Universal Journal of Applied Mathematics & Computation **4 (2016), 23-31**
www.papersciences.com

Laplace Transform for Fractional Integro-differential Equations and Time Fractional Heat Equation

A. Aghili
Department of Applied Mathematics,
Faculty of Mathematical Sciences
University of Guilan, P.O. Box, 1841,
Rasht, Iran
armanaghili@yahoo.com

Abstract
By the current paper, the solution of partial fractional differential equations of time fractional Heat-equation is given. The author uses also certain theorems and corollaries on the Laplace transform for the solution of system of fractional singular integro-differential equations of convolution-type with the Bessel kernel and system of fractional differential equation, where the fractional derivative is described in the Caputo sense. Constructive examples are also provided.

2010 Mathematics Subject Classification : 26A33, 34A08, 34K37, 35R11, 44A10.
Key Word and Phrases
One-dimensional Laplace Transform, Caputo Fractional Derivative, Time Fractional Heat Equation, Singular Integral Equation of Comvolution Type.

1. Introduction

The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing first time derivative by a fractional derivative of order a, where 0<a<1. In recent years, many authors including Poldlubny [5], Beyer and Kempfle [7], Schneider and Wyss [9] Huang and Liu [8] discussed about some problems of homogeneous fractional ordinary differential equations and homogeneous fractional diffusion equations.

Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method [1]-[3], the Fourier transform method. In section 2, we recall the definition of the Caputo fractional integral operator of order $\alpha > 0$, and Efors's theorem. In section 3, we use the complex inversion formula for the Laplace transform and also we solve several examples of integral equation and system of fractional diffusion equations. In section 4, the author solved time fractional heat-equation in the Caputo sense by using theorems, corollaries and methods of the Laplace transform.

2. Definitions

Definition 2.1

The fractional derivatives of order $\alpha > 0$ in the Caputo sense defined as the operator ${}_t^cD_0^\alpha f(t)$:

$$ {}_t^cD_0^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau \quad ,m-1<\alpha\le m \tag{2.1}$$

Theorem 2.1
The Laplace transform of the Caputo fractional derivative is:

$$ L\{{}_t^cD_0^\alpha f(t);p\} = p^\alpha F(s) - \sum_{k=0}^{m-1} p^{\alpha-k-1} f^{(k)}(0^+) \quad ,m-1<\alpha\le m \tag{2.2}$$

Proof

See [5].

Theorem 2.2 (Effros's theorem):
Let $L_t\{f(t)\} = F(s)$ and assuming that $\phi(s), q(s)$ are analytic functions and:

$$L\{\varphi(t,\tau); s\} = \Phi(s)e^{-\tau q(s)}, \tag{2.3}$$

we then have:

$$L_t\{\int_0^\infty f(\eta)\,\varphi(t,\eta)\,d\eta\} = F(q(s))\,\phi(s). \tag{2.4}$$

Proof
See [2].

Corollary 2.1
If $q(s) = \sqrt{s}$, $\phi(s) = \dfrac{1}{\sqrt{s}}$, $L\{f(t)\} = F(s)$ then:

1. $$\frac{F(\sqrt{s})}{\sqrt{s}} = L_t\{\frac{1}{\sqrt{\pi t}}\int_0^\infty \exp(-\frac{\eta^2}{4t}) f(\eta)d\eta; s\}$$

2. $$F(\sqrt{s}) = L_t\{\frac{1}{2t\sqrt{\pi t}}\int_0^\infty \eta \exp(-\frac{\eta^2}{4t}) f(\eta)d\eta; s\}$$

Proof

See [2].

3. Theorems of Laplace Transform

Theorem 3.1
The Complex Inversion Formula of The Laplace transform:
If $L\{f(t); s\} = F(s)$, then $L^{-1}\{F(s); t\}$ is given by:

$$f(t) = L^{-1}\{F(s); t\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds \tag{3.1}$$

and $f(t) = 0$ for $t < 0$, it is also known as Bromwich's integral formula.

Theorem 3.2 (Convolution Theorem)
The convolution of f and g is defined by:

$$(f * g)(t) = \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du \tag{3.2}$$

If $L\{f(t); s\} = F(s)$, $L\{g(t); s\} = G(s)$ then $L\{(f * g)(t); s\} = F(s).G(s)$.

Problem 3.3
Let us solve the following system of fractional singular integro-differential equations of convolution-type with the modified Bessel function of first kind as kernel:

$$\begin{cases} {}_x^c D_0^\alpha g_1(x) = f_1(x) - \lambda\int_0^x (\frac{x-t}{a})^{\frac{\kappa}{2}} I_\kappa(2\sqrt{a(x-t)})\, g_2(t)dt \\ {}_x^c D_0^\alpha g_2(x) = f_2(x) + \lambda\int_0^x (\frac{x-t}{a})^{\frac{\kappa}{2}} I_\kappa(2\sqrt{a(x-t)})\, g_1(t)dt \end{cases}$$

where $g_1(0)=g_2(0)=0\,,\nu>-1,0<\left|\lambda\right|<1$ and $f_1(x),f_2(x)$ are known functions.

Solution
In order to solve the above system, by introducing:
$g(x)=g_1(x)+i\,g_2(x)\,,f\,(x)=f_1(x)+i\,f_2(x)$ we can rewrite the above system of fractional integro- differential equations in the form:

$$ {}_x^c D_0^{\alpha}\, g(x)=f\,(x)+\lambda i\int_0^x \left(\frac{x-t}{a}\right)^{\frac{\kappa}{2}} I_{\kappa}(2\sqrt{a(x-t)})\,g(t)\,dt \tag{3.3}$$

where $g(0)=0\,,\nu>-1,a>0,0<\left|\lambda\right|<1$.

By applying the Laplace transform on both sides of the above equation term - wise we obtain:

$$s^{\alpha}G(s)=F(s)+\lambda i\,\frac{\exp(\frac{a}{s})}{s^{\kappa+1}}G(s)$$

$$G(s)=\frac{1}{s^{\alpha}-\lambda i\,\frac{\exp(\frac{a}{s})}{s^{\kappa+1}}}F(s)=\frac{1}{s^{\alpha}\{1-\frac{\lambda i}{s^{\alpha+\kappa+1}\exp(-\frac{a}{s})}\}}F(s) \tag{3.4}$$

$$=\frac{F(s)}{s^{\alpha}}\sum_{m=0}^{\infty}\frac{(\lambda i)^m}{s^{(\alpha+\kappa+1)\mathrm{m}}\exp(-\frac{ma}{s})}=sF(s)\sum_{m=0}^{\infty}\frac{(\lambda i)^m}{s^{(\alpha+\kappa+1)\mathrm{m}+\alpha+1}\exp(-\frac{ma}{s})}$$

Besides, by using the fact that:

$$L^{-1}\{\frac{\exp(\frac{a}{s})}{s^{\mu+1}}\}=(\frac{x}{a})^{\frac{\mu}{2}}I_{\mu}(2\sqrt{ax}),(i)^{2k}=(-1)^k\,,(i)^{2k+1}=(-1)^k\,i$$

by taking the inverse Laplace transform formula of the above term (3.4) yields:

$$g(x)=\int_0^x\sum_{m=0}^{\infty}(\lambda i)^m\left(\frac{x-t}{am}\right)^{\frac{(\alpha+\kappa+1)\mathrm{m}+\alpha}{2}} I_{(\alpha+\kappa+1)\mathrm{m}+\alpha}(2\sqrt{m\,a(x-t)})f\,'(t)\,dt$$
$$+f\,(0)\sum_{m=0}^{\infty}(\lambda i)^m\left(\frac{x}{am}\right)^{\frac{(\alpha+\kappa+1)\mathrm{m}+\alpha}{2}} I_{(\alpha+k+1)\mathrm{m}+\alpha}(2\sqrt{m\,a\,x}) \tag{3.5}$$

Finally, by taking the real and imaginary part of the above relation we finally obtain the solutions of the system in the following forms:

$$g_1(x)=\sum_{m=0}^{\infty}(-\lambda^2)^m\int_0^x\left(\frac{x-t}{2am}\right)^{m(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{2m(\alpha+\kappa+1)+\alpha}(2\sqrt{2m\,a(x-t)})f_1'(t)dt$$
$$-\sum_{m=0}^{\infty}(-1)^m\lambda^{(2m+1)}\int_0^x\left(\frac{x-t}{(2m+1)a}\right)^{\frac{(2m+1)}{2}(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{(2m+1)(\alpha+\kappa+1)+\alpha}(2\sqrt{a(2m+1)(x-t)})f_2'(t)dt$$
$$+f_1(0)\sum_{m=0}^{\infty}(-\lambda^2)^m\left(\frac{x}{2am}\right)^{m(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{2m(\alpha+\kappa+1)+\alpha}(2\sqrt{2max})$$
$$-f_2(0)\sum_{m=0}^{\infty}(-1)^m\lambda^{(2m+1)}\left(\frac{x}{(2m+1)a}\right)^{\frac{(2m+1)}{2}(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{(2m+1)(\alpha+\kappa+1)+\alpha}(2\sqrt{a(2m+1)x})$$

similarly we get :

$$g_2(x)=\sum_{m=0}^{\infty}(-1)^m\lambda^{(2m+1)}\int_0^x\left(\frac{x-t}{(2m+1)a}\right)^{\frac{(2m+1)}{2}(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{(2m+1)(\alpha+\kappa+1)+\alpha}(2\sqrt{a(2m+1)(x-t)})f_1'(t)dt$$
$$+\sum_{m=0}^{\infty}(-\lambda^2)^m\int_0^x\left(\frac{x-t}{2am}\right)^{m(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{2m(\alpha+\kappa+1)+\alpha}(2\sqrt{2ma(x-t)})f_2'(t)dt$$
$$+f_1(0)\sum_{m=0}^{\infty}(-1)^m\lambda^{(2m+1)}\left(\frac{x}{(2m+1)a}\right)^{\frac{(2m+1)}{2}(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{(2m+1)(\alpha+\kappa+1)+\alpha}(2\sqrt{a(2m+1)x})$$
$$+f_2(0)\sum_{m=0}^{\infty}(-\lambda^2)^m\left(\frac{x}{2am}\right)^{m(\alpha+\kappa+1)+\frac{\alpha}{2}}I_{2m(\alpha+\kappa+1)+\alpha}(2\sqrt{2max})$$

4. Main Results

The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing first time derivative by a fractional derivative of order α ,where $0<\alpha\le 1$.

Problem 4.1

Solution to non-homogenous partial fractional differential equation (Heat equation):

$$ {}_t^cD_0^{\alpha}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+\lambda\,u(x,t)+\mu\,g(t)+\beta, \tag{4.1}$$

where $t>0,\ 0<x<1,\ 0<\alpha\le 1,\ \lambda,\beta$ are constants with boundary conditions:

$$u_x(0,t)=0\,,\ u(1,t)=t\,,\ u(x,0)=0$$

Solution

By using the one dimensional Laplace transform of equation (4.1) with respect to t, we obtain:

$$F_{xx}(x,s)-(s^{\alpha}-\lambda)F(x,s)=-\mu\,G(s)-\frac{\beta}{s}.$$

Solving the above equation, leads to:

$$F(x,s)=A\cosh\sqrt{s^{\alpha}-\lambda}\,x+B\sinh\sqrt{s^{\alpha}-\lambda}\,x+\frac{\mu G(s)}{s^{\alpha}-\lambda}+\frac{\beta}{s(s^{\alpha}-\lambda)}. \tag{4.2}$$

Now, by using the boundary conditions one obtains the unknown constants A , B as follows:

$$F_x(0,s)=0\,; \quad B=0$$

$$F(1,s)=\frac{1}{s^2}\,; \quad A\cosh\sqrt{s^\alpha-\lambda}+\frac{\mu G(s)}{s^\alpha-\lambda}+\frac{\beta}{s(s^\alpha-\lambda)}=\frac{1}{s^2}$$

$$A=\frac{1}{s^2\cosh\sqrt{s^\alpha-\lambda}}-\frac{\mu G(s)}{(s^\alpha-\lambda)\cosh\sqrt{s^\alpha-\lambda}}-\frac{\beta}{s(s^\alpha-\lambda)\cosh\sqrt{s^\alpha-\lambda}}$$

At this point, assuming, $(\alpha=0.5)$, relation (4.2) takes the following form:

$$F(x,s)=\frac{\cosh\sqrt{\sqrt{s}-\lambda}x}{s^2\cosh\sqrt{\sqrt{s}-\lambda}}+\mu\frac{G(s)}{\sqrt{s}-\lambda}\left(\frac{\cosh\sqrt{\sqrt{s}-\lambda}-\cosh\sqrt{\sqrt{s}-\lambda}x}{\cosh\sqrt{\sqrt{s}-\lambda}}\right)$$
$$-\frac{\beta\cosh\sqrt{\sqrt{s}-\lambda}x}{s(\sqrt{s}-\lambda)\cosh\sqrt{\sqrt{s}-\lambda}}+\frac{\beta}{s(\sqrt{s}-\lambda)} \tag{4.3}$$

now, we take the inverse Laplace transform of relation (4.3) term wise as follows.

First, we invert the following term:

$$H_1(s)=\frac{\cosh\sqrt{\sqrt{s}-\lambda}x}{s^2\cosh\sqrt{\sqrt{s}-\lambda}}=\frac{1}{s}\times\frac{1}{\sqrt{s}}\left\{\frac{\cosh\sqrt{\sqrt{s}-\lambda}x}{\sqrt{s}\cosh\sqrt{\sqrt{s}-\lambda}}\right\}. \tag{4.4}$$

By setting $F_1(\sqrt{s})=\dfrac{\cosh\sqrt{\sqrt{s}-\lambda}x}{\sqrt{s}\cosh\sqrt{\sqrt{s}-\lambda}}$ and using corollary (2.1) one has:

$$F_1(s)=\frac{\cosh\sqrt{s-\lambda}x}{s\cosh\sqrt{s-\lambda}} \qquad ;\ s\cosh\sqrt{s-\lambda}=0\to\begin{cases}s=0\\ \sqrt{s_k-\lambda}=(\frac{2k+1}{2})\pi i\,;k\in Z\end{cases}$$

Let us calculate the residue at $s=0$, that is:

$$\mathrm{Res}(F_1(s)\exp(st);s=0)=\frac{\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}}$$

and the residue at $s=s_k$, $k=0,1,2,3,\ldots$

$$\mathrm{Res}(F_1(s)\exp(st);\, s=s_k)=\frac{(-1)^k(2k+1)\pi\cos(\frac{2k+1}{2}\pi x)\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)t)}{\lambda-(\frac{2k+1}{2})^2\pi^2};(k\in Z)$$

therefore:

$$L^{-1}\{\frac{F_1(\sqrt{s})}{\sqrt{s}}\}=\int_0^\infty f(v)\frac{\exp(-\frac{v^2}{4t})}{\sqrt{\pi t}}dv$$

$$=\frac{2}{\sqrt{\pi t}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi\cos(\frac{2k+1}{2}\pi x)}{\lambda-(\frac{2k+1}{2})^2\pi^2}\int_0^\infty \exp(-\frac{v^2}{4t}+(\lambda-(\frac{2k+1}{2})^2\pi^2)v)dv+\frac{1}{\sqrt{\pi t}}\frac{\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}}\int_0^\infty \exp(-\frac{v^2}{4t})dv$$

$$=\frac{\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}}+\frac{2}{\sqrt{\pi t}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi\ \cos(\frac{2k+1}{2}\pi x)}{\lambda-(\frac{2k+1}{2})^2\pi^2}\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2t)\int_0^\infty \exp(-(\frac{v}{2\sqrt{t}}-(\lambda-(\frac{2k+1}{2})^2\pi^2)\sqrt{t})^2)dv$$

Now, by letting $w=-\left(\frac{v}{2\sqrt{t}}-(\lambda-(\frac{2k+1}{2})^2\pi^2)\sqrt{t}\right)$ in the integral above ,we get:

$$L^{-1}\{\frac{F_1(\sqrt{s})}{\sqrt{s}}\}=$$

$$\frac{\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}}+2\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi}{\lambda-(\frac{2k+1}{2})^2\pi^2}\cos(\frac{2k+1}{2}\pi x)\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2t)\,Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t})$$

Finally, we obtain the inverse Laplace transform of the $H_1(s)$ as follows:

$$L^{-1}\{\frac{1}{s}\frac{1}{\sqrt{s}}(\frac{F_1(\sqrt{s})}{\sqrt{s}})\}$$

$$=\int_0^t\{\frac{\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}}+2\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi}{\lambda-(\frac{2k+1}{2})^2\pi^2}\cos(\frac{2k+1}{2}\pi x)\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2t)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t})\}dt$$

$$=\frac{t\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}}+2\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi}{\lambda-(\frac{2k+1}{2})^2\pi^2}\cos(\frac{2k+1}{2}\pi x)\int_0^t\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2t)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t})dt$$

For the second term in (4.3) we can rewrite it as follows:

$$\mu\frac{G(s)}{\sqrt{s}-\lambda}\{\frac{\cosh\sqrt{\sqrt{s}-\lambda}-\cosh\sqrt{\sqrt{s}-\lambda}x}{\cosh\sqrt{\sqrt{s}-\lambda}}\}$$

$$=\mu(sG(s)-g(0))\frac{1}{\sqrt{s}}\{\frac{\cosh\sqrt{\sqrt{s}-\lambda}-\cosh\sqrt{\sqrt{s}-\lambda}x}{\sqrt{s}(\sqrt{s}-\lambda)\cosh\sqrt{\sqrt{s}-\lambda}}\}+ \tag{4.5}$$

$$+\mu g(0)\{\frac{\cosh\sqrt{\sqrt{s}-\lambda}-\cosh\sqrt{\sqrt{s}-\lambda}x}{s(\sqrt{s}-\lambda)\cosh\sqrt{\sqrt{s}-\lambda}}\}$$

Similarly, by using corollary (2.4) we get:

$$F_2(\sqrt{s}) = \frac{\cosh\sqrt{\sqrt{s}-\lambda} - \cosh\sqrt{\sqrt{s}-\lambda}\,x}{\sqrt{s}\,(\sqrt{s}-\lambda)\cosh\sqrt{\sqrt{s}-\lambda}}$$

$$L^{-1}\{F_2(s)\} = \frac{\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}}{\lambda\cos\sqrt{\lambda}} + \frac{\exp(\lambda t) - \exp(\lambda t)}{\lambda} +$$

$$+8\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\left\{\cos(\frac{2k+1}{2}\pi) - \cos(\frac{2k+1}{2}\pi x)\right\}. \quad (4.6)$$

Following the same procedure, one gets:

$$L^{-1}\{\frac{F_2(\sqrt{s})}{\sqrt{s}}\} = \frac{\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}}{\lambda\cos\sqrt{\lambda}} +$$

$$+8\sum_{k=0}^{\infty} \frac{(-1)^{k}\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2 t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t}) \quad (4.7)$$

We may calculate the inverse Laplace transform of the second term as follows:

$$L^{-1}\{\mu(sG(s)-g(0))\frac{F_2(\sqrt{s})}{\sqrt{s}} + \mu g(0)\frac{F_2(\sqrt{s})}{\sqrt{s}}\} = \mu g'(t) * L^{-1}\{\frac{F_2(\sqrt{s})}{\sqrt{s}}\} + \mu g(0)L^{-1}\{\frac{F_2(\sqrt{s})}{\sqrt{s}}\} =$$

after simplifying the above term, we get:

$$= \mu\frac{\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}}{\lambda\cos\sqrt{\lambda}}g(t)$$

$$+8\mu\sum_{k=0}^{\infty}\frac{(-1)^k}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)\int_0^t g'(t-\eta)\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2\eta)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{\eta})d\eta$$

$$+8\mu g(0)\sum_{k=0}^{\infty}\frac{(-1)^k\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2 t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t}$$

By following the same procedure, and using Effros theorem for the term:

$$F_3(s) = \frac{\cosh\sqrt{\sqrt{s}-\lambda}\,x}{\sqrt{s}\,(\lambda-\sqrt{s})\cosh\sqrt{\sqrt{s}-\lambda}} \quad (4.8)$$

one gets,

$$L^{-1}\{F_3(s)\} = \frac{\cos\sqrt{\lambda}x}{\lambda\cos\sqrt{\lambda}} - \frac{\exp(\lambda t)}{\lambda} + 8\sum_{k=0}^{\infty}\frac{(-1)^k\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)$$

$$L^{-1}\{\frac{F_3(\sqrt{s})}{\sqrt{s}}\} = \frac{\cos\sqrt{\lambda}x}{\lambda\cos\sqrt{\lambda}} - \frac{1}{\lambda}\exp(\lambda^2 t)\,Erfc(-\lambda\sqrt{t})$$

$$+8\sum_{k=0}^{\infty}\frac{(-1)^k\ \exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2 t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t}) \quad (4.9)$$

Furthermore, for the term:

$$L^{-1}\{\frac{1}{s(\sqrt{s}-\lambda)}\} = \frac{1}{\lambda}\exp(\lambda^2 t)Erfc(-\sqrt{\lambda t}) - \frac{1}{\lambda}$$

finally, the inverse Laplace transform of (4.3) is:

$$u(x,t) = \frac{t\cosh\sqrt{\lambda}x}{\cosh\sqrt{\lambda}} + \mu\frac{\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}}{\lambda\cos\sqrt{\lambda}}g(t)$$

$$+2\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi}{\lambda-(\frac{2k+1}{2})^2\pi^2}\cos(\frac{2k+1}{2}\pi x)\int_0^t \exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2 t)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t})dt$$

$$+8\mu\sum_{k=0}^{\infty}\frac{(-1)^k}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)\int_0^t g'(t-\eta)\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2\eta)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{\eta})d\eta$$

$$+8\mu g(0)\sum_{k=0}^{\infty}\frac{(-1)^k\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2 t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)Erfc((-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t})$$

$$+\frac{\beta\cos\sqrt{\lambda}x}{\lambda\cos\sqrt{\lambda}}(\int_0^1 w^2 h(w)dw) - \frac{\beta}{\lambda}\exp(\lambda^2 t)Erfc(-\lambda\sqrt{t})$$

$$+8\beta\sum_{k=0}^{\infty}\frac{(-1)^k\exp((\lambda-(\frac{2k+1}{2})^2\pi^2)^2 t)}{(\lambda-(\frac{2k+1}{2})^2\pi^2)(2k+1)\pi}\cos(\frac{2k+1}{2}\pi x)Erfc(-\lambda+(\frac{2k+1}{2})^2\pi^2)\sqrt{t} + \frac{\beta}{\lambda}\left[\exp(\lambda^2 t)\ Erfc(-\lambda\sqrt{t}) - 1\right]$$

5. Conclusions

Engineering and other areas of sciences can be successfully modeled by the use of fractional derivatives. That is because of the fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus.

By the present research, we considered the systems of time fractional differential equation (time fractional in the Caputo sense). The transform method provides powerful method for analyzing linear systems.We note that within such a new frame work as we have described and developed in this article, the extensive usage of the integral transform method opens up new and powerful possibilities, which be more deeply explored in the future publications.

References

1. *Aghili A., Zeinali H.,* 'Advances in Laplace type integral transforms with applications', *Indian Journal Science & Technology,* ***7*** *(2014), 877- 890.*
2. *Aghili A., Zeinali H.,* 'Solution to time fractional wave equation in the presence of friction via integral transform', *Commun. Applied Nonlinear Analysis,* ***21*** *(2014), 67-88.*
3. *Aghili A., Zeinali H.,* 'Integral transform method for solving Volterra singular integral equations and non homogenous time Fractional PDEs', *Gen. Math. Notes,* ***14*** *(2013), 6-20..*
4. *Kilbas A., Trujillo J.J.,* 'Differential equation of fractional order, Methods, results and problems II', *Appl. Analysis.,* ***81*** *(2002), 435-493.*
5. *Podlubny I., 'Fractional differential Equation',* Academic press, San Diego, CA, *1999.*
6. *Samko G., Kilbas A., Marichev O., 'Fractional integrals and derivatives: Theory and applications',* Gordon & Breach, Amsterdam, *1993.*
7. *Beyer H., Kempfle S.,* 'Definition of physically consistent damping laws with fractional derivatives', *Z. Angew Math. Mech.,* ***75*** *(1995), 623-635.*
8. *Huang H., Liu F.,*'The time Fractional diffusion equation and fractional advection dispersion equation, *ANZIAM J.,* ***46*** *(2005), 1-14.*
9. *Schneider W., Wyss W.,* 'Fractional diffusion and wave equations', *J. Math. Phys.,* ***30*** *(1989), 134-144.*