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The Behavior of Discontinuous Kernel of Mixed Integral Equation

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Abstract
By the current research, a singular integral equation of mixed type (Volterra-Fredholm integral equation) in position and time is considered. The Fredholm integral term is considered in position $L_2[-1,1]$, and has a general singular kernel, while the Volterra integral term is considered in time, in the class $C[0,T]$, $T<1$. Using a numerical method we have a finite system of Fredholm integral equations (SFIE) of the second kind. This system is numerically solved, by using a Chebyshev polynomials method. Much new information for the physical meaning of this work are discussed and obtained.

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Mixed Integral Equation (MIE), Singular Integral Equation, Logarithmic Kernel, Linear System, Chebyshev Polynomials (CP).

1. Introduction

Singular integral equations have received considerable interest in mathematical applications in different areas of science, for example see [1]-[6]. The solution of these problems can be obtained analytically or numerically, see [7]-[10].

Consider the following mixed integral equation (MIE):

$$\int_0^t\int_{-1}^1 k\left(\left|\frac{y-x}{\lambda}\right|\right)F(t,\tau)\varphi(y,\tau)dy\,d\tau+\int_0^t G(t,\tau)\varphi(x,t)d\tau=\pi\theta[\gamma(t)-f_r(x)]=f(x,t),$$

$$(|x|<1,\quad \lambda\in(0,\infty),\quad \theta=G(1-\nu)^{-1},\ f_r(x)\in L_2[-1,1]),\tag{1.1}$$

$$k(z)=\int_0^\infty\frac{L(u)\cos u\,z}{u}\,du\,,L(u)=\frac{u+m}{1+u},\ z=(\frac{y-x}{\lambda}),\ (m\geq 1).\tag{1.2}$$

under the condition

$$\int_{-1}^1\varphi(x,t)dx=P(t),\ t\in[0,t],\ T<1.\tag{1.3}$$

The MIE (1.1) with the badly kernel of position (1.2), under the pressure condition (1.3), is investigated from the contact problem of an elastic material of a strip (G_1,ν_1) of thickness h that occupies a region $0\leq y\leq h$, and lies without friction on an elastic surface (G_2,ν_2) of equation $f_r(x)\in L_2[-1,1]$. Here $G_i;\nu_i$, $i=1,2$ are called the displacement magnitude and Poisson's ratio for the upper and lower surface respectively. Also, we consider a rigid rectangular stamp of length 2a impressed into the boundary of the strip $y=h$ by a variable force $P(t)$, $t\in[0,T]$ that causes displacement $\gamma(t)$ against the force of material of the contact region $F(t,\tau)$. In addition, we consider that the contact region has the resistance force $G(t,\tau)$ for all $t,\tau \in [0,T]$, $T<1$.

In order to guarantee the existence of a unique solution of eq. (1.1), under the pressure condition (1.3), we assume the following conditions:

(i) The kernel of the position $k\left(\left|\frac{x-y}{\lambda}\right|\right)$ satisfies the Fredholm condition

$$\left\{\int_{-1}^{1}\int_{-1}^{1} k^2\left(\left|\frac{y-x}{\lambda}\right|\right)dxdy\right\}^{1/2} = A, \quad A \text{ is a constant.}$$

(ii) The two kernels of time $F(t,\tau)$ and $G(t,\tau)$ for $t,\ \tau \in [0,T]$, $T<1$ belong to the class $C[0,T]$ and satisfies $\left|F(t,\tau)\right| \le B$, $\left|G(t,\tau)\right| \le D$, $B,\ D$ are constants.

(iii) The given function $f(x,t)$ with its partial derivatives is continuous in the space $L_2[-1,1]\times C[0,T]$ and its norm is defined as $\|f\| = \max\limits_{0\le t\le T}\left\{\int_{-1}^{1} f^2(x,t)dx\right\}^{1/2}$.

(iv) The unknown function $\phi(x,t)$ satisfies Lipschitz condition with respect to its first argument and Holder to the second argument.

In the remainder of this work, the general kernel of position, using suitable assumptions and method, will be transformed to a logarithmic kernel form. Then, by using a suitable numerical method, the MIE will be reduced to a linear system of Fredholm integral equations (SFIE) of the second kind with logarithmic kernel. By using Chebyshev Polynomials (CP), the solution of the linear SFIE will be solved. The convergence of the linear SFIE will be discussed. Finally numerical results are obtained and the error estimate is computed.

2. The Behavior of Position Kernel

The function $L(u)$ of eq. (1.2) is continuous and positive, for $u \in (0,\infty)$, then it satisfies the following asymptotic equalities:

$$L(u) = m-(m-1)u + O(u^3),\ \ u \to 0,$$

$$L(u) = 1-\frac{m-1}{u} + O(u^{-2}), \quad u\to\infty, \quad m \ge 1. \tag{2.1}$$

When $m=1$ in (4) and $\lambda\to\infty$ in (1), such that the term $(\frac{y-x}{\lambda})$ is very small, we have from [3], [9] that:

$$\int_0^\infty \frac{\cos uz}{u}du = -\ln|x-y| + d \ , \qquad \left(d = \ln\frac{4\lambda}{\pi}\right). \tag{2.2}$$

In this case, the kernel of position takes a logarithmic function form.

If $u\to 0$ in (2.1), then by considering the first and the second approximation of $L(u)$ and using the following famous relations: [3]

$$\frac{1}{\pi}\int_0^\infty \cos vxdv = \delta(x), \quad \delta(x) \text{ is the Durak function,}$$

$$\int_a^b \phi(y)\delta(y-x)dy = \begin{cases} 0, & b < x < a, \\ \tfrac{1}{2}[h(x-a)+h(x+a)], & a < x < b. \end{cases} \tag{2.3}$$

The integral equation (1.1) will be reduced to the following mixed integral equation:

$$\int_0^t H(t,\tau)\phi(x,\tau)\,d\tau - \lambda\int_0^t\int_{-1}^1 F(t,\tau)\ln|x-y|\phi(y,\tau)\,dy\,d\tau = g(x,t)\,. \tag{2.4}$$

Here, we assumed:

$$H(t,\tau) = \frac{G(t,\tau)}{m-1} - F(t,\tau), \qquad (m>1), \tag{2.5}$$

and

$$g(x,t) = \frac{f(x,t)}{m-1} - \lambda d\int_0^t P(\tau)F(t,\tau)d\tau, \left(d = \ln\frac{4\lambda}{\pi}, \quad \lambda = \frac{m}{m-1}, \quad m>1\right). \tag{2.6}$$

From the value of λ, we obtain the physical meaning between λ, m and the logarithmic kernel, where we have: [9]

$$\left\{\int_{-1}^1\int_{-1}^1 \ln^2|x-y|dx\,dy\right\}^{1/2} < \frac{1}{\lambda}, \qquad \lambda = 1+\frac{1}{m}+\frac{1}{m^2}+\ldots.. \tag{2.7}$$

Also, from (2.5), we can establish that for large values of m the total resistance force $H(t,\tau)$ decreases that is, the external resistance force, for $m\to\infty$, is not a available and the total resistance, in this case, is the resistance of material only. The physical meaning of the logarithmic kernel with some applications can be found in [11]. In addition, some different methods for solving the integral equations with logarithmic kernel have been discussed in [12] - [15].

3. System of Fredholm Integral Equations

To obtain the solution of (2.4), under the condition (1.3), we divide the interval $[0,t]$, $0\le t\le T<1$ as $0=t_o<t_1<t_2<\ldots<t_N=T$, where we let $t=t_i$, $0\le i\le N$. Then we approximate the Volterra integral terms, after using the quadrature formula w_j, $j=0,1,2,\ldots,N$, see [13], [14] to have:

$$\sum_{j=0}^{i}\omega_j H_{i,j}\,\phi_j(x) - \lambda\sum_{j=0}^{i}\omega_j F_{i,j}\int_{-1}^{1}\ln|x-y|\phi_j(y)dy + O\left(\hbar_i^{\ell+1}\right) = g_i(x), \tag{3.1}$$

under the condition:

$$\int_{-1}^{1} \phi_i(x)dx = P_i, \quad (0 \le i \le N). \tag{3.2}$$

where, $\hbar_i = \max_{0 \le j \le N} h_j$, $h_j = t_{j+1} - t_j$, ω_j are called the characteristic points and ℓ is the quadratic coefficients. The values of ω_j and ℓ depend on the number of derivatives of $F(t,\tau)$ and $G(t,\tau)$. More information for the characteristic points and the quadratic coefficients are found in [9], [10].

The above formula (3.1) can be adapted in the following form:

$$\mu_i\, \phi_i(x) - \ell_i \int_{-1}^{1} \ln|x-y|\, \phi_i(y)\, dy = m_i(x), \; (\mu_i = \omega_i H_{i,i}, \;\; \ell_i = \lambda \omega_i F_{i,i}). \tag{3.3}$$

where:

$$m_i(x) = g_i(x) + \sum_{j=0}^{i-1} \omega_i H_{i,j} \phi_j(x) + \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} \int_{-1}^{1} \ln|x-y|\, \phi_j(y) dy$$

The formula (3.3) represents a linear system of Fredholm integral equations (SFIE) of the second kind with logarithmic kernel, and its solution can be obtained by using the recurrence relations.

For $i = 0$, we have

$$\mu_o \phi_o(x) - \ell_o \int_{-1}^{1} \ln|x-y|\, \phi_o\, dy = g_o(x). \tag{3.4}$$

Differentiating (3.4) with respect to x, one has:

$$\mu_o \frac{d\phi_o(x)}{dx} - \ell_o \int_{-1}^{1} \frac{\phi_o(y)}{x-y} dy = h(x), \quad h(x) = \frac{dg_o(x)}{dx} \tag{3.5}$$

Here, in (3.5), $\int_{-1}^{1}$ denotes integration in the sense of Cauchy Principal value, the unknown function $\phi_o(x)$ with its derivatives are continuous in $L_2[-1,1]$, $x \in [-1,1]$. Taking the transformations $y = 2u - 1$, $x = 2v - 1$, the integro differential (3.5), on noting the difference notations, becomes:

$$\frac{d\Theta}{dv} - \tilde{\lambda} \int_0^1 \frac{\Theta(u)}{v-u} du = z(v). \tag{3.6}$$

This equation has appeared in both combined infrared gaseous radiations and molecular conduction, where $\tilde{\lambda}$, in (3.6), is known as the radiation conduction number for the large path length limit, and represents the single parameter of the dimensionless system. The formula (3.6) is considered and discussed with its special cases and solved, when $z(v) = \frac{1}{2} - v$, under the conditions $\Theta(0) = \Theta(1) = 0$ by Frankel in [2], where Θ represents the unknown temperature.

4. Chebysev Polynomials

In order to obtain the solution of (3.3), we use the Chebysev Polynomials (CP) with its famous relations. For this, we write the unknown functions $\phi_i(x)$, for each values $i, 0 \le i \le N$, in the form of the weight function of CP of the first kind $\left(1-x^2\right)^{\frac{-1}{2}}$ multiplying by unknown function $B_i(x), \quad 0 \le i \le N$. Then we write $B_i\left(x\right)$ in the CP to get:

$$\phi_i\left(x\right) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} a_n^{(i)} T_n\left(x\right) \tag{4.1}$$

Here, $T_n\left(x\right)$ are the **CPs** and $a_n^{(i)}$ are the unknown coefficients of $T_n\left(x\right)$, which will be determined.

It is difficult to obtain the solution of equation (3.3) numerically in the form of equation (4.1). For this, the formula (4.1) can be truncated to:

$$\phi_i^{(M)} = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{M} a_n^{(i)} T_n\left(x\right). \tag{4.2}$$

By using (4.2) and the following famous relationships: [10]

$$\int_{-1}^{1} \frac{\ln|x-y| T_n\left(y\right) dy}{\sqrt{1-y^2}} = \begin{cases} \pi \ln 2, & n=0 \\ \frac{\pi}{n} T_n\left(x\right), & n \ge 1, \end{cases} \tag{4.3}$$

then formula (3.3) yields:

$$\mu_i \sum_{n=0}^{M} \frac{a_n^{(i)} T_n\left(x\right)}{\sqrt{1-x^2}} - \ell_i \begin{cases} \pi\, a_0^{(i)} \ln 2, & n=0 \\ \pi \sum_{n=1}^{M} \frac{a_n^{(i)} T_n\left(x\right)}{n}, & n \ge 1 \end{cases}$$

$$= \sum_{n=0}^{M} \frac{g_n^{(i)} T_n\left(x\right)}{\sqrt{1-x^2}} + \sum_{j=0}^{i-1} \sum_{n=0}^{M} \omega_j H_{i,j} \frac{a_n^{(i)} T_n\left(x\right)}{\sqrt{1-x^2}} - \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} \begin{cases} \pi \ln 2\, a_0^{(i)}, & n=0 \\ \pi \sum_{n=1}^{M} \frac{a_n^{(i)} T_n\left(x\right)}{n}, & n \ge 1 \end{cases} \tag{4.4}$$

$$\left(g_n^{(i)} = \frac{2}{\pi} \int_{-1}^{1} \frac{g_{(i)}\left(x\right) T_n\left(x\right)}{\sqrt{1-x^2}} dx\right).$$

The formula (4.4) leads us to discuss two important cases, the first case at $n=0$ and the second at $n \ge 1$. For the first case, one has:

$$\mu_i \frac{a_0^{(i)}}{\sqrt{1-x^2}} - \pi \ell_i a_0^{(i)} \ln 2 = \frac{g_0^{(i)}}{\sqrt{1-x^2}} + \sum_{j=0}^{i-1} \omega_j H_{i,j} \frac{a_0^{(j)}}{\sqrt{1-x^2}} - \lambda \pi \ln 2 \sum_{j=0}^{i-1} \omega_j F_{i,j} a_0^{(j)} . \quad (4.5)$$

Integrating (4.5) with respect to x from -1 to 1 we get:

$$a_0^{(i)} = \frac{1}{(\mu_i - 2\ell_i \ln 2)} [2g_0^{(i)} + \sum_{j=0}^{i-1} \omega_j H_{i,j} a_0^{(j)} + 2\ln 2 \, \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} a_0^{(j)}],$$
$$(\mu_i \neq 2 \ell_i \ln 2, \quad 0 \leq i \leq N). \quad (4.6)$$

Also, for the second case $n \geq 1$, formula (4.4) yields:

$$\mu_i \sum_{n=1}^{M} \frac{a_n^{(i)} T_n(x)}{\sqrt{1-x^2}} - \pi \ell_i \sum_{n=1}^{M} \frac{a_n^{(i)} T_n(x)}{n} = \sum_{n=1}^{M} \frac{g_n^{(i)} T_n(x)}{\sqrt{1-x^2}} + \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j H_{i,j} \frac{a_n^{(j)} T_n(x)}{\sqrt{1-x^2}}$$
$$+ \pi \lambda \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j F_{i,j} \frac{a_n^{(j)} T_n(x)}{n}. \quad (4.7)$$

Multiplying both sides of (4.7) by the term $T_m(x)dx$ and integrating from $x = -1$ to $x = 1$, and then by using the following famous relations: [14]

$$T_n(x) T_m(x) = \tfrac{1}{2}\left[T_{n+m}(x) + T_{|n-m|}(x)\right],$$

$$\int_{-1}^{1} T_n(x) dx = \begin{cases} 0 & , n = 1, 3, 5, \ldots \\ \dfrac{2}{1-n^2} & , n = 0, 2, 4, \ldots \end{cases} , \quad (4.8)$$

we obtain the following linear system of algebraic equations:

$$\mu_i a_m^{(i)} - 2\ell_i \sum_{n=1}^{M} \frac{A_{n,m}}{n} a_n^{(i)} = C_m^{(i)}, \qquad (m \geq 1, \ 0 \leq i \leq N), \quad (4.9)$$

where:

$$A_{n,m} = \begin{cases} \dfrac{1}{1-(n+m)^2} + \dfrac{1}{1-(n-m)^2} , & (n+m) \text{ even} \\ 0 & (n+m) \text{ odd}, \end{cases} \quad (4.10)$$

and:

$$C_m^{(i)} = g_m^{(i)} + \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j H_{i,j} a_n^{(j)} + 2\lambda \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j F_{i,j} \frac{A_{n,m}}{n} a_n^{(j)}. \quad (4.11)$$

5. Covergence of the Linear Algebraic System

In order to prove this, we state the following:

Theorem (3.1) For $M \to \infty$, the infinite linear system of algebraic equations (4.9) is bounded and has a unique solution.

Proof: Consider the metric space of real bounded set k is defined as:

$$\rho(x_1, x_2) = \sup_{\ell} \left| x_\ell^{(1)} - x_\ell^{(2)} \right|, \qquad x^{(p)} = \{x_\ell^{(p)}\}_{\ell=1}^{\infty} \quad (p = 1, 2). \tag{5.1}$$

Furthermore, consider an operator $K \subset \{k\}$ such that:

$$y = Kx \ , \ \ y = \{y_\ell\}_{\ell=1}^{\infty}, \qquad x = \{x_\ell\}_{\ell=1}^{\infty}. \tag{5.2}$$

Also, for $C = \{C_\ell\}_{\ell=1}^{\infty} \in k$, we assume the bounded and continuous as infinite system: [12]

$$y_\ell = C_\ell + \gamma \sum_{n=1}^{\infty} K_{n,\ell} x_n \ , \qquad \gamma \text{ is a constant.} \tag{5.3}$$

Hence, under the condition $\sup_{n,\ell} \left| K_{n,\ell} \right| < \infty$, the operator K satisfies $K : k \to k$, that is, the system (5.3) has a unique solution.

So, by the same way, when $M \to \infty$ we rewrite (4.9) to be:

$$a_m^{(i)} - \tilde{\lambda}_i \sum_{n=1}^{\infty} R_{n,m} a_n^{(i)} = L_m^{(i)}, \quad \left(\tilde{\lambda}_i = \frac{2\ell_i}{\mu_i}, \ \ L_m^{(i)} = C_m^{(i)} \mu_i^{(-1)}, \quad R_{n,m} = \frac{A_{n,m}}{n} \right) \tag{5.4}$$

.

Furthermore, we assume the following assumption, $S_m = \tilde{\lambda}_i \sum_{n=1}^{\infty} \left| R_{n,m} \right| = \tilde{\lambda}_i \sum_{n=1}^{\infty} \frac{1}{n} \left| A_{n,m} \right|$, then apply Cauchy inequality, we get:

$$S_m = \tilde{\lambda}_i \left| \sum_{n=1}^{\infty} \frac{1}{n^2} \right|^{\frac{1}{2}} \left| \sum_{n=1}^{\infty} (A_{n,m})^2 \right|^{\frac{1}{2}} \leq 1. \tag{5.5}$$

By using the values of the convergence series $[\sum_{n=1}^{\infty} (A_{n,m})^2] \to 1, \ as \ \ m \to \infty$, we get $\ell_i \leq 0.39 \mu_i$ for all values of i, $0 \leq i \leq N$.

6. Numerical Results

For the analytical solution of equation (3.4) $\phi(x, t) = x^2 t$, we assume:

$$m=2,\quad \lambda=1,\quad F(t,\tau)=\tau^2,\quad G(t,\tau)=\tau^3,\quad k\left(\left|\frac{x-y}{\lambda}\right|\right)=\ln|y-x|,$$

Hence, one has $H(t,\tau)=\tau^3-\tau^2$, and :

$$g(x,t)=\int_0^t[\tau^4-\tau^3]x^2\,d\tau-\lambda\int_0^t\int_{-1}^1\tau^3\ln|y-x|\,y^2\,dy\,d\tau\,.$$

By integrating the above equation we obtain:

$$g(x,t)=\frac{1}{5}t^5-\frac{1}{4}t^4\left(x^2+I(x)\right),\quad I(x)=\frac{1}{3}\left(1-x^3\right)\ln|1-x|+\frac{1}{3}\left(1+x^3\right)ln|1+x|-\frac{2}{9}-\frac{2}{3}x^2$$

The corresponding numerical solution of the linear algebraic system (4.9) for the previous data is obtained through the following table:

Table 1

t	t^2	t^3	x	x^2	$\phi=x^2t^2$	$g_{ana.}(x,t)$	$g_{num}(x,t)$	Error
0	0	0	-0.9	0.81	0	0	0	0
0.1	0.01	0.001	-0.8	0.64	0.0064	-4.5E-06	-4.4E-06	-1.3E-07
0.2	0.04	0.008	-0.6	0.36	0.0144	3.83E-05	3.82E-05	7.06E-08
0.3	0.09	0.027	-0.4	0.16	0.0144	0.000619	0.000617	1.64E-06
0.4	0.16	0.064	-0.2	0.04	0.0064	0.00304	0.00302	2.01E-05
0.5	0.25	0.125	0	0	0	0.009722	0.009720	2.22E-06
0.6	0.36	0.216	0.2	0.04	0.0144	0.024512	0.024510	1.97E-06
0.7	0.49	0.343	0.4	0.16	0.0784	0.052601	0.052600	9.99E-07
0.8	0.64	0.512	0.6	0.36	0.2304	0.098571	0.098570	9.24E-07
0.9	0.81	0.729	0.8	0.64	0.5184	0.157635	0.157634	1.1E-06
1	1	1	0.9	0.81	0.81	0.230803	0.230802	1.28E-06

We note that from the last column in the previous table that the error between the analytical solution and the numerical one is very small which assure the accuracy of the numerical and the analytical techniques considered in this paper.

6. Conclusions

From the previous discussion, we can establish the following:

- The mixed integral equation of the first kind, in time and position, after using quadratic method leads to a system of integral equations of the second kind in position.
- The integro-differential equation with Cauchy kernel, under certain conditions, that appears in both combined infrared gaseous radiations and molecular conduction is considered as special case of this work.

- The system of integral equations, using Chebysev polynomials leads to a linear algebraic system. The convergence of the system is discussed.

Future work:

The discussion of the mixed integral equation when the term of position in the nonlinear case and the term of time in the linear case will be considered.

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