpha, \beta, z \in C$. We have the following relationship:

$$L\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha} \mp a} \quad (\mathrm{Re}(s) > |a|^{\frac{1}{\alpha}}).$$

Theorem 1.1

Let $L\{f(t)\} = F(s)$ and $L\{u(t,\tau)\} = U(s)\exp(-\tau q(s))$ and by assuming $\phi(s)$, $q(s)$ are analytic, then one has:

$$L\left(\int_0^\infty f(\tau)\,u(t,\tau)\,d\tau\right) = U(s)\,F(q(s)).$$

Proof. [8]

Lemma 1.1

In the above Theorem, if we set: $U(s) = \frac{1}{s^{\alpha}}$ and $q(s) = s^{\alpha}$,

$$L\{u(t,\tau)\} = \frac{\exp(-\tau s^{\alpha})}{s^{\alpha}},$$

It leads to:

$$u(t,\tau) = t^{\alpha-1} W(-\alpha,\alpha;-\tau t^{-\alpha}).$$

Then, we obtain:

$$L\left(\frac{1}{t^{1-\alpha}}\int_0^\infty f(\tau) W(-\alpha,\alpha;-\tau t^{-\alpha})\,d\tau\right) = \frac{F(s^{\alpha})}{s^{\alpha}}.$$

provided that the integral in the bracket converges absolutely.

Proof. [8]

2. Evaluation of Certain Integrals via the Laplace Transforms

Like Fourier transform, the Laplace transform is used in a variety of applications. Perhaps the most common usage of the Laplace transform is in the solution of boundary value problems.

However, there are other situations for which the properties of the Laplace transform are also very useful, such as in the evaluation of integrals.

Lemma 2.1

The following integral relation holds true:

$$I(\xi)=\int_0^{+\infty}(\frac{x}{a})^{\frac{\nu-1}{2}}J_{\nu-1}(2\sqrt{ax})J_0(2\sqrt{\xi x})dx=\frac{(a-\xi)^{-\nu}}{\Gamma(1-\nu)},\quad 0<\nu<1,a>0 \tag{2.1}$$

Proof.
Let us find the Laplace transform of $I(\xi)$, to obtain:

$$L\{I(\xi);\xi\to s\}=\int_0^{+\infty}e^{-s\xi}\{\int_0^{+\infty}(\frac{x}{a})^{\frac{\nu-1}{2}}J_{\nu-1}(2\sqrt{ax})J_0(2\sqrt{\xi x})dx\}d\xi \tag{2.2}$$

Changing the order of integration which is permissible leads to:

$$L\{I(\xi);\xi\to s\}=\int_0^{+\infty}(\frac{x}{a})^{\frac{\nu-1}{2}}J_{\nu-1}(2\sqrt{ax})\{\int_0^{+\infty}e^{-s\xi}J_0(2\sqrt{x\xi})d\xi\}dx, \tag{2.3}$$

After evaluation of the inner integral one gets the following relationship:

$$L\{I(\xi);\xi\to s\}=\int_0^{+\infty}(\frac{x}{a})^{\frac{\nu-1}{2}}J_{\nu-1}(2\sqrt{ax})\frac{e^{-\frac{x}{s}}}{s}dx=\frac{1}{s}\int_0^{+\infty}e^{-(\frac{1}{s})x}(\frac{x}{a})^{\frac{\nu-1}{2}}J_{\nu-1}(2\sqrt{ax})dx \tag{2.4}$$

F inally, the last integral can be evaluated easily to get:

$$L\{I(\xi);\xi\to s\}=\frac{e^{-as}}{s^{1-\nu}}$$

By taking the inverse Laplace transform of the above relation, we arrive at:

$$I(\xi)=L^{-1}\{\frac{e^{-as}}{s^{1-\nu}};s\to\xi\}=\delta(\xi-a)*\frac{\xi^{-\nu}}{\Gamma(1-\nu)}=\frac{(\xi-a)^{-\nu}}{\Gamma(1-\nu)}. \tag{2.5}$$

Special case $\nu=0.5, a=1$, we get:

$$I(\xi)=\int_0^{+\infty}(\frac{1}{\sqrt[4]{x}})J_0(2\sqrt{\xi x})J_{-\frac{1}{2}}(2\sqrt{x})dx=\frac{1}{\sqrt{\pi(\xi-1)}}, \tag{2.6}$$

Lemma 2.2

The following integral relation holds true:

$$\int_0^{+\infty}\frac{ber(2\sqrt{\lambda x})}{x^2+\mu^2}dx=\frac{\pi}{2\mu}J_0(2\sqrt{\mu\lambda}). \tag{2.7}$$

Proof .
Let us consider the following integral:

$$I(\lambda)=\int_0^{+\infty}\frac{ber(2\sqrt{\lambda x})}{x^2+\mu^2}dx.$$

Taking the Laplace transform of the above function with respect to λ, results in:

$$L\{I(\lambda);\lambda\to s\}=\int_0^{+\infty}e^{-s\lambda}\{\int_0^{+\infty}\frac{ber(2\sqrt{\lambda x})}{x^2+\mu^2}dx\}d\lambda, \tag{2.8}$$

By changing the order of integration which is permissible leads to:

$$L\{I(\lambda);\lambda\rightarrow s\}=\int_0^{+\infty}\frac{1}{x^2+\mu^2}\{\int_0^{+\infty}e^{-s\lambda}ber(2\sqrt{x\lambda})d\lambda\}dx. \tag{2.9}$$

The value of the inner integral is $\frac{1}{s}\cos\frac{x}{s}$, therefore we get:

$$L\{I(\lambda);\lambda\rightarrow s\}=\frac{1}{s}\int_0^{+\infty}\frac{\cos(s^{=1})x}{x^2+\mu^2}dx, \tag{2.10}$$

The above integral can be evaluated by means of the calculus of residues to obtain:

$$L\{I(\lambda);\lambda\rightarrow s\}=\frac{1}{s}(\frac{\pi}{2\mu})e^{-\frac{\mu}{s}}. \tag{2.11}$$

At this point, taking the inverse Laplace transform of the above relation to get:

$$I(\lambda)=(\frac{\pi}{2\mu})L^{-1}\{\frac{1}{s}e^{-\frac{\mu}{s}};s\rightarrow\lambda\}=(\frac{\pi}{2\mu})J_0(2\sqrt{\mu\lambda}). \tag{2.12}$$

3. Solution to System of Time Fractional Heat Equations.

The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing the first time derivative of a fractional derivative of order $\alpha, 0<\alpha\leq 1$. In this study, we considered methods and results for the system of partial fractional diffusion equations which arise in applications. In fact, the author generalized the problem of the influence of the earth's rotation on ocean currents [6], [9].

Theorem 3.1

Let us consider the following system of partial fractional differential equations subject to the given boundary conditions:

$${}^cD_t^\delta u(x,t)-\alpha v=\beta\frac{\partial^2 u}{\partial x^2}+\lambda_1\phi(t) \qquad \circ<\delta\leq 1 \tag{3.1}$$

$${}^cD_t^\delta v(x,t)+\alpha u=\beta\frac{\partial^2 v}{\partial x^2}+\lambda_2\psi(t) \tag{3.2}$$

B.C. $\begin{cases}\lim_{x\to\infty}u(x,t)=\circ\\ \lim_{x\to\infty}v(x,t)=\circ\end{cases}$ $\begin{cases}u_x(\circ,t)=-g(t)\\ v_x(\circ,t)=-k(t)\end{cases}$ $\begin{cases}u(x,\circ)=\beta_1\\ v(x,\circ)=\beta_2\end{cases}$

where $\alpha,\beta,\lambda_1,\lambda_2,\beta_1,\beta_2$ are real constants.

The above system of partial fractional differential equations has the following formal solutions:

$$u(x,t)=\int_0^t\int_0^{+\infty}\frac{\sqrt{\xi}}{2\pi\beta\sqrt{\beta(t-\eta)^3}}\exp-(\frac{\xi^2}{4(t-\eta)}+\frac{x^2}{4\beta\xi})\{g(\eta)\cos\alpha\xi+k(\eta)\sin\alpha\xi\}d\xi d\eta+$$
$$(\beta_1e^{-\alpha^2t}-\alpha\beta_2(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2t}))-\{\lambda_1\phi(t)\}*\{(\frac{1}{\sqrt{\pi t}}-\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2t}\int_0^{\alpha\sqrt{t}}e^{u^2}du)\}+\{\lambda_2\psi(t)\}*\{\alpha e^{-\alpha^2t}\}, \quad (3.3)$$

and

$$v(z,t)=\int_0^t\int_0^{+\infty}\frac{\sqrt{\xi}}{2\pi\beta\sqrt{\beta(t-\eta)^3}}\exp-(\frac{\xi^2}{4(t-\eta)}+\frac{x^2}{4\beta\xi})\{k(\eta)\cos\alpha\xi-g(\eta)\sin\alpha\xi\}d\xi d\eta-$$
$$-(\beta_2e^{-\alpha^2t}+\alpha\beta_1(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2t}))+\lambda_2\psi(t)*(\frac{1}{\sqrt{\pi t}}-\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2t}\int_0^{\alpha\sqrt{t}}e^{u^2}du)-\lambda_1\phi(t)*(\alpha e^{-\alpha^2t}). \quad (3.4)$$

Proof:

Let us multiply both sides of the relation (3.2) by i and add up to relation (3.1) to get:

$$ {}^cD_t^\delta u(x,t)+i\,{}^cD_t^\delta v(x,t)-\alpha v+i\alpha u=\beta\frac{\partial^2u}{\partial x^2}+i\beta\frac{\partial^2v}{\partial x^2}+\lambda_1\varphi(t)+i\lambda_2\psi(t). \quad (3.5)$$

By introducing the new variable $w(x,t)=u(x,t)+iv(x,t)$, we get the following:

$$ {}^cD_t^\delta w(x,t)+i\alpha(u+iv)=\beta(\frac{\partial^2u}{\partial x^2}+i\frac{\partial^2v}{\partial x^2})+(\lambda_1\phi(t)+i\lambda_2\psi(t))$$

or,

$$ {}^cD_t^\delta w(x,t)+i\alpha w(x,t)=\beta(\frac{\partial^2w}{\partial x^2})+(\lambda_1\phi(t)+i\lambda_2\psi(t)).$$

With the new boundary conditions as follows:

$$\lim_{x\to\infty}w(x,t)=\lim_{x\to\infty}u(x,t)+i\lim_{x\to\infty}v(x,t)=\circ \quad (3.6)$$
$$w_x(\circ,t)=u_x(\circ,t)+iv_x(\circ,t)=-g(t)-ik(t)=-(g(t)+ik(t))$$
$$w(x,\circ)=u(x,\circ)+iv_x(x,\circ)=\beta_1+i\beta_2.$$

Taking the Laplace transform of the both sides of the above relation (3.3) leads to:

$$L\{{}^cD_t^\delta w(x,t)\}+i\,\alpha\,L\{w(x,t)\}=\beta L\{\frac{\partial^2w}{\partial x^2}\}+\lambda_1L\{\phi(t)\}+i\lambda_2L\{\psi(t)\}, \quad (3.7)$$

At this point, using the following relation for the Laplace transform of fractional derivatives,

$$L\{{}^cD_t^\vartheta f(t);t\to s\}=s^\vartheta L\{f(t)\}-\sum_{k=\circ}^{m-1}s^{\vartheta-k-1}f^{(k)}(\circ)\qquad m-1<\beta\le m)$$
$$m=1\to L\{{}^cD_t^\vartheta f(t);t\to s\}=s^\vartheta L\{f(t)\}-s^{\vartheta-1}f(\circ),$$

to obtain:

$$s^\delta L\{w(x,t)\}-s^{\delta-1}w(x,\circ)+i\alpha L\{w(x,t)\}=\beta\frac{\partial^2}{\partial x^2}L\{w(x,t)\}+\lambda_1L\{\varphi(t)\}+i\lambda_2L\{\psi(t)\} \quad (3.8)$$

or:

$$s^{\delta}W(x,s)-s^{\delta-1}(\beta_1+i\beta_2)+i\alpha W(x,s)=\beta\frac{d^2W(x,s)}{dx^2}+\lambda_1F(s)+i\lambda_2H(s)\,.$$

Let us assume that : $\beta_0=(\beta_1+i\beta_2)$

$$\beta\frac{d^2W(x,s)}{dz^2}-(s^{\delta}+i\alpha)W(x,s)+(\lambda_1F(s)+i\lambda_2H(s))+s^{\delta-1}\beta_0=\circ$$

$$\frac{d^2W(x,s)}{dz^2}-\frac{(s^{\delta}+i\alpha)}{\beta}W(x,s)=-\frac{\lambda_1F(s)+i\lambda_2H(s)}{\beta}-\frac{s^{\delta-1}}{\beta}\beta_0 \qquad (3.9)$$

In order to find the solution of the above differential equation, first we solve the homogeneous equation:

$$\frac{d^2W(x,s)}{dx^2}-\frac{(s^{\delta}+i\alpha)}{\beta}W(x,s)=\circ \qquad (3.10)$$

which has the following solution:

$$W(x,s)=C_1e^{x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}}}+C_2e^{-x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}}}$$

$$w_x(\circ,t)=u_x(\circ,t)+iv_x(\circ,t)=-g(t)-ik(t)=-(g(t)+ik(t))\,,$$

Besides, by taking the Laplace transform of the both sides of the above relation we get:

$$\begin{cases}\lim_{x\to\infty}W(x,s)=\circ\\ W_x(\circ,s)=-(G(s)+iK(s))\end{cases} \qquad (3.11)$$

so that, the solution of equation gets the following form:

$$W(x,s)=c_1\exp(x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}})+c_2\exp(-x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}})+\frac{\lambda_1F(s)+i\lambda_2H(s)}{s^{\delta}+i\alpha}+\frac{s^{\delta-1}}{s^{\delta}+i\alpha}\beta_0\,,$$

According to the relation (3.11), we have:

$$\text{if } x\to\infty\to\exp(-x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}})=\circ \quad \text{if } x\to\infty\to\exp(x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}})\to\infty$$

$$\lim_{x\to\infty}W(x,s)=\circ\to C_1=\circ$$

$$W(x,s)=c_2\exp(-x\sqrt{\frac{s^{\delta}+i\alpha}{\beta}})+\frac{\lambda_1F(s)+i\lambda_2H(s)}{s^{\delta}+i\alpha}+\frac{\beta_0s^{\delta-1}}{s^{\delta}+i\alpha}$$

$$W'(x,s)=-c_2\sqrt{\frac{s^{\delta}+i\alpha}{\beta}}\exp(-x\sqrt{\frac{s^{\alpha}+if}{\beta}})\ ,\ W'(0,s)=c_2\sqrt{\frac{s^{\delta}+i\alpha}{\beta}}=-(G(s)+ik(s))$$

$$c_2 = \frac{(G(s)+iK(s))}{\sqrt{\frac{s^{\delta}+i\alpha}{\beta}}}$$

At this point, let us consider the case , $\delta = 0.5$, we get the following:

$$W(x,s) = \frac{G(s)+iK(s)}{\sqrt{\frac{\sqrt{s}+i\alpha}{\beta}}}\exp(-x\sqrt{\frac{\sqrt{s}+i\alpha}{\beta}}) + \frac{\lambda_1 F(s)+i\lambda_2 H(s)}{\sqrt{s}+i\alpha} + \frac{\beta_0 s^{-\frac{1}{2}}}{\sqrt{s}+i\alpha} \qquad (3.12)$$

Using the following relations related to convolution and the Efros's theorem to invert the Laplace transforms:

1. $\int_o^t f(t-x)g(x)dx = L^{-1}\{F(s)*G(s)\}$

2. $L^{-1}\{\frac{1}{\sqrt{s}}\exp(-\sqrt{as})\} = \frac{1}{\sqrt{\pi x}}\exp(-\frac{a}{4x})$

3. $L\{\int_\circ^{+\infty}\frac{\tau}{2\sqrt{\pi t^3}}\exp(-\frac{\tau^2}{4t})f(\tau)d\tau\} = F(\sqrt{s})$

Therefore:

$$L^{-1}\{W(x,s)\} = L^{-1}\{\frac{G(s)+iK(s)}{\sqrt{\frac{\sqrt{s}+i\alpha}{\beta}}}\exp(-x\sqrt{\frac{\sqrt{s}+i\alpha}{\beta}})\} + L^{-1}\{\frac{\lambda_1 F(s)+i\lambda_2 H(s)}{\sqrt{s}+i\alpha}\} + L^{-1}\{\frac{\beta_0 s^{-\frac{1}{2}}}{\sqrt{s}+i\alpha}\}$$

We may find inverse Laplace transform of the above relation term-wise, that is:

$$L^{-1}\{\frac{G(s)+iK(s)}{\sqrt{\frac{\sqrt{s}+i\alpha}{\beta}}}\mathrm{e}^{-x\sqrt{\frac{\sqrt{s}+i\alpha}{\beta}}})\} =$$

$$= \int_0^t (g(\mu)+ik(\mu))\{\int_0^{+\infty}\frac{\sqrt{\beta\tau}}{2\pi\sqrt{(t-\mu)^3}}e^{-\frac{t^2}{4(t-\mu)}}e^{-i\alpha\tau}\exp(-\frac{x^2}{4\beta(\tau-\mu)})d\tau\}d\mu$$

(3.13)

$$L^{-1}\{\frac{\lambda_1 F(s)+i\lambda_2 H(s)}{\sqrt{s}+i\alpha}\} = L^{-1}\{\frac{1}{\sqrt{s}+i\alpha}\}*\{\lambda_1\phi(t)+i\lambda_2\psi(t)\} =$$

$$= \beta_0 L^{-1}\{\frac{s^{-\frac{1}{2}}}{\sqrt{s}+i\alpha}\} = \beta_0 L^{-1}\{\frac{\sqrt{s}+i\alpha}{\sqrt{s}(s+\alpha^2)}\} = \beta_0(e^{-\alpha^2 t} + i\alpha(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t})).$$

$$= (\beta_1 e^{-\alpha^2 t} - \alpha\beta_2(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t})) + i(\beta_2 e^{-\alpha^2 t} + \alpha\beta_1(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t})).$$

Finally, we obtain:

$$W(x,t)=\int_0^t (g(\eta)+ik(\eta))\int_0^{+\infty}\frac{\sqrt{\tau}}{2\pi\beta\sqrt{\beta(t-\eta)^3}}\exp(-\frac{u^2}{4(t-\eta)}-i\alpha\tau-\frac{x^2}{4\beta\tau})d\tau\}d\eta+$$
$$.(\beta_1 e^{-\alpha^2 t}-\alpha\beta_2(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t}))+i(\beta_2 e^{-\alpha^2 t}+\alpha\beta_1(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t}))+ \tag{3.13}$$
$$+\{\lambda_1\phi(t)+i\lambda_2\psi(t)\}*\{(\frac{1}{\sqrt{\pi t}}-\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2 t}\int_0^{\alpha\sqrt{t}} e^{u^2}du)-i\alpha e^{-\alpha^2 t}\}.$$

Taking real and imaginary part of the above relationship (3.13) to obtain:

$$u(x,t)=\int_0^t\int_0^{+\infty}\frac{\sqrt{\xi}}{2\pi\beta\sqrt{\beta(t-\eta)^3}}\exp-(\frac{\xi^2}{4(t-\eta)}+\frac{x^2}{4\beta\xi})\{g(\eta)\cos\alpha\xi+k(\eta)\sin\alpha\xi\}d\xi d\eta+$$
$$(\beta_1 e^{-\alpha^2 t}-\alpha\beta_2(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t}))-\{\lambda_1\phi(t)\}*\{(\frac{1}{\sqrt{\pi t}}-\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2 t}\int_0^{\alpha\sqrt{t}} e^{u^2}du)\}+\{\lambda_2\psi(t)\}*\{\alpha e^{-\alpha^2 t}\}, \tag{3.14}$$

and:

$$v(z,t)=\int_0^t\int_0^{+\infty}\frac{\sqrt{\xi}}{2\pi\beta\sqrt{\beta(t-\eta)^3}}\exp-(\frac{\xi^2}{4(t-\eta)}+\frac{x^2}{4\beta\xi})\{k(\eta)\cos\alpha\xi-g(\eta)\sin\alpha\xi\}d\xi d\eta-$$
$$-(\beta_2 e^{-\alpha^2 t}+\alpha\beta_1(\frac{1}{\sqrt{\pi t}}*e^{-\alpha^2 t}))+\lambda_2\psi(t)*(\frac{1}{\sqrt{\pi t}}-\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2 t}\int_0^{\alpha\sqrt{t}} e^{u^2}du)-\lambda_1\phi(t)*(\alpha e^{-\alpha^2 t}). \tag{3.15}$$

4. Conclusions

The current paper is devoted to studying and application of Laplace transform for solving certain system of time fractional partial differential equations and evaluation of certain integrals of Bessel’s functions. The author considered a generalization of the problem of the influence of the earth’s rotation on ocean currents first posed by V.W. Ekman [9].

One dimensional Laplace transform provides a powerful method for analyzing linear systems. The transform method introduces a significant improvement in this field over existing techniques.

So, the main purpose of this study is to develop a method for finding analytic solutions of the system of time fractional heat equations.

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